The critical 2d Stochastic Heat Flow

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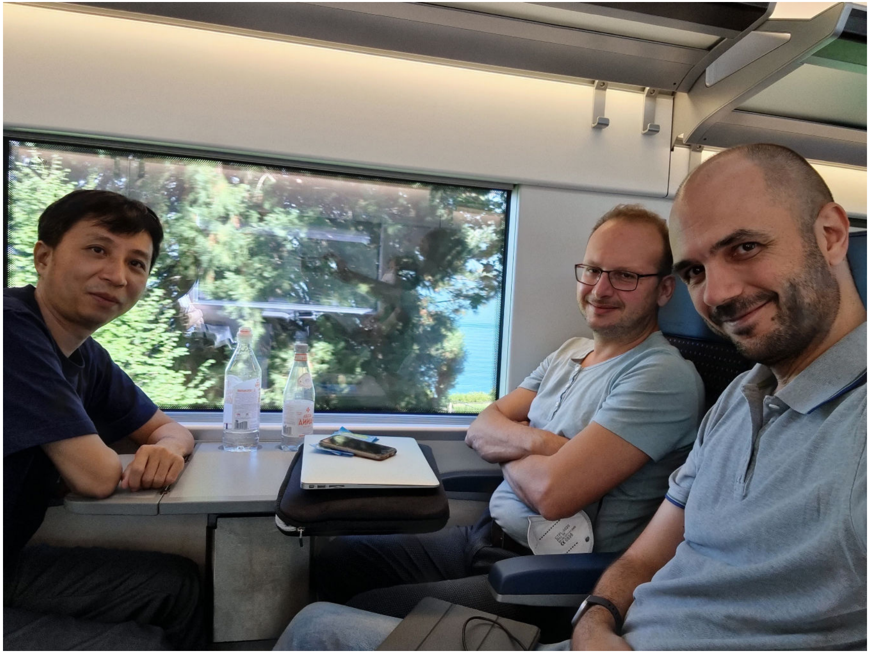
Based on joint works with



Rongfeng Sun and Nikos Zygouras

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THE CRITICAL 2D STOCHASTIC HEAT FLOW
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I. Introduction

THE STOCHASTIC HEAT EQUATION

$$(\mathrm{SHE}) \quad \begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \beta\, \xi(t,x)\, u(t,x) & t>0,\ x\in\mathbb{R}^d \\ u(0,x) \equiv 1 \end{cases}$$

- $\beta > 0$ coupling constant
- $\xi(t,x)$ "space-time white noise" (δ-correlated Gaussian)

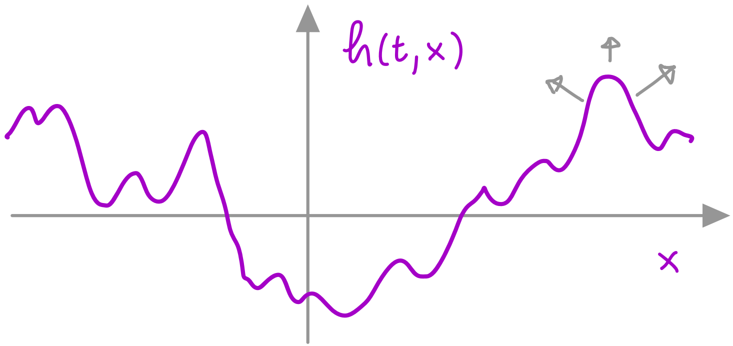
GOAL: Construct the solution $u(t,x)$ for $d=2$

THE KARDAR - PARISI - ZHANG EQUATION

[PRL 1986]

COLE-HOPF TRANSFORMATION: formally $h(t,x) := \log U(t,x)$ solves

$$\text{(KPZ)} \qquad \partial_t h(t,x) = \underbrace{\Delta h(t,x)}_{\text{SMOOTHING}} + \underbrace{|\nabla h(t,x)|^2}_{\perp \text{ GROWTH}} + \underbrace{\beta\, \xi(t,x)}_{\text{NOISE}}$$



SHE can help us make sense of KPZ

SINGULARITY

(SHE) and (KPZ) are ill-defined due to singular products

$$\xi(t,x)\; u(t,x) \qquad\qquad |\nabla h(t,x)|^2$$

$\xi(t,x)$ is a distribution $\rightsquigarrow$ $u(t,x)$ and $h(t,x)$ expected to be:

- non-smooth functions $(d=1)$
- genuine distributions $(d\geq 2)$

Henceforth we focus on (SHE)

