HAIRER'S RECONSTRUCTION THEOREM WITHOUT REGULARITY STRUCTURES

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11 JUNE 2020

BERLIN / ZOOM SPDE SEMINAR

0 - OUTLINE

- PRESENT AN ENHANCED VERSION OF THE RECONSTRUCTION THEOREM
- OPTIMAL ASSUMPTIONS & ELEMENTARY APPROACH

  (NO REFERENCE TO REGULARITY STRUCTURES)
- DISCUSS A COUPLE OF APPLICATIONS
- SKETCH THE MAIN IDEAS OF THE PROOF

1 - INTRODUCTION

$$\mathcal{D} := \{ \varphi : \mathbb{R}^d \to \mathbb{R},\ \varphi \in C^\infty_c \} = \{ \text{TEST FUNCTIONS} \}$$

$$\mathcal{D}' := \{ T : \mathcal{D} \to \mathbb{R} \text{ LINEAR \& "CONTINUOUS"} \} = \{ \text{DISTRIBUTIONS} \} \quad [\text{NON TEMPERED}]$$

$$\forall K \in \mathbb{R}^d \text{ COMPACT} \quad \exists r = r_K \in \mathbb{N}$$

$$|T(\varphi)| \lesssim \|\varphi\|_{C^r} := \max_{|k| \leq r} \|\partial^k \varphi\|_\infty \qquad \forall \varphi \in \mathcal{D} : \ \mathrm{supp}(\varphi) \subseteq K$$

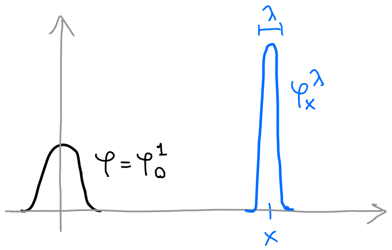
$\forall x \in \mathbb{R}^d$ A DISTRIBUTION $F_x \in \mathcal{D}'$ IS GIVEN $\rightsquigarrow$ GERM $F = (F_x)_{x \in \mathbb{R}^d}$

PROBLEM: CAN WE FIND A SINGLE DISTRIBUTION $f \in \mathcal{D}'$ WHICH IS "LOCALLY WELL APPROXIMATED" BY F_x ?

2 - UNIQUENESS

f IS "LOCALLY WELL APPROXIMATED" BY F_x ?

RESCALED TEST FUNCTION $\varphi_x^\lambda(z) := \frac{1}{\lambda^d}\varphi\left(\frac{z-x}{\lambda}\right)$



(*) FOR SOME $\gamma > 0$: $\begin{cases} |(f - F_x)(\varphi_x^\lambda)| \lesssim \lambda^\gamma \\ \text{UNIF. FOR } x \in \text{COMPACT}, \ \lambda \in (0,1] \end{cases}$

LEMMA (UNIQUENESS). FIX ANY $\varphi \in \mathcal{D}$ WITH $\int \varphi \neq 0$. FOR ANY GERM $F = (F_x)$, THERE IS AT MOST ONE $f \in \mathcal{D}'$ SUCH THAT (*) HOLDS.

PROOF: $$(f_1 - f_2)(\psi) = \lim_{\lambda \downarrow 0} (f_1 - f_2)(\varphi_0^\lambda * \psi)$$
$$= \lim_{\lambda \downarrow 0} \int_{\mathbb{R}^d} (f_1 - f_2)(\varphi_z^\lambda)\, \psi(z)\, dz = 0 \qquad \forall\, \psi \in \mathcal{D}$$

3- COHERENCE

THE UNIQUE f WHICH SATISFIES ⊛ IS CALLED RECONSTRUCTION OF $F=(F_x)$

PROVIDED IT EXISTS! WE MUST IMPOSE CONDITIONS ON THE GERM $F=(F_x)$

DEFINITION (COHERENCE). FIX $\gamma \in \mathbb{R}$. A GERM $F=(F_x)$ IS γ-COHERENT IF THERE IS ONE TEST FUNCTION $\varphi \in \mathcal{D}$ WITH $\int \varphi \neq 0$ SUCH THAT:

$$\forall K \subseteq \mathbb{R}^d \text{ COMPACT}: \quad \begin{cases} |(F_x - F_y)(\varphi_y^\lambda)| \lesssim \lambda^\alpha \left(|x-y| + \lambda\right)^{\gamma-\alpha} \\ \text{UNIF. FOR } x,y \in K \text{ AND } \lambda \in (0,1] \end{cases} \qquad ✪$$

