The critical 2d Stochastic Heat Flow

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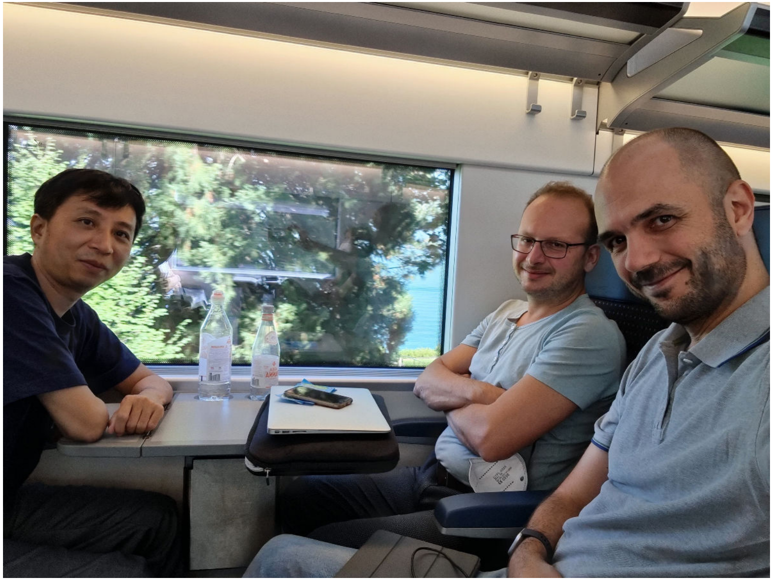
Based on joint works with



Rongfeng Sun and Nikos Zygouras

REFERENCES

- [CSZ 21] F. Caravenna, R. Sun, N. Zygouras
  THE CRITICAL 2D STOCHASTIC HEAT FLOW
  arXiv (2021)

- [CSZ 22] F. Caravenna, R. Sun, N. Zygouras
  THE CRITICAL 2D S.H.F. IS NOT A G.M.C.
  arXiv (2022)

Overview

I. Introduction and main results

II. Ideas and Techniques

III. Conclusions & Perspectives

I. INTRODUCTION AND MAIN RESULTS

The Stochastic Heat Equation

For $t > 0$, $x \in \mathbb{R}^d$:

coupling constant

$$(\text{SHE}) \qquad \begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \beta\, \xi(t,x)\, u(t,x) \\ u(0,x) \equiv 1 \end{cases}$$

- $\xi(t,x)$ "space-time white noise" (δ-correlated Gaussian)

GOAL: Construct the natural candidate solution $u(t,x)$ for $d=2$:

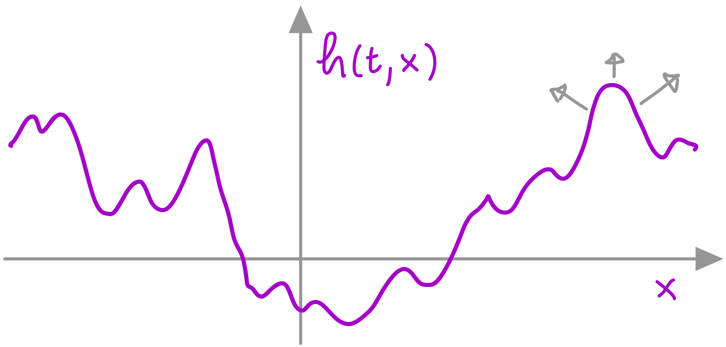
$\rightsquigarrow$ CRITICAL 2d STOCHASTIC HEAT FLOW

THE KARDAR-PARISI-ZHANG EQUATION

[PRL 1986]

If $\xi(t,x)$ is regular, then $h(t,x) := \log u(t,x)$ solves

$$\partial_t h(t,x) = \underbrace{\Delta h(t,x)}_{\text{SMOOTHING}} + \underbrace{|\nabla h(t,x)|^2}_{\perp \text{ GROWTH}} + \beta \underbrace{\xi(t,x)}_{\text{NOISE}} \qquad \text{(KPZ)}$$



If $\xi(t,x)$ is NOT regular?

