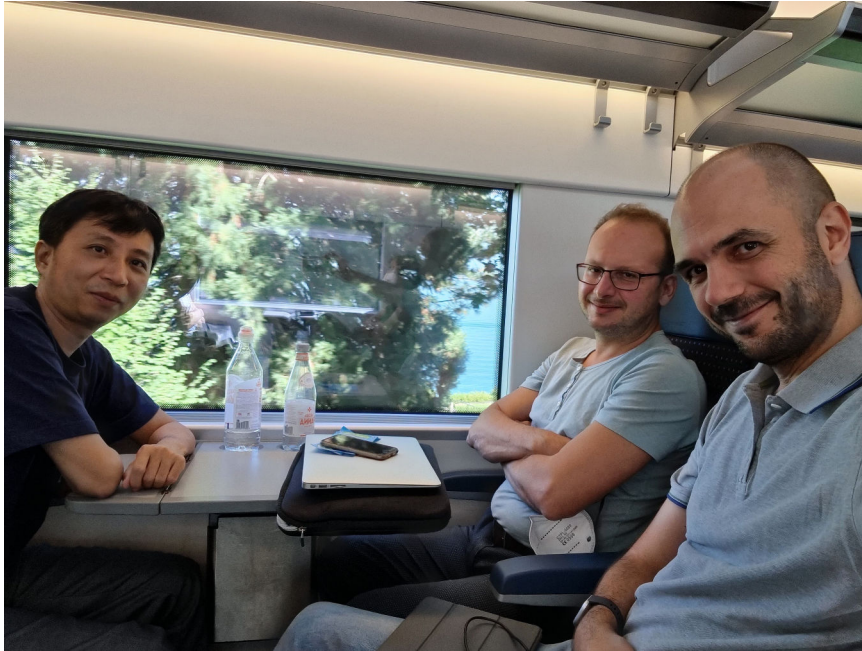


The critical 2d Stochastic Heat Flow

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Based on joint works with



Rongfeng Sun and Nikos Zygouras

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I. INTRODUCTION

THE STOCHASTIC HEAT EQUATION

$$(SHE) \quad \begin{cases} \partial_t U(t,x) = \Delta U(t,x) + \beta \xi(t,x) U(t,x) \\ U(0,x) \equiv 1 \end{cases} \quad t > 0, x \in \mathbb{R}^d$$

- $\beta > 0$ coupling constant
- $\xi(t,x)$ "space-time white noise" (δ -correlated Gaussian)

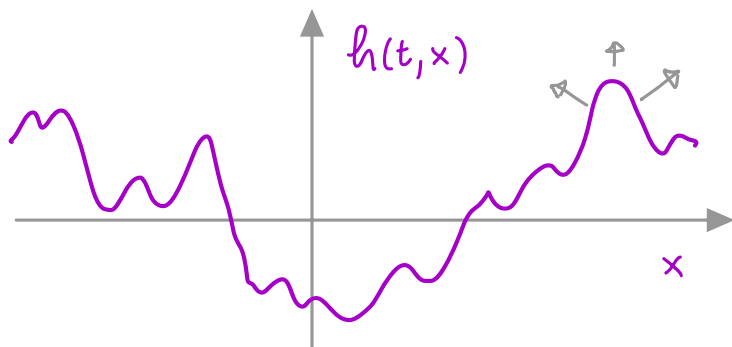
GOAL: Construct the solution $U(t,x)$ for $d=2$

THE KARDAR - PARISI - ZHANG EQUATION

[PRL 1986]

Formally $h(t,x) := \log U(t,x)$ solves

$$(KPZ) \quad \partial_t h(t,x) = \underbrace{\Delta h(t,x)}_{\text{SMOOTHING}} + \underbrace{|\nabla h(t,x)|^2}_{\perp \text{ GROWTH}} + \underbrace{\beta \xi(t,x)}_{\text{NOISE}}$$