FOR SOME $\alpha = \alpha_K$ SUCH THAT $\alpha \leq 0$ AND $\alpha \leq \gamma$

IF $\alpha_K \equiv \alpha$ $(\forall K)$ WE SAY THAT $F=(F_x)$ IS (α,γ)-COHERENT

COHERENCE IS A PRECISE WAY TO REQUIRE THAT

"F_x AND F_y ARE SUITABLY CLOSE WHEN x AND y ARE CLOSE"

EXAMPLE. LET $F=(F_x)$ BE (α,γ)-COHERENT WITH $\gamma>0$ AND $\alpha<0$

CONSIDER $|x-y|=\lambda^t$, $t\in[0,1]$ WHICH INTERPOLATES BETWEEN 1 AND λ

$$|(F_x-F_y)(\varphi_y^\lambda)| \lesssim \lambda^{t\gamma+(1-t)\alpha} = \begin{cases} \lambda^{\alpha<0} & \text{IF } |x-y|=1 \quad \text{DIVERGES AS } \lambda\downarrow 0 \\ \lambda^{\gamma>0} & \text{IF } |x-y|=\lambda \quad \text{VANISHES AS } \lambda\downarrow 0 \end{cases}$$

LEMMA (HOMOGENEITY). IF $F=(F_x)$ IS COHERENT, THEN

$$\forall\, K\subseteq\mathbb{R}^d \text{ COMPACT}: \quad |F_x(\varphi_x^\lambda)|\lesssim\lambda^\beta \quad \text{UNIF. FOR } x\in K \text{ AND } \lambda\in(0,1]$$

FOR SOME $\beta=\beta_K<\gamma$. WE CALL β_K HOMOGENEITY OF F ON K.

4- RECONSTRUCTION THEOREM

THEOREM (RECONSTRUCTION). LET $F = (F_x)$ BE γ-COHERENT ($\gamma \in \mathbb{R}$).

THERE EXISTS A DISTRIBUTION $f \in \mathcal{D}'$ WITH THE FOLLOWING PROPERTY:

$\forall\, K \subseteq \mathbb{R}^d$ COMPACT, $\forall\, \psi \in \mathcal{D}$ SUPPORTED IN $B(0,1)$,

$$|(f - F_x)(\psi_x^\lambda)| \leqslant C \begin{cases} \lambda^\gamma & \text{IF } \gamma \neq 0 \\ (1 + |\log \lambda|) & \text{IF } \gamma = 0 \end{cases} \qquad \text{UNIF. FOR } x \in K \text{ AND } \lambda \in (0,1]$$

$$C = C(K, F, \psi) = \underbrace{c(K,F)}_{\text{EXPLICIT}} \cdot \|\psi\|_{C^r} \qquad \text{FOR ANY } r > \max\{-\overset{\overset{\text{COHERENCE}}{\uparrow}}{\alpha}_{\underset{\hookrightarrow \text{2-FATTENING}}{\overline{K}_2}}, -\overset{\overset{\text{HOMOGENEITY}}{\uparrow}}{\beta}_{\overline{K}_2}\}$$

IF $\gamma > 0$, $f = \mathcal{R}F$ IS UNIQUE AND LINEAR

IF $\gamma \leqslant 0$, f IS NOT UNIQUE BUT CAN BE CHOSEN SO THAT THE MAP

$F \mapsto f = \mathcal{R}F$ IS LINEAR ON (α, γ)-COHERENT GERMS (WITH $\alpha_K \equiv \alpha$)

- THE FAMILY OF COHERENT GERMS IS A VECTOR SPACE $\left(\varphi \rightsquigarrow \text{ANY } \xi \in \mathcal{D}\right)$

- FOR $\gamma = 0$, $|\log \lambda|$ CANNOT BE DROPPED (COUNTEREXAMPLE)