SHE can help make sense of KPZ

SHE AND KPZ

They are both ill-defined due to singular products

$$\xi(t,x)\, u(t,x) \qquad\qquad |\nabla h(t,x)|^2$$

(No Banach space of functions or distributions to set fixed point)

We regularize / discretize the noise $\xi_\varepsilon(t,x)$ on scale $\varepsilon > 0$

Do regularized/discretized solutions converge as $\varepsilon \downarrow 0$?

$$u_\varepsilon(t,x) \longrightarrow u(t,x) \qquad\qquad h_\varepsilon(t,x) \longrightarrow h(t,x) \quad ?$$

THE CASE $d=1$

- SHE solution $u(t,x)$ is classically well-posed [Ito - Walsh]

- SHE and KPZ well understood for $d=1$ via robust solution theories for "sub-critical" singular PDEs

  - REGULARITY STRUCTURES [Hairer]
  - PARACONTROLLED CALCULUS [Gubinelli, Imkeller, Perkowski]
  - ENERGY SOLUTIONS [Goncalves, Jara]
  - RENORMALIZATION [Kupiainen]

- This breaks down for SHE / KPZ higher dimensions $d \geq 2$

SHE IN THE CRITICAL DIMENSION $d=2$

Formally: if $u(t,x)$ solves SHE, then $\tilde{u}(t,x) := u(\delta^2 t, \delta x)$

$$\partial_t \tilde{u}(t,x) = \Delta \tilde{u}(t,x) + \beta\, \delta^{\frac{2-d}{2}}\, \tilde{\xi}(t,x)\, \tilde{u}(t,x)$$

As $\delta \downarrow 0$, the noise term $\begin{cases} \text{vanishes} & (d<2) \\ \text{stays constant} & (d=2) \\ \text{diverges} & (d>2) \end{cases}$

$d=2$ is CRITICAL DIMENSION for SHE : no solution theory

(no clear physical picture)

DISCRETIZED SHE

($\longrightarrow$ LINK TO DIRECTED POLYMERS)

Henceforth we focus on $d = 2$

We restrict (t,x) in the lattice $\mathbb{T}_N = \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$ $\quad (N \in \mathbb{N})$

$$\partial_t^N u_N(t,x) = \frac{1}{4} \Delta^N u_N(t,x) + N \, \xi_N^\beta\left(t+\tfrac{1}{N}, x\right) \langle u_N(t,x) \rangle \qquad \text{(D-SHE)}$$

TIME DIFFERENCE

$N\left\{u\left(t+\frac{1}{N},x\right) - u(t,x)\right\}$

LATTICE LAPLACIAN

$\frac{N}{4} \sum_{x' \sim x} \left\{u(t,x') - u(t,x)\right\}$

I.I.D. RVs

MEAN ZERO
VARIANCE β^2

SPACE AVERAGE

$\frac{1}{4} \sum_{x' \sim x} u(t,x')$

Solution well-defined $u_N(t,x) \geq 0$ $\quad$ (with $u_N(0,\cdot) \equiv 1$)

CONVERGENCE ?

Does $U_N(t,x)$ converge to a non-trivial limit $\mathcal{U}$ as $N \to \infty$?

YES, but we first need to do 1 + 2

1 Look for convergence as (random) distributions on $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} \varphi(x)\, U_N(t,x)\, dx \xrightarrow[N\to\infty]{d} \int_{\mathbb{R}^2} \varphi(x)\, \mathcal{U}(t,dx) \quad ?$$

i.e. $U_N(t,x)\,dx \xrightarrow{d} \mathcal{U}(t,dx)$ as (random) measures on $\mathbb{R}^2$

② Rescale the coupling constant

$$\beta = \beta_N \sim \frac{\hat\beta}{\sqrt{\log N}} \longrightarrow 0$$

$$\mathbb{E}\left[\int \varphi(x)\, U_N(t,x)\, dx\right] = \int \varphi(x)\, dx$$

$$\mathbb{V}\mathrm{AR}\left[\int \varphi(x)\, U_N(t,x)\, dx\right] \longrightarrow \begin{cases} 0 & \text{if } \hat\beta < \sqrt{\pi} \\ \infty & \text{if } \hat\beta > \sqrt{\pi} \end{cases}$$

PHASE TRANSITION

For $\hat\beta < \sqrt{\pi}$: $\quad U_N(t,x)\, dx \longrightarrow dx$ = Lebesgue measure (LLN)

$\sqrt{\log N}\,\{U_N(t,x) - 1\}\, dx \longrightarrow V(t,x)\, dx$ log-correlated Gaussian (CLT)

For $\hat\beta = \sqrt{\pi}$ does $U_N(t,x)$ converge to a non-trivial limit $\mathcal{U}$?

