Central Limit Theorems in Disordered Systems and Stochastic PDEs

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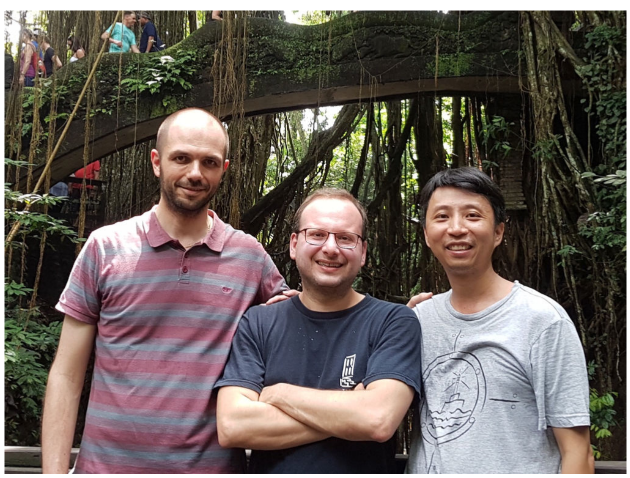

Based on joint works with
Nikos Zygouras and Rongfeng Sun

Overview

1. STOCHASTIC PDEs
   - 1.1 KPZ
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   - 3.1 NEW: 4^{TH} MOMENT
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References

[CSZ 17] F. Caravenna, R. Sun, N. Zygouras

UNIVERSALITY IN MARGINALLY RELEVANT DISORDERED SYSTEMS

Ann. Appl. Probab. 27 (2017), 3050-3112.

[CSZ 20] F. Caravenna, R. Sun, N. Zygouras

THE TWO-DIMENSIONAL KPZ EQ. IN THE ENTIRE SUBCRITICAL REGIME

Ann. Probab. 48 (2020), 1086-1127.

1. STOCHASTIC PDEs

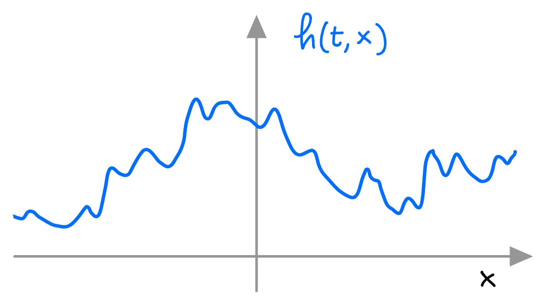
1.1 KPZ

KPZ equation [Kardar, Parisi, Zhang '86]

$$\underbrace{\partial_t h(t,x) = \frac{1}{2}\Delta h(t,x)}_{\substack{\text{HEAT EQUATION} \\ \text{(smoothing)}}} + \underbrace{\frac{1}{2}|\nabla h(t,x)|^2}_{\substack{\text{LATERAL GROWTH} \\ \text{(orthogonal)}}} + \underbrace{\beta\, \xi(t,x)}_{\substack{\text{DISORDER} \\ \text{(random)}}}$$

$h(t,x)$ = interface height at time $t \geq 0$, space $x \in \mathbb{R}^d$.

β = coupling constant



Choice of $\xi(t,x)$: **white noise** on $(0,\infty) \times \mathbb{R}^d$ (space-time)

Irregular: not a function, only a distribution (Schwartz)

[White noise on $(0,\infty)$: $\xi(t) = \frac{d}{dt} B(t)$ BROWNIAN MOTION]

KPZ solution non smooth: meaning of $|\nabla h(t,x)|^2$?

Key problem: well-posedness of KPZ

$$\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}|\nabla h|^2 + \beta\xi \qquad \text{(KPZ)}$$

- ($d=1$) Solved with REGULARITY STRUCTURES [Hairer]

  PARACONTROLLED CALCULUS [Gubinelli, Imkeller, Perkowski]

  ENERGY SOLUTIONS [Gonçalves, Jara] [Gubinelli, Perkowski]

  RENORMALIZATION [Kupiainen]

- ($d \geq 2$) Problem is open! This is our focus.