SHE can help us
make sense of KPZ

SINGULARITY

(SHE) and (KPZ) are ill-defined due to singular products

$$\xi(t, x) \quad U(t, x)$$

$$|\nabla h(t, x)|^2$$

$\xi(t, x)$ is a distribution $\leadsto U(t, x)$ and $h(t, x)$ expected to be:

$$\mathcal{C}_{\text{PAR}}^{-\frac{d}{2}-1-}$$

- non-smooth functions ($d=1$) $\mathcal{C}_{\text{PAR}}^{\frac{1}{2}-}$
- genuine distributions ($d \geq 2$) $\mathcal{C}_{\text{PAR}}^{\frac{2-d}{2}-}$

Henceforth we focus on (SHE)

THE CASE $d=1$

(1980s) $u(t,x)$ well-posed by stochastic integration (Itô-Walsh)

(2010s) Robust solution theories for "sub-critical" singular PDEs

- REGULARITY STRUCTURES [Hairer]
- PARACONTROLLED CALCULUS [Gubinelli, Imkeller, Perkowski]
- ENERGY SOLUTIONS [Goncalves, Jara] • RENORMALIZATION [Kupiainen]

All of this breaks down in higher dimensions $d \geq 2$

THE ROLE OF DIMENSION

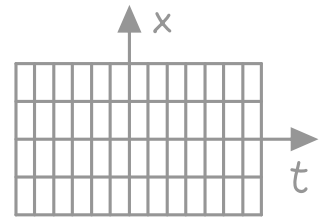
Space-time blow up: $\tilde{U}(t,x) := U(\delta^2 t, \delta x)$ solves

$$\partial_t \tilde{U}(t,x) = \Delta \tilde{U}(t,x) + \beta \delta^{\frac{2-d}{2}} \tilde{\xi}(t,x) \tilde{U}(t,x)$$

As $\delta \downarrow 0$ the noise formally $\left\{ \begin{array}{ll} \text{vanishes} & (d < 2) \\ \text{stays constant} & (d = 2) \\ \text{diverges} & (d > 2) \end{array} \right.$

$d=2$ is CRITICAL DIMENSION for SHE

REGULARIZING SHE VIA DISCRETIZATION



We fix $d=2$. Restrict (t, x) in the lattice $\mathbb{T}_N = \frac{N}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$:

$$\partial_t^N u_N(t, x) = \Delta_N^N u_N(t, x) + \beta \cdot N \cdot \xi_N(t + \frac{1}{N}, x) \langle u_N(t, x) \rangle$$

DISCR. DERIVATIVE

DISCR. LAPLACIAN

I.I.D. RVs

$$N \left\{ u(t + \frac{1}{N}, x) - u(t, x) \right\}$$

$$\frac{N}{4} \sum_{x' \sim x} \{ u(t, x') - u(t, x) \}$$

$$\mathbb{E}[\xi] = 0 \quad \mathbb{E}[\xi^2] = 1$$

$$\beta \xi \geq -1$$



$$\frac{1}{4} \sum_{x' \sim x} u(t, x')$$

Well-defined solution $u_N(t, x) \geq 0$

$$[u_N(0, \cdot) \equiv 1]$$

CONVERGENCE ?

Can we hope that $U_N(t, x) \xrightarrow{N \rightarrow \infty} \mathcal{U}(t, x)$? (non-trivial limit)

① Convergence as (random) distributions : $\varphi \in C_c(\mathbb{R}^2)$

$$\underbrace{\int_{\mathbb{R}^2} \varphi(x) U_N(t, x) dx}_{\text{RANDOM VARIABLE IN } \mathbb{R}} \xrightarrow[N \rightarrow \infty]{d} \int_{\mathbb{R}^2} \varphi(x) \mathcal{U}(t, dx) ?$$

i.e. $U_N(t, x) dx \xrightarrow{d} \mathcal{U}(t, dx)$ as (random) measures on $\mathbb{R}^2$?

