Renewal Theory, Disordered Systems, and Stochastic PDEs

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Workshop on Polymers and self-avoiding walks

A celebration of Philippe Carmona's career

Overview

1. Renewal Theory : ultra-heavy tailed renewal processes

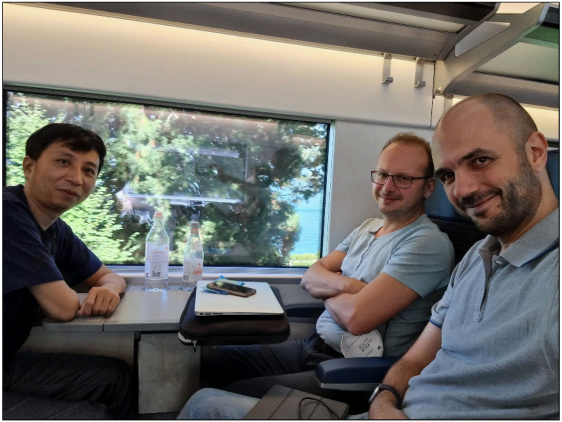
2. Disordered Systems : $2d$ Directed Polymer in Random Environment

3. Stochastic PDEs : $2d$ Stochastic Heat Equation

Main references:

- [CSZ19] *The Dickman subordinator, renewal theorems, and disordered systems*, EJP (2019)
- [CSZ21+] *The critical 2d Stochastic Heat Flow*, Inventiones Math. (to appear)

Based on joint works with

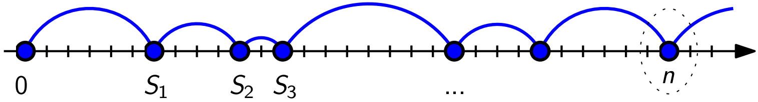


Rongfeng Sun (NUS) and Nikos Zygouras (Warwick)

Renewal process

Random walk $S_k := X_1 + X_2 + \dots X_k$ with positive increments

(X_i) i.i.d. $\quad X_i \in \mathbb{N} = \{1, 2, \dots\}$ $\quad$ aperiodic



Renewal function $\quad u(n) := \mathrm{P}(S \text{ visits } n) = \sum_{k \geq 0} \mathrm{P}(S_k = n)$

Renewal Theorem (Erdos, Feller, Pollard 1949)

$$\lim_{n \to \infty} u(n) = \frac{1}{\mathrm{E}[X]}$$ also when $\mathrm{E}[X] = \infty$

Heavy tails

When $\mathrm{E}[X] = \infty$ we have $u_n \to 0$. At which rate?

Tail Assumption

$$\mathrm{P}(X > n) \underset{n\to\infty}{\sim} \frac{\ell_n}{n^\alpha} \qquad 0 < \alpha < 1 \qquad \ell_\cdot \text{ slowly varying}$$

Theorem

[Garsia, Lamperti 1962] [Doney 1997]

$$u(n) \underset{n\to\infty}{\sim} \frac{c}{\mathrm{E}[X \wedge n]} = \frac{c_\alpha}{\ell_n \, n^{1-\alpha}} \qquad \text{with} \quad c_\alpha := \frac{\sin \pi\alpha}{\pi}$$

+ local assumption for $\alpha \leq \frac{1}{2}$:

$$\mathrm{P}(X = n) \leq C \, \frac{\ell_n}{n^{1+\alpha}}$$

Necessary and sufficient conditions are known [Caravenna, Doney 19]

Ultra-heavy tails

We now focus on the extreme case $\alpha = 0$

$$\mathrm{P}(X = n) = p_n \sim \frac{1}{n}$$

This makes sense via truncation at scale N

$$\mathrm{P}(X^{(N)} = n) = \frac{p_n \, \mathbb{1}_{\{1 \le n \le N\}}}{p_1 + \ldots + p_N} \sim \frac{1}{n} \, \frac{\mathbb{1}_{\{1 \le n \le N\}}}{\log N}$$

Triangular array of renewal processes

$$S_k^{(N)} = X_1^{(N)} + \ldots + X_k^{(N)}$$

Renewal function (exponentially weighted)

$$u^{(N)}(n) = \mathrm{P}(S^{(N)} \text{ visits } n) = \sum_{k \ge 0} \mathrm{P}(S_k^{(N)} = n) \left(1 + \tfrac{\vartheta}{\log N}\right)^k$$