1.2 SHE

We can linearize the KPZ equation with a "trick":

$$u(t,x) := e^{h(t,x)} \qquad \text{(COLE-HOPF)}$$

If $\xi(t,x)$ were smooth, then $u(t,x)$ would solve

(MULTIPLICATIVE) STOCHASTIC HEAT EQUATION

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \beta\,\xi(t,x)\,u(t,x) \qquad \text{(SHE)}$$

- $(d=1)$ SHE well-posed by Ito stochastic integration:

$u(t,x)$ is a function >0 [Walsh '80s] [Bertini, Cancrini '95]

$\rightsquigarrow$ Notion of "KPZ solution" $h(t,x) := \log u(t,x)$

[Bertini, Giacomin 97]

INTEGRABLE PROBABILITY: Borodin, Corwin, Ferrari, Quastel, Spohn, ...

- $(d \geq 2)$ White noise $\xi(t,x)$ too irregular for Ito integration

REGULARIZATION $\xi^\varepsilon(t,x) := \big(\xi(t,\cdot) * \varrho_\varepsilon\big)(x)$ MOLLIFIER ON SCALE $\varepsilon > 0$

SHE well-posed with $\xi^\varepsilon(t,x)$ (Ito) $\rightsquigarrow$ SHE solution $u^\varepsilon(t,x)$

(explicit "Feynman-Kac" representation)

$\rightsquigarrow$ We can define "KPZ solution" $h^\varepsilon(t,x) := \log u^\varepsilon(t,x)$

Why? Ito's formula: h^ε solves a "renormalized" KPZ

$$\partial_t h^\varepsilon = \frac{1}{2}\Delta h^\varepsilon + \frac{1}{2}|\nabla h^\varepsilon|^2 + \beta\,\xi^\varepsilon - \underbrace{\beta^2\,\varepsilon^{-d}}_{\text{DIVERGING CONSTANTS!}}$$

Convergence of $u^\varepsilon(t,x)$ and $h^\varepsilon(t,x)$ as $\varepsilon \downarrow 0$?

"Yes", but things are subtle:

- We need to rescale the coupling constant $\beta = \beta_\varepsilon$ as $\varepsilon \downarrow 0$

$$\beta_\varepsilon \sim \begin{cases} \hat\beta \dfrac{1}{\sqrt{\log \frac{1}{\varepsilon}}} & (d=2) \\ \hat\beta\, \varepsilon^{\frac{d-2}{2}} & (d>2) \end{cases} \qquad \text{with } \hat\beta > 0$$

(Since $\beta_\varepsilon \to 0$, formally randomness disappears!)

- Convergence of u^ε and h^ε as distributions: for $\varphi \in C_c(\mathbb{R}^d)$

$$\langle u^\varepsilon(t), \varphi \rangle := \int_{\mathbb{R}^d} u^\varepsilon(t,x)\, \varphi(x)\, dx \longrightarrow \dots$$

1.3 MAIN RESULTS IN $d=2$

Summarising:

$$\begin{cases} \partial_t u^\varepsilon = \frac{1}{2} \Delta u^\varepsilon + \dfrac{\hat\beta}{\sqrt{\log \frac{1}{\varepsilon}}} \xi^\varepsilon u^\varepsilon \\ u^\varepsilon(0,x) \equiv 1 \end{cases} \qquad \text{(SHE)}$$

$$\begin{cases} \partial_t h^\varepsilon = \frac{1}{2} \Delta h^\varepsilon + \frac{1}{2} |\nabla h^\varepsilon|^2 + \dfrac{\hat\beta}{\sqrt{\log \frac{1}{\varepsilon}}} \xi^\varepsilon - \dfrac{\hat\beta^2}{\varepsilon^2 \log \frac{1}{\varepsilon}} \\ h^\varepsilon(0,x) \equiv 0 \end{cases} \qquad \text{(KPZ)}$$

There is a phase transition in $\hat{\beta}$ with "critical value" $\sqrt{2\pi}$:

$$\mathbb{E}\left[h^{\varepsilon}(t,x)\right] \xrightarrow[\varepsilon \downarrow 0]{} \begin{cases} -\frac{1}{2}\log\frac{2\pi}{2\pi-\hat{\beta}^2} & \text{if } \hat{\beta}<\sqrt{2\pi} \\ -\infty & \text{if } \hat{\beta}\geq\sqrt{2\pi} \end{cases}$$