② Rescale the coupling constant $\beta = \beta_N \sim \frac{\hat{\beta}}{\sqrt{\log N}} \xrightarrow{N \rightarrow \infty} 0$

$$\mathbb{E} \left[\int \varphi(x) U_N(t, x) dx \right] \equiv \int \varphi(x) dx$$

$$\text{VAR} \left[\int \varphi(x) U_N(t, x) dx \right] \longrightarrow \begin{cases} 0 & \text{if } \hat{\beta} < \sqrt{\pi} \\ \sigma^2(\varphi) & \text{if } \hat{\beta} = \sqrt{\pi} \\ \infty & \text{if } \hat{\beta} > \sqrt{\pi} \end{cases} \quad \text{PHASE TRANSITION}$$

For $\hat{\beta} < \sqrt{\pi}$: $U_N(t, x) dx \xrightarrow{d} dx = \text{Lebesgue measure ("trivial")}$

For $\hat{\beta} = \sqrt{\pi}$ do we have $U_N(t, x) dx \xrightarrow{d} \mathcal{U}(t, dx)$?

(non trivial)

II. MAIN RESULTS

THEOREM

[CSZ 23+]

Rescale $\beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$, more precisely

$$\textcircled{\star} \quad \beta = \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{g}{\log N} \right) \quad g \in \mathbb{R}$$

Then $U_N(t, x)$ converges to a non-trivial limit:

$$(U_N(t, x) \, dx)_{t \geq 0} \xrightarrow[N \rightarrow \infty]{\text{f.d.d.}} \mathcal{U}^g = (\mathcal{U}_t^g(dx))_{t \geq 0}$$

We call $\mathcal{U}^g$ critical 2d STOCHASTIC HEAT FLOW (SHF)

SHF & SHE

We have built a candidate solution of the Critical 2d SHE:

the SHF $\mathcal{U}^{\mathcal{J}} = (\mathcal{U}_t^{\mathcal{J}}(dx))_{t \geq 0}$

$$\beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$$

with initial condition $\mathcal{U}_0^{\mathcal{J}}(dx) \equiv dx$ $(U(0, \cdot) \equiv 1)$

Remark. We actually build a two-parameter process

$$\mathcal{U}^{\mathcal{J}} = (\mathcal{U}_{s,t}^{\mathcal{J}}(dy, dx))_{0 \leq s \leq t < \infty}$$

"CANDIDATE SHE SOLUTION
FROM dy AT TIME s "

SOME FEATURES OF THE SHF

- $\mathbb{E}[\mathcal{U}_t^g(dx)] = dx$

- $\mathbb{E}[\mathcal{U}_t^g(dx) \mathcal{U}_t^g(dy)] = K_t^g(x, y) dx dy$

$\sim \log \frac{1}{|x-y|}$

[Bertini, Cancrini 98]

$\Rightarrow \mathcal{U}^g$ non trivial

- $\mathcal{U}_{at}^g(d(\sqrt{a}x)) \stackrel{d}{=} a \mathcal{U}_t^{g+\log(a)}(dx)$

- Formulas for higher moments

[Gu, Quastel, Tsai 21]

[CSZ 19b]

CORRELATION FUNCTION

$$\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{\gamma}{\log N} \right)$$

$$K_N^\beta(t, x, x') := \mathbb{E} [U_N(t, x) \cdot U_N(t, x')] \xrightarrow{N \rightarrow \infty} K_t^\gamma(x, x')$$

It solves the 2-body **delta-Bose gas** discretized in $\frac{\mathbb{Z}^2}{\sqrt{N}}$:

$$\partial_t^N K_N^\beta = -\mathcal{H}_N^\beta K_N^\beta \quad \text{where} \quad \mathcal{H}_N^\beta = -\Delta^N - \beta \cdot N \cdot \mathbb{1}_{\{x=x'\}}$$

$$K_t^\gamma = e^{-t \mathcal{H}^\gamma} \xrightarrow{\text{self-adj. ext. of}} \mathcal{H}^\beta = -\Delta - \beta \delta(x-x')$$