THE ROLE OF DIMENSION

Space-time blow up: $\tilde{U}(t,x) := U(\varepsilon^2 t, \varepsilon x)$ solves

$$\partial_t \tilde{U}(t,x) = \Delta \tilde{U}(t,x) + \beta\, \varepsilon^{\frac{2-d}{2}}\, \tilde{\xi}(t,x)\, \tilde{U}(t,x)$$

As $\varepsilon \downarrow 0$ the noise formally
$$\begin{cases} \text{vanishes} & (d<2) \\ \text{stays constant} & (d=2) \\ \text{diverges} & (d>2) \end{cases}$$

$d=2$ is CRITICAL DIMENSION for SHE

Dimension $d=1$

(1980s) $u(t,x)$ well-posed by stochastic integration (Ito-Walsh)

(2010s) Robust solution theories for "sub-critical" singular PDEs

- REGULARITY STRUCTURES [Hairer]
- PARACONTROLLED CALCULUS [Gubinelli, Imkeller, Perkowski]
- ENERGY SOLUTIONS [Goncalves, Jara]
- RENORMALIZATION [Kupiainen]

Dimensions $d \geq 3$

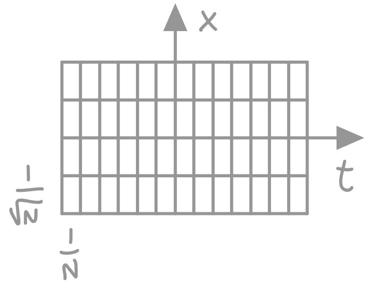
(2015- ...) Results by several authors

Magnen, Unterberger, Chatterjee, Dunlap, Gu, Ryzhik, Zeitouni, Comets, Cosco, Mukherjee, Lygkonis, Zygouras, Nakajima, Nakashima, Junk, ...

Regularizing SHE via discretization

We fix $d=2$ and restrict to the lattice

$$(t,x) = \left(\tfrac{k}{N}, \tfrac{z}{\sqrt{N}}\right) \in \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$$



Discretized SHE

$$\partial_t^N u_N(t,x) = \Delta^N u_N(t,x) + \beta \cdot N \cdot X\left(t+\tfrac{1}{N}, x\right) \langle u_N(t,x) \rangle$$

DISCR. DERIVATIVE

$$N\left\{u\left(t+\tfrac{1}{N},x\right) - u(t,x)\right\}$$

DISCR. LAPLACIAN

$$\frac{N}{4}\sum_{x'\sim x}\left\{u(t,x') - u(t,x)\right\}$$

I.I.D. RVs

$$\mathbb{E}[X]=0 \qquad \mathbb{E}[X^2]=1$$

$$\frac{1}{4}\sum_{x'\sim x} u(t,x')$$

Well-defined solution $u_N(t,x) \geq 0$ (if $\beta X \geq -1$) $\left[u_N(0,\cdot) \equiv 1\right]$

CONVERGENCE ?

Can we hope that $u_N(t,x) \xrightarrow[N\to\infty]{} \mathcal{U}(t,x)$? (non-trivial limit)

YES ! But...

1. Convergence as (random) distributions : $\varphi \in C_c(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} \varphi(x)\, u_N(t,x)\, dx \xrightarrow[N\to\infty]{d} \int_{\mathbb{R}^2} \varphi(x)\, \underbrace{\mathcal{U}(t,dx)}_{\text{random measure on } \mathbb{R}^2}$$

2. Rescale coupling

$$\boxed{\beta = \beta_N = O\left(\frac{1}{\sqrt{\log N}}\right) \xrightarrow[N\to\infty]{} 0}$$

Why do we rescale $\beta = \beta_N \sim \dfrac{\hat\beta}{\sqrt{\log N}}$?

$$\mathbb{E}\left[\int \varphi(x)\, U_N(t,x)\, dx\right] \equiv \int \varphi(x)\, dx$$

$$\mathbb{V}\mathrm{AR}\left[\int \varphi(x)\, U_N(t,x)\, dx\right] \longrightarrow \begin{cases} 0 & \text{if } \hat\beta < \sqrt{\pi} \\ \sigma^2(\varphi) & \text{if } \hat\beta = \sqrt{\pi} \\ \infty & \text{if } \hat\beta > \sqrt{\pi} \end{cases}$$

PHASE TRANSITION

For $\hat\beta < \sqrt{\pi}$: $U_N(t,x)\, dx \xrightarrow{d} dx$ = Lebesgue measure ("trivial")

For $\hat\beta = \sqrt{\pi}$ do we have $U_N(t,x)\, dx \xrightarrow{d} \mathcal{U}(t, dx)$?