We focus on the sub-critical regime $\hat{\beta}<\sqrt{2\pi}$. Define :

$$\mathcal{U}^{\varepsilon}(t,x) := \frac{\sqrt{\log\frac{1}{\varepsilon}}}{\hat{\beta}}\left(u^{\varepsilon}(t,x) - \overbrace{\mathbb{E}[u^{\varepsilon}(t,x)]}^{\equiv 1}\right)$$

CENTERING

+ RESCALING

$$\mathcal{H}^{\varepsilon}(t,x) := \frac{\sqrt{\log\frac{1}{\varepsilon}}}{\hat{\beta}}\left(h^{\varepsilon}(t,x) - \mathbb{E}[h^{\varepsilon}(t,x)]\right)$$

Theorem (Edwards-Wilkinson fluctuations) [CSZ 17-20]

$$\langle \mathcal{U}^{\varepsilon}(t),\varphi\rangle \text{ or } \langle \mathcal{H}^{\varepsilon}(t),\varphi\rangle \xrightarrow[\varepsilon\downarrow 0]{d} \langle X(t),\varphi\rangle$$

$\langle X(t),\varphi\rangle$ centered Gaussian process (~ GFF)

$$\mathrm{Cov}\left[\langle X(t),\varphi\rangle, \langle X(t),\psi\rangle\right] = \iint \varphi(x)\, K_t(x,y)\, \psi(y)\, dx\, dy$$

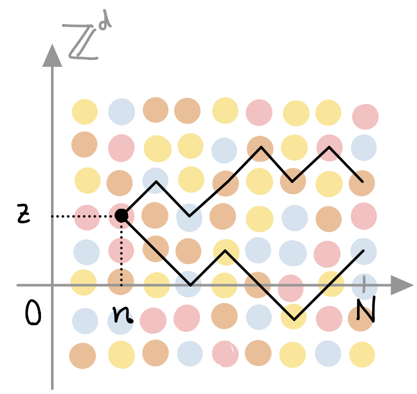
$$K_t(x,y) = \sqrt{\frac{2\pi}{2\pi-\hat{\beta}^2}} \int_0^t \frac{e^{-\frac{|x-y|^2}{4s}}}{4\pi s}\, ds \sim \log\frac{1}{|x-y|}$$

2. DISORDERED SYSTEMS

2.1 DIRECTED POLYMERS

We now switch to discrete space-time $\mathbb{N} \times \mathbb{Z}^d$ and define the model of directed polymer in random environment.

Two independent ingredients:



- $(S_i)_{i \geq 0}$ simple random walk
  "POLYMER"

- $(\omega(i,y))_{i \geq 0,\, y \in \mathbb{Z}^d}$ i.i.d. $\mathcal{N}(0,1)$
  "ENVIRONMENT" or "DISORDER"

Partition functions of directed polymer: $N \in \mathbb{N}$, $\beta \geq 0$

$$\underbrace{Z_N(n,z)}_{\text{STARTING POINT IN } \mathbb{N} \times \mathbb{Z}^d} := \underbrace{E}_{\text{SRW}}\left[e^{\sum_{i=n}^{N} \{\beta\, \omega(i, S_i) - \frac{\beta^2}{2}\}} \,\middle|\, S_n = z \right]$$

$\rightsquigarrow$ $Z_N(n,z)$ is a random variable, function of disorder ω.

Partition function solve a discretized SHE:

$$Z_N(n-1,z) - Z_N(n,z) = \frac{1}{2d}\underset{\text{LATTICE LAPLACIAN}}{\underbrace{\Delta}} Z_N(n,z) + \beta\, \underbrace{\tilde{\omega}(n,z)\, \tilde{Z}_N(n,z)}_{\text{SLIGHT MODIFICATIONS}}$$

Theorem: $Z_N(n,z)$ is close to SHE solution $u^{\varepsilon}(t,x)$

$$u^{\varepsilon}(t,x) \overset{d}{\approx} Z_N(tN, x\sqrt{N}) \qquad h^{\varepsilon}(t,x) \overset{d}{\approx} \log Z_N(tN, x\sqrt{N})$$

with $\varepsilon = \frac{1}{\sqrt{N}}$ and $\beta_{SHE} = \beta_{POLY}\, \varepsilon^{\frac{d-2}{2}}$

2.2 POLYNOMIAL CHAOS

We can express $Z_N(n,z)$ as an explicit function of modified disorder and random walk transition kernel:

- $$\tilde{\omega}(i,x) := \frac{e^{\beta\omega(i,x) - \frac{\beta^2}{2}} - 1}{\beta} \simeq \omega(i,x) \text{ as } \beta \to 0$$
- $$q(n,z\,;\, i,x) := P\left(S_i = x \mid S_n = z\right) \simeq \frac{e^{-\frac{|z-x|^2}{n-i}}}{n-i}$$

Polynomial chaos expansion

RANDOM WALK TRANSITION KERNEL

MODIFIED DISORDER

$$Z_N(n,z) = 1 + \beta \sum_{\substack{n \le i \le N \\ x \in \mathbb{Z}^d}} q(n,z;i,x)\, \tilde{\omega}(i,x)$$

$$+ \beta^2 \sum_{\substack{n \le i < j \le N \\ x,y \in \mathbb{Z}^d}} q(n,z;i,x)\, q(i,x;j,y)\, \tilde{\omega}(i,x)\, \tilde{\omega}(j,y)$$

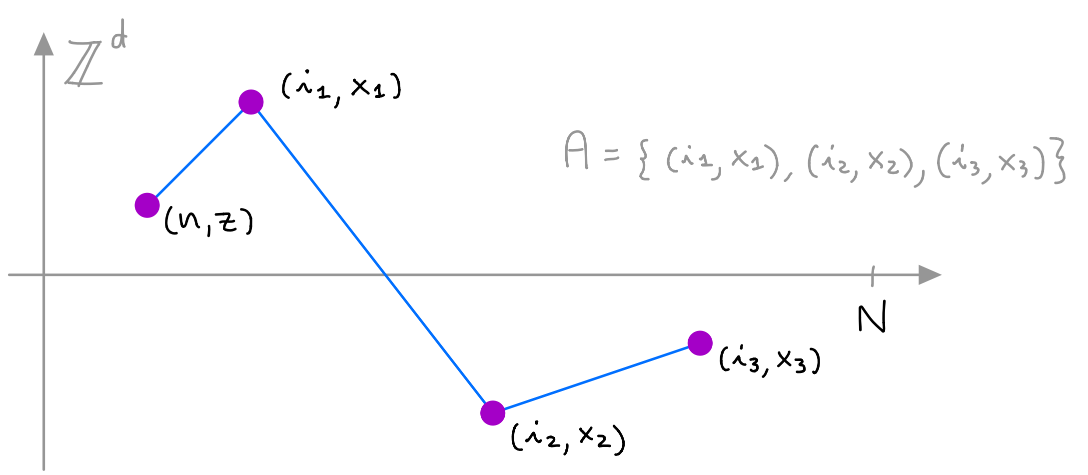
$$+ \dots$$

(FINITE SUM)

With compact notation:

$$Z_N(n,z) = \sum_{A \subseteq \{n+1,\dots,N\} \times \mathbb{Z}^d} q^\beta(n,z;A) \cdot \prod_{(i,x) \in A} \tilde{\omega}(i,x)$$

PROBABILITY THAT SRW FROM (n,z) VISITS ALL POINTS IN A



- $Z_N(n,z)$ is a multi-linear polynomial in the $\tilde{\omega}$'s

- Discrete analogue of Wiener chaos (→ Maurizia's talk)

  Analogous "Feynman-Kac formula" available for $u^\varepsilon(t,x)$ with Wiener chaos expansion (multiple Wiener-Ito integrals)

- Ideal for L^2 approximations:

$$\mathbb{E}\left[Z_N(n,z)^2\right] = \sum_{A \subseteq \{n,\dots,N\}\times\mathbb{Z}^d} q^\beta(A)^2$$

How to get the polynomial chaos?

$$Z_N(n,z) = E\left[e^{\sum_{i=n}^{N}\{\beta\omega(i,S_i) - \frac{\beta^2}{2}\}} \,\middle|\, S_n = z\right]$$

$$= E\left[\prod_{i=n}^{N}\left(1+\beta\,\tilde{\omega}(i,S_i)\right) \,\middle|\, S_n = z\right]$$

$$= 1 + \sum_{k=1}^{N-n}\beta^k \sum_{n<i_1<\dots<i_k\le N} E\left[\prod_{\ell=1}^{k}\tilde{\omega}(i_\ell, S_{i_\ell}) \,\middle|\, S_n = z\right]$$

$$= 1 + \sum_{k=1}^{N-n}\beta^k \sum_{\substack{n<i_1<\dots<i_k\le N \\ x_1,\dots,x_k\in\mathbb{Z}^d}} \prod_{\ell=1}^{k} q(i_{\ell-1},x_{\ell-1};i_\ell,x_\ell)\cdot\tilde{\omega}(i_\ell,x_\ell)$$

3. CENTRAL LIMIT THEOREMS

Main Theorem (E-W fluctuations) is a generalized CLT.