**EXPLICIT
FORMULA**

[Albeverio, Gesztesy, Høegh-Krohn, Holden 87]

HIGHER ORDER CORRELATIONS

$$K_N^\beta(t, x_1, \dots, x_n) := \mathbb{E} [U_N(t, x_1) \cdots U_N(t, x_n)] \xrightarrow{N \rightarrow \infty} K_t^\gamma(x_1, \dots, x_n)$$

It solves the n -body delta-Bose gas discretized in $\frac{\mathbb{Z}^2}{\sqrt{N}}$:

$$\partial_t^N K_N^\beta = -\mathcal{H}_N^\beta K_N^\beta \quad \text{where} \quad \mathcal{H}_N^\beta = -\Delta^N - \beta \cdot N \cdot \sum_{1 \leq i < j \leq n} \mathbb{1}_{\{x_i = x_j\}}$$

$$K_t^\gamma = e^{-t \mathcal{H}^\gamma} \rightarrow \text{self-adj. ext. of } \mathcal{H}^\beta = -\Delta - \beta \sum_{1 \leq i < j \leq n} \delta(x_i - x_j)$$

EXPLICIT FORMULAS

[dell'Antonio, Figari, Teta 94] [Dimock, Rajeev 04] [Gu, Quastel, Tsai 19]

GAUSSIAN MULTIPLICATIVE CHAOS

Generalized Gaussian field $\chi \sim \mathcal{N}(0, \kappa)$ on $\mathbb{R}^d$

$$\iint \varphi(x) \varphi(y) \mathbb{E}[\chi(x) \chi(y)] dx dy = \iint \kappa(x, y) \varphi(x) \varphi(y) dx dy$$

Gaussian Multiplicative Chaos (GMC) (random measure)

$$\mathcal{M}(dx) = e^{\chi(x) - \frac{1}{2} \kappa(x, x)} dx$$

$$\mathbb{E}[\mathcal{M}(dx)] = dx \quad \mathbb{E}[\mathcal{M}(dx) \mathcal{M}(dy)] = e^{\kappa(x, y)} dx dy$$

GMCs are "canonical": many explicit features

THEOREM The SHF $\mathcal{U}_t^g(dx)$ is NOT a GMC [CSZ 22+]

Recall: formally $h(t, x) = \log U(t, x)$ solves (KPZ)

CONJECTURE The critical 2d KPZ equation should have a NON GAUSSIAN SOLUTION $\mathcal{H}_t(dx)$

(KPZ) solution yet to be constructed! Cannot take $\log \mathcal{U}_t^g(dx)$

III. IDEAS AND TECHNIQUES

FEYNMAN-KAC FORMULA

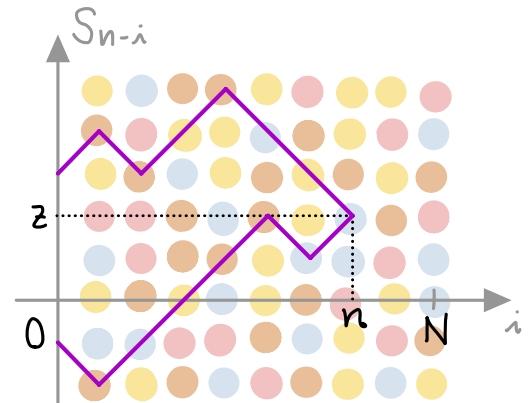
$$(t, x) = \left(\frac{n}{N}, \frac{z}{\sqrt{N}} \right) \in \frac{N}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$$

$$u_N(t, x) = Z_N(n, z) = E \left[e^{\sum_{i=1}^N \beta \omega(i, S_{n-i}) - \lambda(\beta)} \mid S_0 = z \right]$$