Strong Renewal Theorem

Since $\mathrm{E}[X^{(N)}] \sim \frac{N}{\log N}$ we expect $u^{(N)}(n) \approx \frac{\log N}{N}$ as $n \approx N \to \infty$

Theorem [CSZ19]

$$u^{(N)}(n) \sim \frac{\log N}{N} \, G_\vartheta\left(\tfrac{n}{N}\right) \qquad \text{uniformly for } \delta N \le n \le N$$

where $$G_\vartheta(t) := \int_0^\infty \frac{\mathrm{e}^{(\vartheta-\gamma)s} \, s \, t^{s-1}}{\Gamma(s+1)} \, \mathrm{d}s$$

- Renewal process $S^{(N)} = (S_k^{(N)})_{k\in\mathbb{N}} \underset{N\to\infty}{\longrightarrow}$ Lévy process $Y = (Y_s)_{s\ge 0}$
  (suitably rescaled) "Dickman subordinator"
- $G_\vartheta(t)$ is the renewal function of Y

The Dickman subordinator

Our renewal process $S^{(N)}$ is attracted to a pure jump Lévy process Y

$$\left(\frac{S^{(N)}_{\lfloor s \log N \rfloor}}{N} \right)_{s \geq 0} \xrightarrow[N\to\infty]{d} Y = \left(Y_s \right)_{s \geq 0}$$

called the Dickman subordinator

- Lévy measure $$\nu^Y(\mathrm{d}t) := \frac{1}{t} \, \mathbb{1}_{(0,1)}(t) \, \mathrm{d}t$$
- Explicit density $$\frac{\mathrm{P}(Y_s \in \mathrm{d}t)}{\mathrm{d}t} = \frac{\mathrm{e}^{-\gamma s} \, s \, t^{s-1}}{\Gamma(s+1)} \qquad \text{for } t \in (0,1)$$

$G_\vartheta(t)$ is the (exponentially weighted) renewal function of Y

$$G_\vartheta(t) = \int_0^\infty \mathrm{e}^{\vartheta s} \, \frac{\mathrm{P}(Y_s \in \mathrm{d}t)}{\mathrm{d}t} \, \mathrm{d}s$$

Directed Polymer in Random Environment

Disordered model in statistical mechanics

"random walk interacting with a random medium" (Gibbs)

Introduced in the 1980s to describe interfaces in Ising model

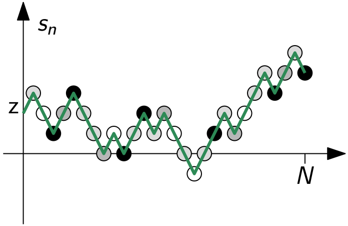
[Imbrie, Spencer JSP 88] [Bolthausen CMP 89]

A stream of mathematical research in the last 25 years

- Localization phenomena
- Super-diffusivity
- KPZ universality class

St. Flour 2016 Lecture notes by Francis Comets

Partition Functions



- $s = (s_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^d$
- $\omega(n, z)$ independent $\mathcal{N}(0, 1)$ (disorder)
- $H_N(s, \omega) := \sum_{n=1}^{N} \omega(n, s_n) \qquad \sim \mathcal{N}(0, N)$

Directed Polymer Partition Functions $(N \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z(N, z) := \frac{\mathrm{E}\left[e^{\beta H_N(s,\omega)}\right]}{e^{\frac{1}{2}\beta^2 N}} = \frac{1}{(2d)^N} \sum_{\substack{s=(s_0,\ldots,s_N) \\ \text{s.r.w. path with } s_0 = z}} e^{\beta H_N(s,\omega)} \frac{e^{\beta H_N(s,\omega)}}{e^{\frac{1}{2}\beta^2 N}}$$

Hidden (but deep) connection to the renewal function $u^{(N)}(n)$!