THEOREM [CSZ 21]

Let $u_N(t,x)$ solve (D-SHE). Fix $\vartheta \in \mathbb{R}$ and rescale

$$(\star) \qquad \beta = \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \frac{\vartheta}{\log N}\right)$$

As $N \to \infty$ we have the convergence to a non-trivial limit

$$\big(u_N(t,x)\,dx\big)_{t \geq 0} \xrightarrow{\text{f.d.d.}} \mathcal{U}^{\vartheta} = \big(\mathcal{U}^{\vartheta}_t(dx)\big)_{t \geq 0}$$

which we call the CRITICAL 2D STOCHASTIC HEAT FLOW

SOME FEATURES

- $\mathbb{E}\left[\mathcal{U}_t^{\vartheta}(dx)\right] = dx$

- $\mathbb{E}\left[\mathcal{U}_t^{\vartheta}(dx)\,\mathcal{U}_t^{\vartheta}(dy)\right] = K_t^{\vartheta}(x,y)\,dx\,dy$ $\dashrightarrow \sim \log \frac{1}{|x-y|}$

  [Bertini, Cancrini 98]

  $\rightsquigarrow$ $\mathcal{U}^{\vartheta}$ is random

- $\mathcal{U}_{at}^{\vartheta}(d(\sqrt{a}\,x)) \overset{d}{=} a\,\mathcal{U}_t^{\vartheta + \log(a)}(dx)$

- Formulas for higher moments

  [Gu, Quastel, Tsai 21]
  [CSZ 19]

Gaussian Multiplicative Chaos (GMC)

Consider a Gaussian random field $X \sim \mathcal{N}(0, K)$:

$$\mathbb{C}\mathrm{ov}\left[X(\varphi), X(\psi)\right] = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi(x) \underbrace{K(x,y)}_{\text{POSSIBLY SINGULAR}} \psi(y) \, dx \, dy$$

Gaussian Multiplicative Chaos $M(dx)$ is the random measure

$$M(dx) = \text{“} e^{X(x) - \frac{1}{2} K(x,x)} \, dx \text{”} \qquad \text{(FORMALLY)}$$

$$\mathbb{E}\left[M(dx)\right] = dx \qquad \mathbb{E}\left[M(dx)\, M(dy)\right] = e^{K(x,y)} \, dx \, dy$$

- Is the 2d Stochastic Heat Flow $\mathcal{U}_t^\vartheta(dx)$ a GMC $\mathcal{M}(dx)$?

(only possible with $\kappa(x,y) = \log K_t^\vartheta(x,y) \sim \log\log \frac{1}{|x-y|}$)

THEOREM $\mathcal{U}_t^\vartheta(dx)$ is NOT a GMC [CSZ 22+]

This suggests that the "solution of critical 2d KPZ"

(yet to be constructed!) should be NON GAUSSIAN

WE CANNOT TAKE $\log \mathcal{U}_t^\vartheta(dx)$

INITIAL CONDITION

$$\beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1+\frac{\vartheta}{\log N}\right)$$

We built a candidate solution of the Critical 2d SHE

$$\mathscr{U}^{\vartheta} = \left(\mathscr{U}^{\vartheta}_t(dx)\right)_{t\geq 0}$$

with initial condition $\mathscr{U}^{\vartheta}_0(dx) \equiv dx$ (that is $u(0,\cdot) \equiv 1$)

We actually build a two-parameter process

$$\mathscr{U}^{\vartheta} = \left(\mathscr{U}^{\vartheta}_{s,t}(dy, dx)\right)_{0\leq s\leq t<\infty}$$

where $\mathscr{U}^{\vartheta}_{s,t}(\varphi, dx)$ corresponds to the initial condition $u(s,\cdot) = \varphi(\cdot)$

II. Ideas and Techniques

A link with Directed Polymers

Recall the discretized SHE in the lattice $\mathbb{T}_N = \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$

$$\partial_t^N u_N(t,x) = \frac{1}{4}\Delta^N u_N(t,x) + N\, \xi_N^\beta\left(t+\tfrac{1}{N}, x\right) \langle u_N(t,x) \rangle \qquad \text{(D-SHE)}$$

I.I.D. RVs MEAN ZERO, VARIANCE β^2

Assume $\xi^\beta \geqslant -1 \rightsquigarrow \xi^\beta = \dfrac{e^{\beta\omega}}{\mathbb{E}[e^{\beta\omega}]} - 1$ ω: MEAN 0, VARIANCE 1