II. Main Results

MAIN THEOREM

[CSZ 23]

Rescale $\beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$, more precisely

$$\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \frac{\vartheta}{\log N}\right) \qquad \vartheta \in \mathbb{R}$$

Then $u_N(t,x)$ converges to a unique & non-trivial limit:

$$\left(u_N(t,x)\,dx\right)_{t\geq 0} \xrightarrow[N\to\infty]{\text{f.d.d.}} \mathscr{U}^{\vartheta} = \left(\mathscr{U}_t^{\vartheta}(dx)\right)_{t\geq 0}$$

$\mathscr{U}^{\vartheta}$ is a stochastic process of random measures on $\mathbb{R}^2$:

critical 2d STOCHASTIC HEAT FLOW (SHF)

SHF & SHE

We have built a candidate solution of the Critical 2d SHE : $\beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$

the SHF $\quad \mathcal{U}^{\vartheta} = \left(\mathcal{U}^{\vartheta}_t(dx)\right)_{t \geq 0}$

with initial condition $\mathcal{U}^{\vartheta}_0(dx) \equiv dx \qquad (u(0,\cdot) \equiv 1)$

Remark. We actually build a two-parameter process

$$\mathcal{U}^{\vartheta} = \left(\mathcal{U}^{\vartheta}_{s,t}(dy, dx)\right)_{0 \leq s \leq t < \infty}$$

"CANDIDATE SHE SOLUTION FROM dy AT TIME s"

KEY FEATURES OF THE SHF

- $\mathbb{E}\left[\mathcal{U}_t^{\vartheta}(dx)\right] = dx$

- $\mathbb{E}\left[\mathcal{U}_t^{\vartheta}(dx)\,\mathcal{U}_t^{\vartheta}(dy)\right] = K_t^{\vartheta}(x,y)\,dx\,dy$ [Bertini, Cancrini 98]

  $\sim \log \frac{1}{|x-y|}$ $\Rightarrow \mathcal{U}^{\vartheta}$ non trivial

- $\mathcal{U}_{at}^{\vartheta}(d(\sqrt{a}\,x)) \overset{d}{=} a\,\mathcal{U}_t^{\vartheta+\log(a)}(dx)$

- Formulas for higher moments [Gu, Quastel, Tsai 21] [CSZ 19b]

Gaussian Multiplicative Chaos ?

Random measure $\quad \mathcal{M}(dx) = $ " $e^{\mathcal{X}(x) - \frac{1}{2}\kappa(x,x)}\, dx$ "

$$\mathcal{X} \sim \mathcal{N}(0, \kappa) \quad \text{generalized Gaussian field}$$

$$\iint \varphi(x)\varphi(y)\, \mathbb{E}[\mathcal{X}(x)\,\mathcal{X}(y)]\, dx\, dy = \iint \kappa(x,y)\, \varphi(x)\varphi(y)\, dx\, dy$$

GMCs are "canonical": many explicit features

$$\mathbb{E}[\mathcal{M}(dx)] = dx \qquad \mathbb{E}[\mathcal{M}(dx)\,\mathcal{M}(dy)] = e^{\kappa(x,y)}\, dx\, dy$$

Is the SHF $\mathcal{U}_t^{\vartheta}(dx)$ a GMC $\mathcal{M}(dx)$?

(With $\kappa(x,y) \sim \log\log\frac{1}{|x-y|}$)

THEOREM The SHF $\mathscr{U}_t^{\vartheta}(dx)$ is NOT a GMC [CSZ 23+]

Recall: formally $h(t,x) = \log u(t,x)$ solves (KPZ)

CONJECTURE The critical 2d KPZ equation should have a NON GAUSSIAN SOLUTION $\mathscr{H}_t(dx)$

(KPZ) solution yet to be constructed! Cannot take $\log \mathscr{U}_t^{\vartheta}(dx)$

FURTHER FEATURES OF THE SHF

THEOREM [CSZ 23++]

The SHF $\mathscr{U}_t^{\vartheta}(dx)$ is a.s. non atomic & singular w.r.t. Lebesgue:

$$\text{for Leb-a.e. } x \in \mathbb{R}^2: \quad \frac{\mathscr{U}_t^{\vartheta}(B(x,\delta))}{\underbrace{|B(x,\delta)|}_{\pi\delta^2}} \xrightarrow[\delta \downarrow 0]{a.s.} 0$$

$$\mathbb{E}\left[\frac{\mathscr{U}_t^{\vartheta}(B(x,\delta))}{|B(x,\delta)|}\right] \equiv 1 \qquad \mathbb{V}\mathrm{AR}\left[\frac{\mathscr{U}_t^{\vartheta}(B(x,\delta))}{|B(x,\delta)|}\right] \xrightarrow[\delta \downarrow 0]{} \infty$$