Moments

The random variables $\big(Z(N,z)\big)_{z\in\mathbb{Z}^d}$ depend on disorder ω

- They are stationary with unit mean:

$$\mathbb{E}\big[Z(N,z)\big] = 1$$

- They are not independent: explicit covariance

$$\mathbb{C}\mathrm{ov}\left[Z(N,z), Z(N,z')\right] = \sum_{1\le\ell\le N} \tilde\beta^2\, q(2\ell, z-z')\cdot v(N-\ell)$$

$q(n,z) := \mathrm{P}(s_n = z)$ SRW transition kernel $\qquad \tilde\beta^2 := \mathrm{e}^{\beta^2} - 1$

$$v(n) = 1 + \sum_{1\le\ell\le n} \tilde\beta^2\, q(2\ell,0) + \sum_{1\le\ell<m\le n} \tilde\beta^2\, q(2\ell,0)\, \tilde\beta^2\, q(2(m-\ell),0) + \ldots$$

Renewal theory

Now fix $d = 2$

$$v(n) = 1 + \sum_{1 \le \ell \le n} \tilde{\beta}^2 \, q(2\ell, 0) + \sum_{1 \le \ell < m \le n} \tilde{\beta}^2 \, q(2\ell, 0) \, \tilde{\beta}^2 \, q(2(m-\ell), 0) + \ldots$$

- Local CLT: $q(2\ell, 0) \sim \frac{1}{\pi} \frac{1}{\ell}$
- Critical rescaling: $\tilde{\beta}^2 = \frac{\pi}{\log N}$

Renewal theory interpretation: $\mathrm{P}(X^{(N)} = \ell) := \tilde{\beta}^2 \, q(2\ell, 0)$

$$v(n) = 1 + \mathrm{P}(S_1^{(N)} \le n) + \mathrm{P}(S_2^{(N)} \le n) + \ldots = \sum_{m=0}^{n} u^{(N)}(m)$$

The renewal function $u^{(N)}(\cdot)$ sheds light on directed polymers

Stochastic Heat Equation

Singular stochastic PDE on $\mathbb{R}^d$

$$\partial_t u(t,x) = \Delta u(t,x) + \beta\, u(t,x)\, \xi(t,x) \qquad \text{(SHE)}$$

$u(0,x) \equiv 1$ for simplicity

$\xi(t,x)$ = space-time white noise

ξ is very irregular $\rightsquigarrow$ product $u\,\xi$ is classically ill-defined

- $(d=1)$ Well-posed via stochastic integration (Ito-Walsh 1980s)
  Also pathwise, via Regularity Structures or Paracontrolled Calculus
- $(d \geq 2)$ No solution theory

Critical 2d Stochastic Heat Equation

Now fix $d = 2$. We regularize the noise

- mollification in space: $\xi_\delta = \xi * g_\delta \quad \delta > 0$
- discretization in space-time: $\xi(t,x) \rightsquigarrow \omega(\frac{n}{N}, \frac{z}{\sqrt{N}}) \quad$ i.i.d. $\mathcal{N}(0,1)$

$\rightsquigarrow$ (SHE) becomes a difference equation on the rescaled lattice $\frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}}{\sqrt{N}}$

Solution $u_N(t,x)$ of discretized SHE is ...

... the Directed Polymer Partition Function $Z(\lfloor tN \rfloor, \lfloor x\sqrt{N} \rfloor)$!

Dicrete Feynman-Kac formula (up to time reversal)

Does $Z(\lfloor tN \rfloor, \lfloor x\sqrt{N} \rfloor)$ admit a non-trivial limit as $N \to \infty$?

Yes, but

- We must look at $Z(\lfloor tN \rfloor, \lfloor x\sqrt{N} \rfloor)$ as a distribution in x

$$Z(\lfloor tN \rfloor, \varphi) := \int_{\mathbb{R}^2} Z(\lfloor tN \rfloor, \lfloor x\sqrt{N} \rfloor)\, \varphi(x)\, \mathrm{d}x \qquad \varphi \in C_c(\mathbb{R}^2)$$

- To ensures convergence of mean and variance of $Z(\lfloor tN \rfloor, \varphi)$

we critically rescale $\beta^2 \sim \frac{\pi}{\log N}\left(1 + \frac{\vartheta}{\log N}\right)$

$\rightsquigarrow$ renewal theory interpretation

Main result

Theorem [CSZ21]