We prove it for $Z_N^{(u,z)}$ exploiting polynomial chaos.

Let us describe some powerful techniques, old and new.

3.1 NEW: 4^{TH} MOMENT THEOREMS

Criterion for polynomial chaos to converge to a Gaussian.

[de Jong 87, 90] [Nualart, Peccati 05] [Nourdin, Peccati, Reinert 10]

Theorem (4^{TH} Moment for Polynomial Chaos)

Let X_N be a polynomial chaos of fixed order $k \in \mathbb{N}$

$$X_N = \sum_{\substack{0 < i_1 < \dots < i_k \\ x_1, \dots, x_k \in \mathbb{Z}^d}} \underbrace{c_N(i_1, \dots, i_k; x_1, \dots, x_k)}_{= O(1)} \prod_{\ell=1}^{k} \underbrace{\tilde{\omega}(i_\ell, x_\ell)}_{\text{I.I.D. ZERO MEAN UNIT VARIANCE}}$$

Assume $\mathbb{E}[X_N^2] \to \sigma^2 < \infty$ and $\mathbb{E}[X_N^4] \to 3(\sigma^2)^2$.

Then

$$\boxed{X_N \xrightarrow{d} \mathcal{X} \sim N(0, \sigma^2)}$$

(+ QUANTITATIVE)

- Chaos of different orders converge to independent Gaussians

- Extension of the method of moments

$$\mathbb{E}[X_N^k] \longrightarrow \mathbb{E}[\mathcal{X}^k] \quad \forall k \in \mathbb{N} \quad \Longrightarrow \quad X_N \xrightarrow{d} \mathcal{X} \sim N(0, \sigma^2)$$

- Fourth moment of X_N is an explicit sum:

$$\mathbb{E}[X_N^4] = \sum_{\substack{A_1, A_2, A_3, A_4 \\ \subseteq \mathbb{N} \times \mathbb{Z}^d}} \left\{ \prod_{j=1}^{4} c_N(A_j) \right\} \underbrace{\mathbb{E}\left[\prod_{j=1}^{4} \prod_{(i,x) \in A_j} \tilde{\omega}(i,x) \right]}_{= 0 \text{ UNLESS } (i,x) \text{ "MATCH"}}$$

- Checking convergence of 4^{TH} moment is a (hard, but) feasible task ⟵ Provided we can control $c_N(A)$

TOY EXAMPLE $(\omega_n)_{n \in \mathbb{N}}$ i.i.d. zero mean, unit variance

- $X_N := \sum_{i=1}^{N} \frac{1}{\sqrt{N}} \omega_i \xrightarrow{d} \mathcal{X} \sim N(0,1)$
- $Y_N := \sum_{i=1}^{N} \frac{1}{\sqrt{N}} \omega_i \, \omega_{i+1} \xrightarrow{d} \mathcal{Y} \sim N(0,1)$

INDEPENDENT!

On the other hand

- $W_N := \sum_{1 \leq i < j \leq N} \frac{1}{N} \omega_i \omega_j \xrightarrow{d} \int_0^1 B_t \, dB_t = \frac{B_1^2 - 1}{2}$

NON GAUSSIAN !

Indeed $\mathbb{E}[W_N^2] \to \frac{1}{2}$ and $\mathbb{E}[W_N^4] \to \frac{5}{4} > 3 \cdot \left(\frac{1}{2}\right)^2$

RELEVANT EXAMPLE

- $X_N := \frac{1}{\sqrt{\log N}} \sum_{i=1}^{N} \frac{1}{\sqrt{i}} \omega_i \xrightarrow{d} \mathcal{X} \sim \mathcal{N}(0,1)$

- $Y_N := \frac{1}{\log N} \sum_{1 \leq i < j \leq N} \frac{\omega_i \, \omega_j}{\sqrt{i}\sqrt{j-i}} \xrightarrow{d} \frac{\mathcal{X}^2 - 1}{2} + \mathcal{Y}$

$\mathcal{Y} \sim$ INDEP. $\mathcal{N}(0,1)$

- ...