SMALL SCALE LOG-NORMALITY

We can decrease disorder strength ϑ to keep variance finite:

$$\mathrm{VAR}\left[\frac{\mathcal{U}_t^{\vartheta + c \log \delta^2}(B(x,\delta))}{|B(x,\delta)|}\right] \xrightarrow[\delta \downarrow 0]{} \frac{1}{c} < \infty$$

THEOREM

$$\frac{\mathcal{U}_t^{\vartheta + c \log \delta^2}(B(x,\delta))}{|B(x,\delta)|} \xrightarrow[\delta \downarrow 0]{d} e^{N(0, \log \frac{1}{c}) - \frac{1}{2} \log \frac{1}{c}} \xrightarrow[c \downarrow 0]{d} 0$$

- Multiplicative decomposition

  [C., Cottini 22] [Cosco, Donadini 23+]

- Monotonicity of fractional moments

  (FKG)

LONG TIME BEHAVIOUR

We can prove that, as time $t \uparrow \infty$, the SHF locally vanishes:

$$\forall R > 0: \qquad \mathscr{U}_t^{\vartheta}(B(0,R)) \xrightarrow[t\uparrow\infty]{d} 0 \qquad \text{"mass escapes to infinity"}$$

CONJECTURE

$$\frac{\mathscr{U}_t^{\vartheta}(B(0,\sqrt{t}))}{t} \xrightarrow[t\uparrow\infty]{d} 0 \qquad \text{"super-diffusivity"}$$

$$\iff \quad \mathscr{U}_1^{\vartheta}(B(0,1)) \xrightarrow[\vartheta\uparrow\infty]{d} 0 \qquad \text{"strong disorder"}$$

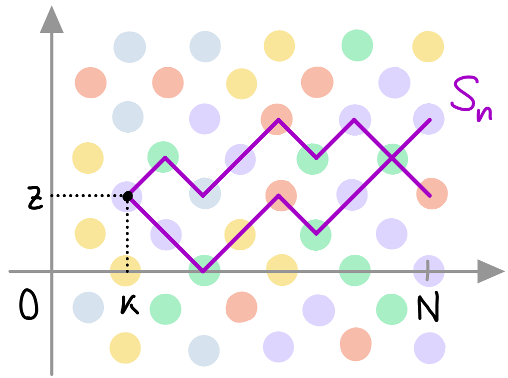
In progress: continuum polymer measure $\mathcal{P}_t^{\vartheta}(x, dy) := \dfrac{\mathscr{U}_t^{\vartheta}(dx, dy)}{\mathscr{U}_t^{\vartheta}(dx, \mathbb{R}^2)}$

III. Ideas and Techniques

Directed Polymer in Random Environment

- $(S_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^2$

$$P\left(S_n - S_{n-1} = \begin{matrix}(\pm 1, 0)\\(0, \pm 1)\end{matrix}\right) = \frac{1}{4}$$



- $(\omega(n,z))_{n \in \mathbb{N}, z \in \mathbb{Z}^2}$ i.i.d. environment

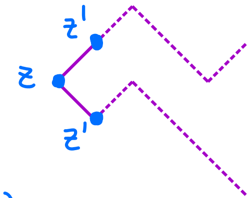
$$\mathbb{E}[\omega] = 0 \qquad \mathbb{E}[\omega^2] = 1 \qquad \lambda(\beta) = \log \mathbb{E}[e^{\beta\omega}] < \infty$$

Partition Functions	$Z_N^{\omega}(k,z) := E\left[e^{\sum_{n=k}^{N} \beta\,\omega(n,S_n) - \lambda(\beta)} \,\middle|\, S_k = z \right]$

Feynman-Kac formula

Discretized SHE solution $u_N(t,x) = Z_N^\omega(N(1-t), \sqrt{N}x)$

Proof. Markov property of SRW:



$$Z_N^\omega(k,z) = e^{\beta\omega(k,z)-\lambda(\beta)} \frac{1}{4}\sum_{z'\sim z} Z_N^\omega(k+1,z')$$

$$\Downarrow$$

$$-\partial_{k,k+1} Z_N^\omega(k,z) = \Delta_z Z_N^\omega(k+1,z) + \underbrace{\left\{e^{\beta\omega(k,z)-\lambda(\beta)}-1\right\}}_{\text{i.i.d. } \beta X(k,z) \;\geq -1} \langle Z_N^\omega(k+1,z)\rangle$$