With the critical rescaling

$$\beta^2 = \frac{\pi}{\log N}\left(1 + \frac{\vartheta}{\log N}\right) \qquad \text{for } \vartheta \in \mathbb{R}$$

we have the joint convergence in distribution over $t \geq 0$, $\varphi \in C_c(\mathbb{R}^2)$

$$Z(tN, \varphi) \xrightarrow[N\to\infty]{d} \mathscr{Z}^{\vartheta}(t, \varphi) = \int_{\mathbb{R}^2} \varphi(x)\, \mathscr{Z}^{\vartheta}(t, \mathrm{d}x)$$

The limiting process $\mathscr{Z}^{\vartheta}(t, \mathrm{d}x)$ is called critical $2d$ Stochastic Heat Flow

$\rightsquigarrow$ It is the natural candidate solution of the critical $2d$ (SHE)

Conclusions

Renewal Theory is remarkably useful in a variety of different contexts (also with seemingly exotic ultra heavy tails!)

Non-obvious application to $2d$ Directed Polymers, where renewal theory sheds light on the covariance of the Partition Functions $Z(N, z)$

These solve of a discretized critical $2d$ (SHE) $\rightsquigarrow$ their scaling limits yield

the critical $2d$ Stochastic Heat Flow $(\mathscr{Z}^{\vartheta}(t, \mathrm{d}x))_{t \geq 0}$

Universal process of random measures on $\mathbb{R}^2$ with many explicit features (but several open questions)

Conclusions

The measure $\mathscr{Z}^\vartheta(t, \mathrm{d}x)$ is a.s. singular w.r.t. Lebesgue, i.e. not a function

This is unlike what happens in the $1d$ case [Alberts, Khanin, Quastel 14] or for heavy-tailed disorder [Berger, Chong, Lacoin 21] [Viveros 21]

We actually build scaling limits of point-to-point partition functions

$$\sqrt{N}\, Z(tN, x\sqrt{N}, y\sqrt{N})\, \mathrm{d}x\, \mathrm{d}y \xrightarrow[N\to\infty]{d} \mathscr{Z}^\vartheta(t, \mathrm{d}x, \mathrm{d}y)$$

Polymer endpoint distribution (with averaged initial condition) for critical intermediate disorder $\beta^2 = \frac{\pi}{\log N}$: diffusive, but non Brownian

For subcritical $\beta^2 = (1-\varepsilon)\frac{\pi}{\log N}$: Brownian, but random LLT [Gabriel 21]

Merci

Polynomial chaos

- Simple random walk kernel on $\mathbb{Z}^2$ $\qquad q(n,z) = \mathrm{P}(s_n = z)$
- New i.i.d. centred random variables $\qquad \tilde{\omega}(n,z) := \dfrac{e^{\beta\omega(n,x) - \frac{1}{2}\beta^2} - 1}{\tilde{\beta}}$

Polynomial chaos

Equivalent rewriting of the partition function

$$Z(N,z) = 1 + \tilde{\beta} \sum_{\substack{1 \le \ell \le N \\ x \in \mathbb{Z}^2}} q(\ell,x)\, \tilde{\omega}(\ell,x) + \tilde{\beta}^2 \sum_{\substack{1 \le \ell < m \le N \\ x,y \in \mathbb{Z}^2}} q(\ell,x)\, q(m-\ell, y-x)\, \tilde{\omega}(\ell,x)\, \tilde{\omega}(m,y) + \dots$$

Variance

Scaling limit of the variance

$$\mathbb{V}\mathrm{ar}[Z(N,\varphi)] \;\approx\; \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\, K_N(x,y)\, \varphi(y)\, \mathrm{d}x\, \mathrm{d}y$$

Explicit kernel

$$K_N(x,y) = \tilde{\beta}^2 \sum_{1\le m<n\le N} \mathrm{P}(s_m = \sqrt{N}(x-y)) \cdot u^{(N)}(n-m)$$

$$u^{(N)}(\cdot) = \text{ renewal function } \;\text{(ultra-heavy tailed renewal process)}$$

$$\lim_{N\to\infty} K_N(x,y) \;=\; \pi \iint_{0<s<t<1} \underbrace{g_s(x-y)}_{\text{heat kernel on } \mathbb{R}^2} \;\cdot\; \underbrace{G_\vartheta(t-s)}_{\substack{\text{renewal function of the}\\ \text{Dickman subordinator}}} \;\mathrm{d}s\, \mathrm{d}t$$