More precisely:

$$X_N^{(k)} = \frac{1}{(\sqrt{\log N})^k} \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ x_1, \dots, x_k \in \mathbb{Z}^2}} \left\{ \prod_{\ell=1}^{k} q(i_{\ell-1}, x_{\ell-1}; i_\ell, x_\ell) \right\} \cdot \prod_{\ell=1}^{k} \tilde{\omega}(i_\ell, x_\ell)$$

3.2 "OLD": FELLER-LINDEBERG CLT

Fourth moment theorems are "optimal tools" for general polynomial and Wiener chaos (necessary and sufficient)

In the specific setting of SHE / KPZ / directed polymer, a more elementary approach is possible as follows (work in progress with F. Cottini)

Theorem (Feller-Lindegerg CLT for triangular arrays)

For $N \in \mathbb{N}$, let $(X_{N,i})_{i=1,\dots,M_N}$ be independent r.v.s with zero mean and finite variance.

Assume:

- VARIANCE CONVERGENCE: $$\sum_{i=1}^{M_N} \mathbb{E}\left[X_{N,i}^2\right] \longrightarrow \sigma^2 < \infty$$
- LINDEBERG: $$\forall \varepsilon > 0 \quad \sum_{i=1}^{M_N} \mathbb{E}\left[X_{N,i}^2 \, \mathbb{1}_{\{|X_{N,i}| > \varepsilon\}}\right] \longrightarrow 0$$

Then
$$\boxed{X_N := \sum_{i=1}^{M_N} X_{N,i} \xrightarrow{d} \mathcal{X} \sim N(0, \sigma^2)}$$

In our setting X_N is a polynomial chaos:

$$X_N = \sum_{A \subseteq \mathbb{N} \times \mathbb{Z}^2} c_N(A) \prod_{(n,x) \in A} \tilde{\omega}(n,x)$$

We partition $\mathbb{N} \times \mathbb{Z}^2$ in disjoint boxes and define

$$\mathbb{N} \times \mathbb{Z}^2 = \bigcup_{i=1}^{M_N} \mathbb{B}_{N,i} \quad \rightsquigarrow \text{DISJOINT}$$

$$X_{N,i} := \sum_{A \subseteq \mathbb{B}_{N,i}} c_N(A) \prod_{(n,x) \in A} \tilde{\omega}(n,x) \qquad \text{INDEPENDENT!}$$

Lemma. We can choose the boxes $\mathbb{B}_{N,i}$ so that

- $X_N - \sum_{i=1}^{M_N} X_{N,i} \xrightarrow{L^2} 0$
- $\max_{i=1,\dots,M_N} \mathbb{E}[X_{N,i}^2] \longrightarrow 0$

($\Rightarrow$ LINDEBERG CONDITION, by hypercontractivity)

$\rightsquigarrow$ By Feller-Lindeberg $\quad X_N \xrightarrow{d} \mathcal{X} \sim \mathcal{N}(0,\sigma^2)$

- Stronger assumptions than 4^{TH} moment (but more elementary)

CONCLUSIONS

- We presented CLTs for singular stochastic PDEs

  (KPZ / SHE in space dimension $d=2$)

- Close link with directed polymers

- Polynomial / Wiener chaos allows for powerful tools

  (4^{TH} Moment Theorem, Hypercontractivity, Concentration, ...)

- Many results in $d \geqslant 3$ and for anisotropic KPZ

  [Chatterjee, Dunlap] [Dunlap, Gu, Ryzhik, Zeitouni]
  [Comets, Cosco, Mukerjee] [Magnen, Unterberger]
  [Lygkonis, Zygouras] [Cannizzaro, Erhard, Toninelli] ...

- Next challenge: critical regime $\hat{\beta} = \sqrt{2\pi}$

  [Bertini, Cancrini 96] [CSZ 20b] [Gu, Quastel, Tsai 20]

Thanks!