(time-reversed) discretized SHE

SECOND MOMENT AND CRITICAL SCALING OF β

$$\mathbb{E}\left[u_N(1,x) \cdot u_N(1,x') \right] = E\left[e^{\beta^2 \underbrace{\sum_{i=0}^{N} \mathbb{1}_{\{S_i = S'_i\}}}_{\mathcal{L}_N}} \,\middle|\, S_0 = x\sqrt{N},\ S'_0 = x'\sqrt{N} \right]$$

$$\parallel$$

$$\mathbb{E}\left[Z_N(x\sqrt{N}) \cdot Z_N(x'\sqrt{N}) \right]$$

"REPLICA OVERLAP"

For $x = x'$: $\qquad \dfrac{\pi}{\log N}\, \mathcal{L}_N \xrightarrow[N\to\infty]{d} Y \sim \mathrm{Exp}(1)$ $\qquad$ [Erdös-Taylor 60]

This explains the CRITICAL SCALING $\qquad \beta \sim \dfrac{\sqrt{\pi}}{\sqrt{\log N}}$

CORRELATION FUNCTION

$$\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \frac{\vartheta}{\log N}\right)$$

$$K_N^\beta(t,x,x') := \mathbb{E}\left[U_N(t,x)\cdot U_N(t,x')\right] \xrightarrow[N\to\infty]{} K_t^\vartheta(x,x')$$

It solves the 2-body delta-Bose gas discretized in $\frac{\mathbb{Z}^2}{\sqrt{N}}$:

$$\partial_t^N K_N^\beta = -\mathcal{H}_N^\beta K_N^\beta \qquad \text{where} \qquad \mathcal{H}_N^\beta = -\Delta^N - \beta\cdot N\cdot \mathbb{1}_{\{x=x'\}}$$

$$K_t^\vartheta = e^{-t\mathcal{H}^\vartheta}$$

self-adj. ext. of $\mathcal{H}^\beta = -\Delta - \beta\,\delta(x-x')$

EXPLICIT FORMULA

[Albeverio, Gesztesy, Høegh-Krohn, Holden 87]

[Y.-T. Chen 22] Probabilistic representation as singular diffusion

HIGHER ORDER CORRELATIONS

$$\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \frac{\vartheta}{\log N}\right)$$

$$K_N^\beta(t, x_1, \dots, x_n) := \mathbb{E}\left[U_N(t, x_1) \cdots U_N(t, x_n) \right] \xrightarrow[N\to\infty]{} K_t^\vartheta(x_1, \dots, x_n)$$

It solves the n-body delta-Bose gas discretized in $\frac{\mathbb{Z}^2}{\sqrt{N}}$:

$$\partial_t^N K_N^\beta = -\mathcal{H}_N^\beta K_N^\beta \qquad \text{where} \qquad \mathcal{H}_N^\beta = -\Delta^N - \beta \cdot N \cdot \sum_{1 \le i < j \le n} \mathbb{1}_{\{x_i = x_j\}}$$

$$K_t^\vartheta = e^{-t\mathcal{H}^\vartheta}$$

self-adj. ext. of $\mathcal{H}^\beta = -\Delta - \beta \sum_{1 \le i < j \le n} \delta(x_i - x_j)$

EXPLICIT FORMULAS

[dell'Antonio, Figari, Teta 94] [Dimock, Rajeev 04] [Gu, Quastel, Tsai 19]

MAIN RESULT: STRATEGY OF THE PROOF

$$u_N(t,x)\,dx \xrightarrow{d} \mathscr{U}_t^{\vartheta}(dx)$$

- Existence of subsequential limits is "easy" [Bertini-Cancrini 98]
- Non-triviality of the limit is non trivial [CSZ 19b]
- Uniqueness is hard [CSZ 23]

(Moments grow too fast to determine the distribution)

HOW TO PROVE UNIQUENESS ?