Then $u_N(t,x)$ admits a Feynman-Kac representation formula

For $(t,x) = \left(\frac{n}{N}, \frac{z}{\sqrt{N}}\right)$ with $(n,z) \in \mathbb{N} \times \mathbb{Z}^2$:

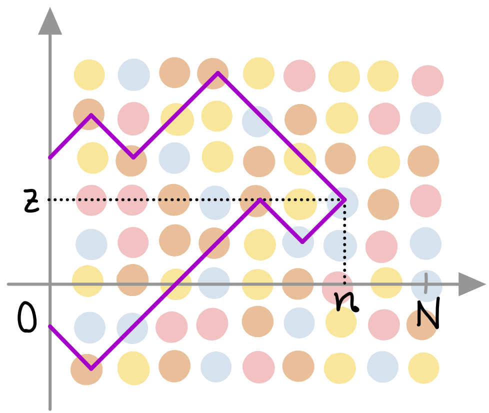
$$u_N(t,x) = Z_N(n,z) = \mathrm{E}\left[e^{\sum_{i=0}^{n-1} \beta\, \omega(n-i, S_i) - \frac{\beta^2}{2}} \,\middle|\, S_0 = z \right]$$

S SIMPLE RANDOM WALK ON $\mathbb{Z}^2$

Partition function of the

DIRECTED POLYMER

IN RANDOM ENVIRONMENT



SECOND MOMENT AND CRITICAL SCALING OF β

$$\mathbb{E}\left[u_N(1,x) \cdot u_N(1,x') \right] = \mathrm{E}\left[e^{\beta^2 \underbrace{\sum_{i=0}^{N} \mathbb{1}_{\{S_i = S'_i\}}}_{\mathcal{L}_N}} \,\middle|\, S_0 = z,\ S'_0 = z' \right]$$

"REPLICA OVERLAP"

Classical result: $\frac{\pi}{\log N} \mathcal{L}_N \xrightarrow[N\to\infty]{d} Y \sim \mathrm{Exp}(1)$ [Erdös-Taylor 60]

This explains the CRITICAL SCALING of $\beta = \beta_N$

$$\beta \sim \hat{\beta} \frac{\sqrt{\pi}}{\sqrt{\log N}} \qquad \text{with} \qquad \hat{\beta} = \hat{\beta}_c = 1 + O\left(\frac{1}{\log N}\right)$$

MAIN RESULT: STRATEGY OF THE PROOF

- Existence of subsequential limits (tightness) is easy:

$$U_N(t,x)\,dx \xrightarrow{d} \mathcal{U}_t^{\vartheta}(dx)$$ [Bertini-Cancrini 98]

- Non-triviality of the limit is harder. [CSZ 19b]

- Uniqueness is very difficult! [CSZ 21]

(Formulas for all moments of $\mathcal{U}_t^{\vartheta}$ are available, [GQT 21]
but moments grow too fast to determine the law)

HOW TO PROVE UNIQUENESS

Problem: we do not have a characterization of the limit

Solution: we use a Cauchy argument:

$$u_N(t,x)\,dx \overset{d}{\approx} u_M(t,x)\,dx \qquad \text{for large } N, M$$

exploiting self-similarity of the model. Four main pillars:

A. COARSE-GRAINING

B. RENEWAL STRUCTURE

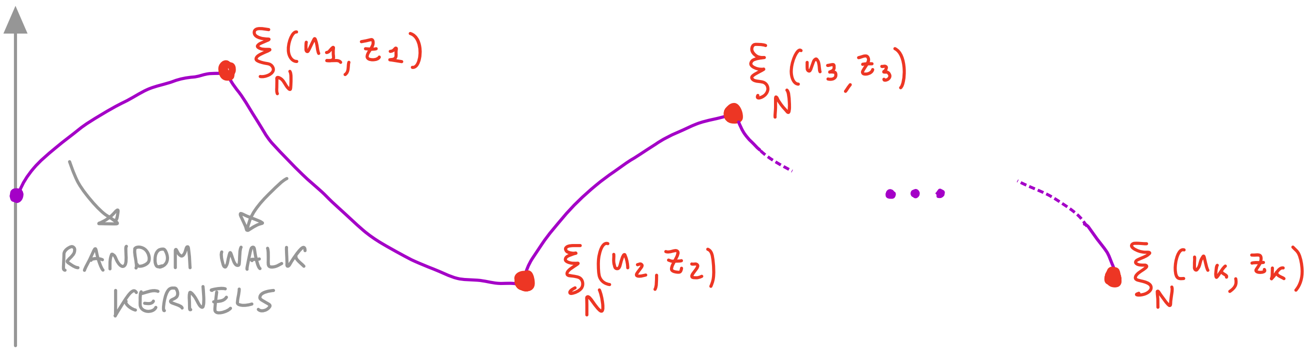
C. LINDEBERG PRINCIPLE

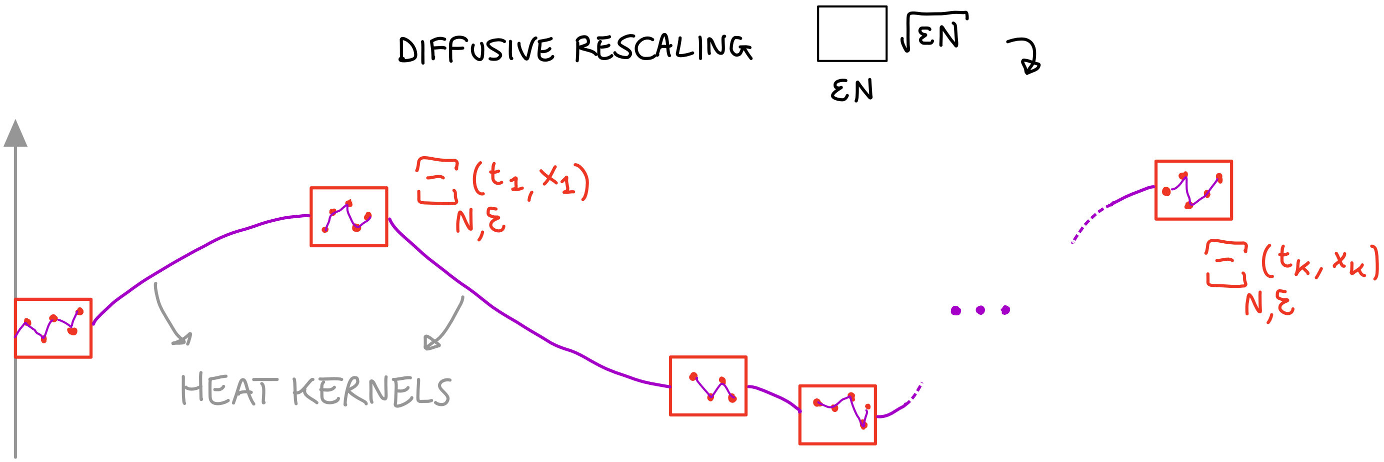
D. FUNCTIONAL INEQUALITIES

A. Coarse-Graining

Polynomial chaos:

$$U_N(t,x) = 1 + \sum_{k \geq 1} \sum_{(n_1,z_1),\dots,(n_k,z_k)} \overbrace{q\big((n_1,z_1),\dots,(n_k,z_k)\big)}^{P(S_{n_1}=z_1,\,\dots,\,S_{n_k}=z_k)} \cdot \prod_{i=1}^{k} \xi_N^{\beta}(n_i,z_i)$$





Sharp L^2 approximation via a coarse-grained model

$$U_N(t,x)\,dx \;\approx\; \mathcal{Z}^{CG}_{\varepsilon}\big(t, dx \,\big|\, \Xi_{N,\varepsilon}\big) \qquad (\text{as } \varepsilon \downarrow 0)$$

MULTI-LINEAR POLYNOMIAL

"COARSE-GRAINED" NOISE

B. RENEWAL STRUCTURE

Probabilistic interpretation of 2ND moment calculations

$$\mathbb{E}\left[u_N(t,x) \cdot u_N(t,x') \right] = \sum \ \cdots \ q\left((n_1,z_1),\dots,(n_k,z_k)\right)^2 \cdots$$

$$\xrightarrow[N\to\infty]{} 2\pi \int_0^t ds \ g_s(x-x') \int_s^t e^{\vartheta u} \, P(Y_u \le t) \, du$$

HEAT KERNEL

"DICKMAN SUBORDINATOR"

[CSZ 19a]

c. LINDEBERG PRINCIPLE

The distribution of coarse-grained model $\mathcal{Z}^{CG}_{\varepsilon}(t, dx \mid \Xi)$

is insensitive to the distribution of Ξ

(as $\varepsilon \downarrow 0$, provided 1ST & 2ND moments are fixed)

$\rightsquigarrow$ We can change $\Xi_{N,\varepsilon}$ to $\Xi_{M,\varepsilon}$ to get our goal:

$$U_N(t,x)\,dx \overset{d}{\approx} U_M(t,x)\,dx$$

(Coarse-grained variables Ξ are dependent) [Röllin 2013]

D. FUNCTIONAL INEQUALITIES

Lindeberg requires higher moment bounds on CG model.