S SIMPLE RANDOM WALK ON $\mathbb{Z}^2$

$$\beta \xi = e^{\beta \omega - \lambda(\beta)} - 1 \geq -1$$

Partition function of the
DIRECTED POLYMER
IN RANDOM ENVIRONMENT



SECOND MOMENT AND CRITICAL SCALING OF β

$$\mathbb{E} \left[u_N(1, x) \cdot u_N(1, x') \right] = \mathbb{E} \left[e^{\beta^2 \underbrace{\sum_{i=0}^N \mathbb{1}\{S_i = S'_i\}}_{L_N}} \mid S_0 = z, S'_0 = z' \right]$$

"REPLICA OVERLAP"

For $x = x'$: $\frac{\pi}{\log N} L_N \xrightarrow[N \rightarrow \infty]{d} Y \sim \text{Exp}(1)$ [Erdős-Taylor 60]

This explains the **CRITICAL SCALING** $\beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$

MAIN RESULT: STRATEGY OF THE PROOF

$$U_N(t, x) dx \xrightarrow{d} \mathcal{U}_t^g(dx)$$

- Existence of subsequential limits is "easy" [Bertini-Cancrini 98]
- Non-triviality of the limit is non trivial [CSZ 19b]
- Uniqueness is **hard** [CSZ 23+]

(Moments grow too fast to determine the distribution)

HOW TO PROVE UNIQUENESS ?

We use a Cauchy argument:

$$U_N(t, x) dx \stackrel{d}{\approx} U_M(t, x) dx \quad \text{for large } N, M$$

exploiting self-similarity of the model

A. COARSE-GRAINING $\rightsquigarrow$ B. RENEWAL STRUCTURE

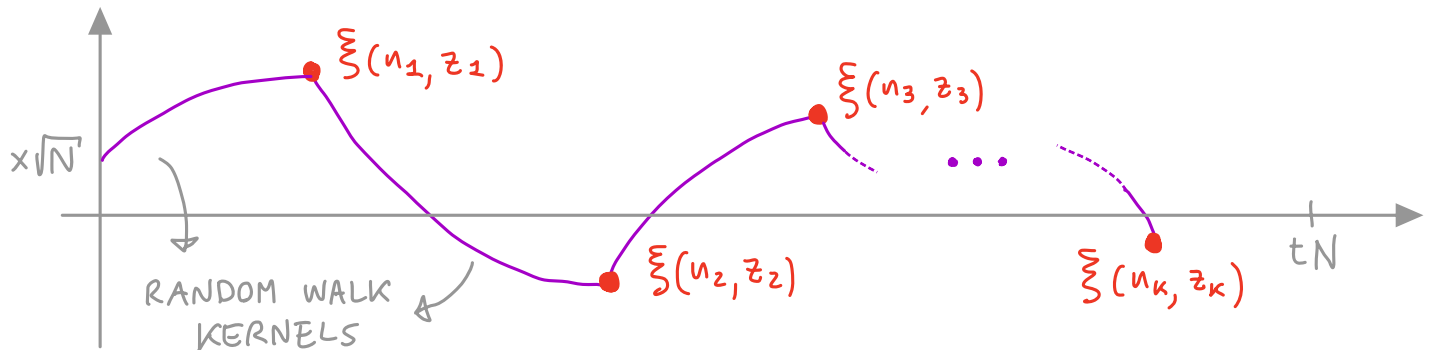
C. LINDBERG PRINCIPLE $\rightsquigarrow$ D. FUNCTIONAL INEQUALITIES

A. COARSE - GRAINING

Polynomial chaos expansion:

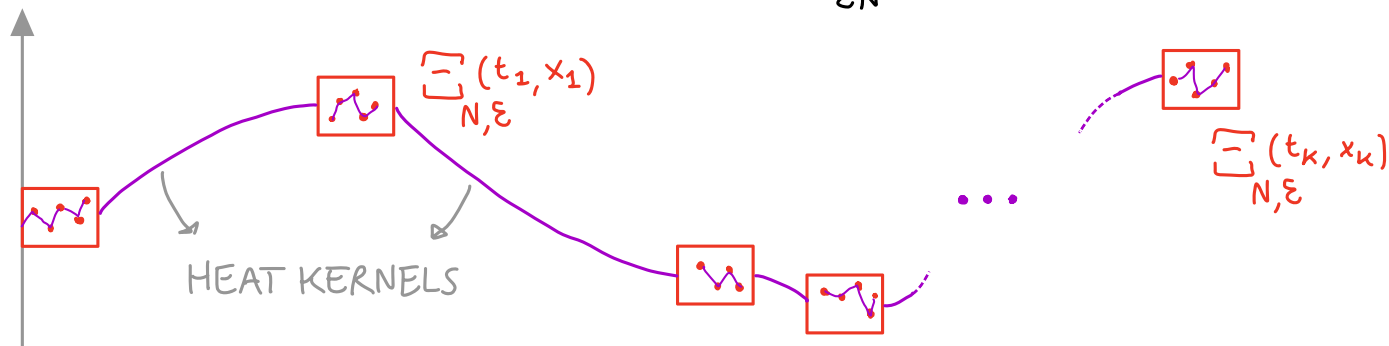
$$U_N(t, x) = 1 + \sum_{k=1}^N \beta^k \sum_{(n_1, z_1), \dots, (n_k, z_k)} q((n_1, z_1), \dots, (n_k, z_k)) \cdot \prod_{i=1}^k \xi(n_i, z_i)$$

$P(S_{n_1}=z_1, \dots, S_{n_k}=z_k)$



DIFFUSIVE RESCALING

$$\frac{\square}{\varepsilon N} \xrightarrow{\sqrt{\varepsilon N}}$$



Sharp L^2 approximation via a coarse-grained model

$$U_N(t, x) dx \approx \mathcal{Z}_\varepsilon^{\text{CG}}(t, dx | \Xi_{N,\varepsilon}) \quad \text{as } \varepsilon \downarrow 0$$

MULTI-LINEAR POLYNOMIAL $\downarrow$ "COARSE-GRAINED" NOISE

B. RENEWAL STRUCTURE

Probabilistic interpretation of 2ND moment

$$\mathbb{E}[U_N(t, x) \cdot U_N(t, x')] = \sum_{k=1}^N \beta^{2k} \sum_{(n_1, z_1), \dots, (n_k, z_k)} q((n_1, z_1), \dots, (n_k, z_k))^2$$

$$\xrightarrow{N \rightarrow \infty} K_t^g(x, x') = 2\pi \int_0^t ds \, g_s(x - x') \int_s^t e^{g_u} P(Y_u \leq t) du$$

[CSZ 19a]

HEAT KERNEL

"DICKMAN SUBORDINATOR"

C. LINDBERG PRINCIPLE

The distribution of coarse-grained model $\mathcal{Z}_\varepsilon^{\text{CG}}(t, dx | \Xi)$ is insensitive to the distribution of Ξ

(as $\varepsilon \downarrow 0$, provided 1st & 2nd moments are fixed) [Röllin 2013]

~> We can switch $\Xi_{N,\varepsilon}$ to $\Xi_{M,\varepsilon}$ and get our goal

$$U_N(t, x) dx \stackrel{d}{\approx} U_M(t, x) dx$$

D. FUNCTIONAL INEQUALITIES

Inequalities for Green's function of multiple random walks

"CRITICAL" HARDY-LITTLEWOOD-SOBOLEV INEQUALITY

$$\int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \frac{f(x, x') \cdot g(y, y')}{(|x - y| + |x' - y'| + |x - y'|)^{2d}} dx dx' dy dy' \leq C \|f\|_{L^p} \|g\|_{L^q}$$

Generalizes an inequality by [dell'Antonio, Figari, Teti 94]

IV. CONCLUSIONS AND PERSPECTIVES

CONCLUSIONS

We introduced the CRITICAL 2D STOCHASTIC HEAT FLOW $\mathcal{U}_t^g(dx)$ as the scaling limit of solutions of discretized SHE

$\longleftrightarrow$ directed polymer partition functions

- Universal process of random measures on $\mathbb{R}^2$ ($\neq$ GMC)
- Natural candidate solution for critical 2d SHE

Many explicit features...