We use a Cauchy argument:

$$u_N(t,x)\,dx \overset{d}{\approx} u_M(t,x)\,dx \qquad \text{for large } N, M$$

exploiting self-similarity of the model

A. COARSE-GRAINING $\rightsquigarrow$ B. RENEWAL STRUCTURE

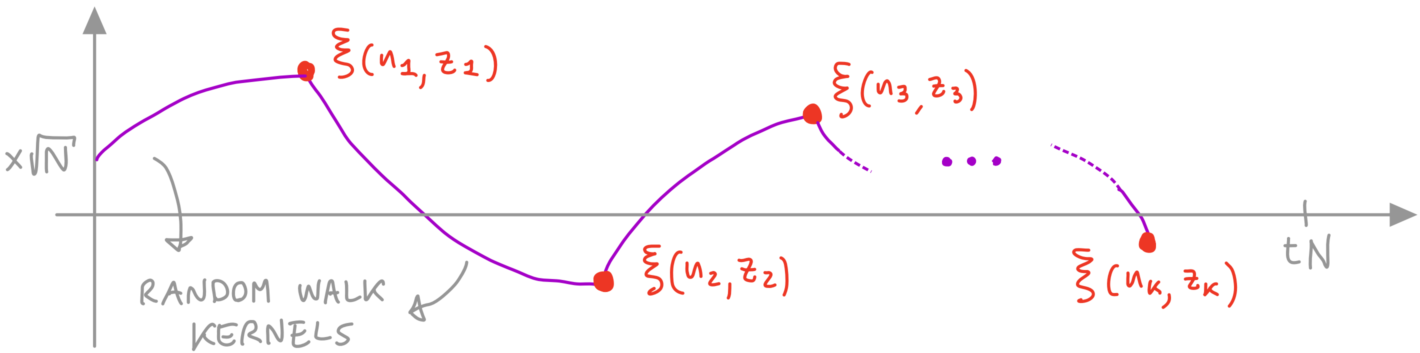
C. LINDEBERG PRINCIPLE $\rightsquigarrow$ D. FUNCTIONAL INEQUALITIES

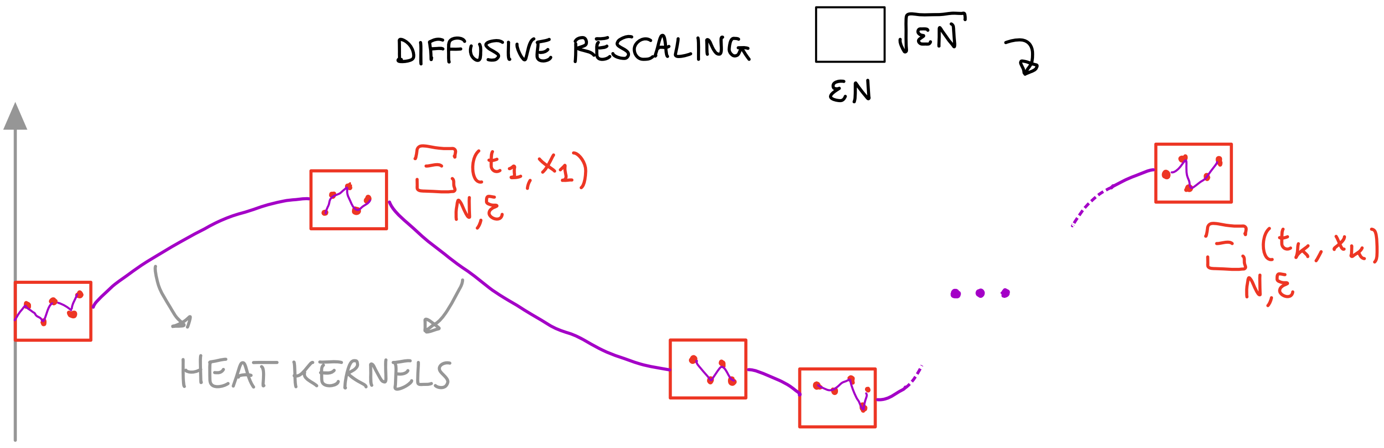
A. Coarse-Graining

Polynomial chaos expansion:

$P(S_{n_1} = z_1, \dots, S_{n_k} = z_k)$

$$U_N(t,x) = 1 + \sum_{k=1}^{N} \beta^k \sum_{(n_1,z_1),\dots,(n_k,z_k)} q\big((n_1,z_1),\dots,(n_k,z_k)\big) \cdot \prod_{i=1}^{k} \xi(n_i,z_i)$$





Sharp L^2 approximation via a coarse-grained model

$$U_N(t,x)\,dx \approx \mathscr{Z}^{CG}_{\varepsilon}(t, dx \mid \Xi_{N,\varepsilon}) \quad \text{as } \varepsilon \downarrow 0$$