$\rightsquigarrow$ Inequalities for Green's function of multiple random walks

$$\int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \frac{f(x,x') \cdot g(y,y')}{\left(|x-y| + |x'-y'| + |x-y'|\right)^{2d}} \, dx\, dx'\, dy\, dy' \leq C \, \|f\|_{L^p} \, \|g\|_{L^q}$$

"CRITICAL" HARDY-LITTLEWOOD-SOBOLEV INEQUALITY

Generalizes an inequality by [dell'Antonio, Figari, Teta 94]

NON GMC-NESS

Consider the 2d Stochastic Heat Flow $\mathcal{U}_t^\vartheta$ for fixed $t>0$, $\vartheta \in \mathbb{R}$

Let $\mathcal{M}(dx)$ be the GMC with matching 1st and 2nd moments

$$\mathbb{E}\left[\mathcal{M}(dx)\,\mathcal{M}(dy)\right] = \mathbb{E}\left[\mathcal{U}_t^\vartheta(dx)\,\mathcal{U}_t^\vartheta(dy)\right] = K_t^\vartheta(x,y)\,dx\,dy$$

We prove that higher moments do not match

- 3rd MOMENT BOUND: For any $R>0$

$$\mathbb{E}\left[\mathcal{U}_t^\vartheta(B_R)^3\right] > \mathbb{E}\left[\mathcal{M}(B_R)^3\right]$$

- HIGHER MOMENT BOUND: there is $\eta > 0$ s.t. for any $K \geqslant 3$

$$\liminf_{\delta \downarrow 0} \frac{\mathbb{E}\left[\mathcal{U}_t^{\vartheta}(g_\delta)^K\right]}{\mathbb{E}\left[\mathcal{M}(g_\delta)^K\right]} \geqslant 1 + \eta$$

HEAT KERNEL AT TIME δ

- 3rd MOMENT BOUND based on explicit diagrammatic expansion for the 3rd moment + Gaussian calculations

- HIGHER MOMENT BOUND based on the Gaussian Correlation Inequality (inspired by an argument in [Feng 16])

III. Conclusions and Perspectives

CONCLUSIONS

We introduced the CRITICAL 2D STOCHASTIC HEAT FLOW $\mathscr{U}_t^{\vartheta}(dx)$

as the scaling limit of solutions of discretized SHE

$\longleftrightarrow$ directed polymer partition functions

- Universal process of random measures on $\mathbb{R}^2$ ($\neq$ GMC)
- Natural candidate solution for critical 2d SHE

Many explicit features ...

... but several interesting questions are open:

- SINGULARITY W.R.T. LEBESGUE MEASURE
- FLOW PROPERTY
- CHARACTERIZING PROPERTIES
- UNIVERSALITY
- TAKING LOG $\rightsquigarrow$ KPZ

Interesting connections:

Statistical Mechanics $\longleftrightarrow$ Singular Stochastic PDEs

(also for heavy-tailed disorder [Berger, Chong, Lacoin])

Thanks !

Moment Formulas

$$\mathbb{E}\left[\mathcal{U}_t^{\vartheta}(dx) \cdot \mathcal{U}_t^{\vartheta}(dy) \cdot \mathcal{U}_t^{\vartheta}(dz) \right] = \underbrace{K^{(3)}(x, y, z)}_{\lim\limits_{N \to \infty} \mathbb{E}\left[U_N(t,x) \cdot U_N(t,y) \cdot U_N(t,z) \right]} dx \, dy \, dz$$

$$K^{(3)}(z_1, z_2, z_3) = \sum_{m \geq 2} \int \cdots \int_{\substack{0 < a_1 < b_1 < \ldots < a_m < b_m < t \\ x_1, y_1, \ldots, x_m, y_m \in \mathbb{R}^2}} d\vec{a} \, d\vec{b} \, d\vec{x} \, d\vec{y} \quad g_{\vec{z}}^{(m)}(\vec{a}, \vec{b}, \vec{x}, \vec{y})$$

Moment Formulas

$g^{(4)} =$