... and several interesting open questions:

- SINGULARITY W.R.T. LEBESGUE MEASURE ✓
- FLOW PROPERTY
- CHARACTERIZING PROPERTIES & UNIVERSALITY
- TAKING LOG $\rightsquigarrow$ KPZ

Statistical Mechanics $\leftrightarrow$ Singular Stochastic PDEs

RELATED WORKS

- ANISOTROPIC KPZ: $\Delta h = (\partial_x^2 + \partial_y^2) h \rightsquigarrow (\partial_x^2 - \partial_y^2) h$
[Erhard, Cannizzaro, Toninelli]
- SHE WITH LÉVY NOISE: $\mathbb{P}(|\xi| > t) \sim \frac{C}{t^\alpha} \quad 0 < \alpha < 2$
[Berger, Chong, Lacoïn]
- DIRECTED POLYMERS, SHE & KPZ IN $d \neq 2$
[Magnen, Unterberger] [Chatterjee, Dunlap] [Dunlap, Gu, Ryzhik, Zeitouni]
[Clark] [Comets, Cosco, Mukherjee] [Lygkonis, Zygouras] [Junk] ...

Thanks

NON GMC-NESS

Consider the 2d Stochastic Heat Flow $\mathcal{U}_t^g$ for fixed $t > 0$, $g \in \mathbb{R}$

Let $\mathcal{M}(dx)$ be the GMC with matching 1st and 2nd moments

$$\mathbb{E}[\mathcal{M}(dx) \mathcal{M}(dy)] = \mathbb{E}[\mathcal{U}_t^g(dx) \mathcal{U}_t^g(dy)] = K_t^g(x, y) dx dy$$

We prove that higher moments do not match

- 3rd MOMENT BOUND: For any $R > 0$

$$\mathbb{E}[\mathcal{U}_t^g(B_R)^3] > \mathbb{E}[\mathcal{M}(B_R)^3]$$

- **HIGHER MOMENT BOUND**: there is $\eta > 0$ s.t. for any $k \geq 3$

$$\liminf_{\delta \downarrow 0} \frac{\mathbb{E}[\mathcal{U}_t^g(g_\delta)^k]}{\mathbb{E}[\mathcal{M}(g_\delta)^k]} \geq 1 + \eta$$

HEAT KERNEL AT TIME δ

- 3rd MOMENT BOUND $\left\{ \begin{array}{l} \text{explicit diagrammatic expansion} \\ \text{Gaussian calculations} \end{array} \right.$

- HIGHER MOMENT BOUND via **Gaussian Correlation Inequality**
(inspired by [Feng 16])

MOMENT FORMULAS

$$\mathbb{E} \left[\mathcal{U}_t^g(dx) \cdot \mathcal{U}_t^g(dy) \cdot \mathcal{U}_t^g(dz) \right] = \underbrace{K^{(3)}(x, y, z)}_{\text{}} dx dy dz$$

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[U_N(t, x) \cdot U_N(t, y) \cdot U_N(t, z) \right]$$

$$K^{(3)}(z_1, z_2, z_3) = \sum_{m \geq 2} \int \cdots \int d\vec{a} d\vec{b} d\vec{x} d\vec{y} \quad g_{\vec{z}}^{(m)}(\vec{a}, \vec{b}, \vec{x}, \vec{y})$$

$0 < a_1 < b_1 < \cdots < a_m < b_m < t$
 $x_1, y_1, \dots, x_m, y_m \in \mathbb{R}^2$

MOMENT FORMULAS

$g^{(4)} =$