MULTI-LINEAR POLYNOMIAL

"COARSE-GRAINED" NOISE

B. Renewal structure

Probabilistic interpretation of 2^{ND} moment

$$\mathbb{E}\left[u_N(t,x) \cdot u_N(t,x') \right] = \sum_{k=1}^{N} \beta^{2k} \sum_{(n_1,z_1),\dots,(n_k,z_k)} q\big((n_1,z_1),\dots,(n_k,z_k)\big)^2$$

$$\xrightarrow[N\to\infty]{} K_t^{\vartheta}(x,x') = 2\pi \int_0^t ds \; g_s(x-x') \int_s^t e^{\vartheta u} \, P(Y_u \le t) \, du$$

[CSZ 19a]

HEAT KERNEL "DICKMAN SUBORDINATOR"

c. LINDEBERG PRINCIPLE

The distribution of coarse-grained model $\mathcal{Z}^{CG}_{\varepsilon}(t, dx \mid \Xi)$ is insensitive to the distribution of Ξ

(as $\varepsilon \downarrow 0$, provided 1^{ST} & 2^{ND} moments are fixed) [Röllin 2013]

$\rightsquigarrow$ We can switch $\Xi_{N,\varepsilon}$ to $\Xi_{M,\varepsilon}$ and get our goal

$$U_N(t,x)\,dx \overset{d}{\approx} U_M(t,x)\,dx$$

D. FUNCTIONAL INEQUALITIES

Inequalities for Green's function of multiple random walks

"CRITICAL" HARDY-LITTLEWOOD-SOBOLEV INEQUALITY

$$\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \frac{f(x,x')\cdot g(y,y')}{\left(|x-y|+|x'-y'|+|x-y'|\right)^{2d}}\, dx\, dx'\, dy\, dy' \leqslant C\, \|f\|_{L^p}\, \|g\|_{L^q}$$

Generalizes an inequality by [dell'Antonio, Figari, Teta 94]

IV. Conclusions and Perspectives

CONCLUSIONS

We introduced the CRITICAL 2D STOCHASTIC HEAT FLOW $\mathscr{U}_t^{\vartheta}(dx)$

as the scaling limit of solutions of discretized SHE

$\longleftrightarrow$ directed polymer partition functions

- Universal process of random measures on $\mathbb{R}^2$ $(\neq \text{GMC})$
- Natural candidate solution for critical 2d SHE

Many explicit features ...

... and several interesting open questions:

- FLOW PROPERTY (CHAPMAN-KOLMOGOROV)
- SHF AS A MARKOV PROCESS
- CHARACTERIZING PROPERTIES & UNIVERSALITY
- TAKING LOG $\rightsquigarrow$ KPZ

$$\boxed{\text{Statistical Mechanics} \longleftrightarrow \text{Singular Stochastic PDEs}}$$

RELATED WORKS

- ANISOTROPIC KPZ: $\Delta h = (\partial_x^2 + \partial_y^2) h \rightsquigarrow (\partial_x^2 - \partial_y^2) h$

  [Erhard, Cannizzaro, Toninelli, Gubinelli]

- SHE WITH LÉVY NOISE: $\mathbb{P}(|\xi| > t) \sim \frac{c}{t^\alpha}$ $\quad 0 < \alpha < 2$

  [Berger, Chong, Lacoin, Viveros]

- DIRECTED POLYMERS, SHE & KPZ

  [Clark] [Comets, Cosco, Mukherjee] [Lygkonis, Zygouras] [Junk] [Nakajima, Nakashima] [Dunlap, Gu] [Tao] [Dunlap, Graham] ...

Thanks

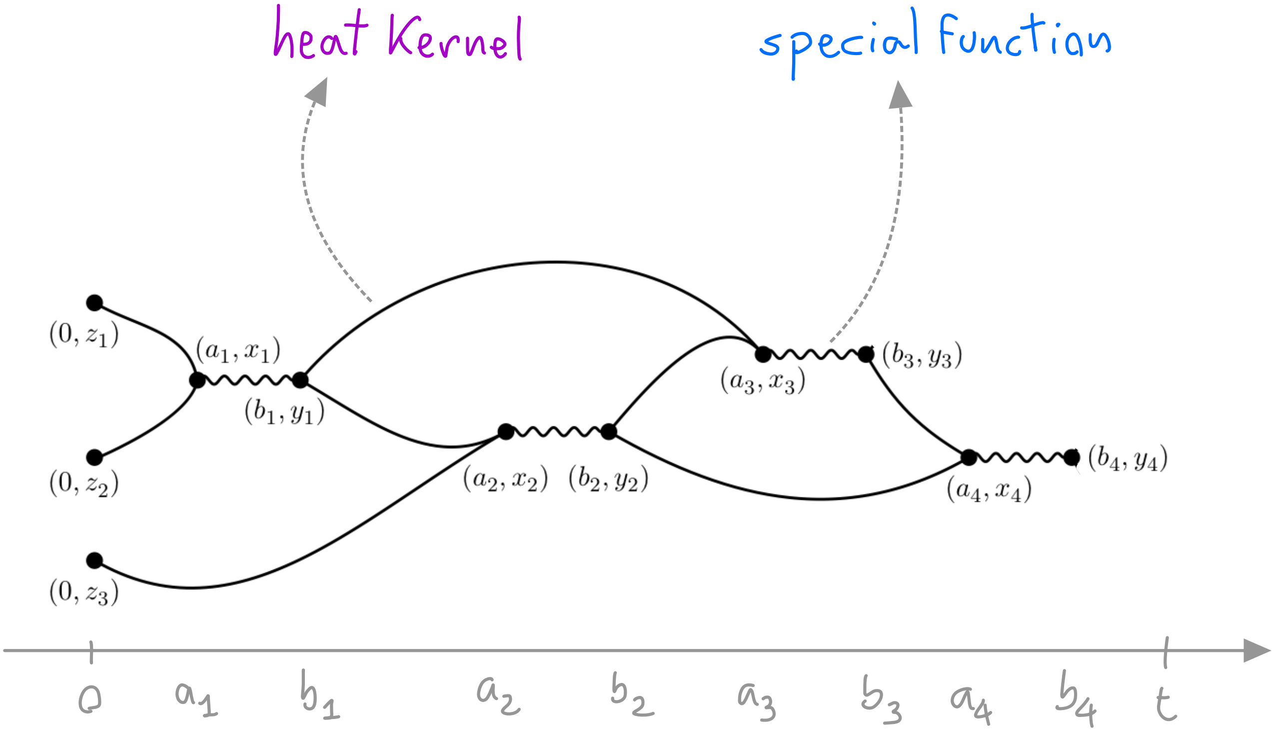
Moment Formulas

$$\mathbb{E}\left[\mathcal{U}_t^{\vartheta}(dx)\cdot\mathcal{U}_t^{\vartheta}(dy)\cdot\mathcal{U}_t^{\vartheta}(dz)\right] = \underbrace{K^{(3)}(x,y,z)}_{\lim\limits_{N\to\infty}\mathbb{E}\left[U_N(t,x)\cdot U_N(t,y)\cdot U_N(t,z)\right]}\,dx\,dy\,dz$$

$$K^{(3)}(z_1,z_2,z_3) = \sum_{m\geq 2}\ \underset{\substack{0<a_1<b_1<\ldots<a_m<b_m<t\\ x_1,\,y_1,\,\ldots,\,x_m,\,y_m\in\mathbb{R}^2}}{\int\cdots\int} d\vec{a}\,d\vec{b}\,d\vec{x}\,d\vec{y}\ \ g^{(m)}_{\vec{z}}(\vec{a},\vec{b},\vec{x},\vec{y})$$

Moment Formulas

$$g^{(4)} =$$



NON GMC-NESS

Consider the 2d Stochastic Heat Flow $\mathscr{U}_t^\vartheta$ for fixed $t > 0$, $\vartheta \in \mathbb{R}$

Let $\mathcal{M}(dx)$ be the GMC with matching 1st and 2nd moments

$$\mathbb{E}\left[\mathcal{M}(dx)\,\mathcal{M}(dy)\right] = \mathbb{E}\left[\mathscr{U}_t^\vartheta(dx)\,\mathscr{U}_t^\vartheta(dy)\right] = K_t^\vartheta(x,y)\,dx\,dy$$

We prove that higher moments do not match

- 3rd MOMENT BOUND: For any $R > 0$

$$\mathbb{E}\left[\mathscr{U}_t^\vartheta(B_R)^3\right] > \mathbb{E}\left[\mathcal{M}(B_R)^3\right]$$

- HIGHER MOMENT BOUND: there is $\eta > 0$ s.t. for any $k \geqslant 3$

$$\liminf_{\delta \downarrow 0} \frac{\mathbb{E}\left[\mathcal{U}_t^{\vartheta}(g_\delta)^k\right]}{\mathbb{E}\left[\mathcal{M}(g_\delta)^k\right]} \geqslant 1 + \eta$$

HEAT KERNEL AT TIME δ

- 3^{rd} MOMENT BOUND $\begin{cases} \text{explicit diagrammatic expansion} \\ \text{Gaussian calculations} \end{cases}$

- HIGHER MOMENT BOUND via Gaussian Correlation Inequality

(inspired by [Feng 16])