- FOR $\gamma \neq 0$, $\exists\, f \in \mathcal{D}'$ SUCH THAT $\forall\, K \subseteq \mathbb{R}^d$ COMPACT $\exists\, r = r_K \in \mathbb{N}$:

$$\left|(f - F_x)(\psi_x^\lambda)\right| \lesssim \|\psi\|_{C^r}\, \lambda^\gamma$$

UNIF. FOR $x \in K$, $\lambda \in (0,1]$ AND $\psi \in \mathcal{D}$ SUPPORTED IN $B(0,1)$ $\Bigg\}$ (RT)

REMARKABLY, COHERENCE IS NECESSARY FOR (RT), HENCE IT IS AN OPTIMAL ASSUMPTION FOR THE RECONSTRUCTION THEOREM.

PROPOSITION. A GERM $F = (F_x)$ SATISFIES (RT) IFF IT IS γ-COHERENT.

[IT SATISFIES (RT) WITH A FIXED $r_K \equiv r$ IFF IT IS (α,γ)-COHERENT FOR SOME α.]

5 - COMMENTS

THE RT WAS FIRST PROVED BY MARTIN HAIRER (2014) IN THE FRAMEWORK OF HIS THEORY OF REGULARITY STRUCTURES.

IT IS AN EXTENSION IN $\mathbb{R}^d$ (AND TO DISTRIBUTIONS) OF THE SEWING LEMMA BY M. GUBINELLI (2004) AND D. FEYEL & A. DE LA PRADELLE (2006).
ORIGINAL MOTIVATION: THE THEORY OF ROUGH PATHS BY T. LYONS (1998).

HAIRER'S ORIGINAL PROOF IS BASED ON WAVELETS. AN ALTERNATIVE PROOF USING SEMIGROUPS WAS GIVEN BY F. OTTO AND H. WEBER (2019)

OUR PROOF IS BASED ON ARBITRARY TEST FUNCTION $\varphi \in \mathcal{D}$ WITH $\int \varphi \neq 0$ (WE WILL EXPLAIN THE KEY IDEAS).

EXAMPLE: A KEY CLASS OF (α,γ)-COHERENT GERMS $F=(F_x)$

THERE ARE $\gamma\in\mathbb{R}$ AND A FINITE SET $A\subseteq\mathbb{R}$ SUCH THAT

(RS) $$|(F_x-F_y)(\varphi_y^\lambda)| \lesssim \sum_{a\in A,\, a<\gamma} \lambda^a |x-y|^{\gamma-a}$$ ⇝ "GRADED CONTINUITY"

(EXERCISE) $$\lesssim \lambda^\alpha\left(|x-y|+\lambda\right)^{\gamma-\alpha}$$ WHERE $\alpha := \min(A)$

THIS INCLUDES ALL GERMS ARISING IN REGULARITY STRUCTURES:

- (A, T, G) REGULARITY STRUCTURE
- (Π_x, Γ_{xy}) MODEL ON $\mathbb{R}^d$
- $f\in\mathcal{D}^\gamma$ MODELLED DISTRIBUTION

⇝ THE GERM $F=\left(F_x := \Pi_x f(x)\right)_{x\in\mathbb{R}^d}$ SATISFIES (RS)

6 - NEGATIVE HÖLDER SPACES

DEFINITION. A DISTRIBUTION $T \in \mathcal{D}'$ IS HÖLDER WITH EXPONENT $\alpha \leq 0$, WRITTEN $T \in \mathcal{C}^{\alpha}$, IF FOR SOME (HENCE ANY) $r > -\alpha$ WE HAVE

$$|T(\psi_x^{\lambda})| \lesssim \lambda^{\alpha} \|\psi\|_{C^r} \quad \begin{cases} \text{UNIF. FOR } x \in \text{ COMPACT SETS, } \lambda \in (0,1] \\ \text{AND } \psi \in \mathcal{D} \text{ SUPPORTED IN } B(0,1) \end{cases}$$

AS A COROLLARY OF OUR APPROACH, WE OBTAIN THE FOLLOWING

THEOREM. $T \in \mathcal{C}^{\alpha} \iff |T(\varphi_x^{\lambda})| \lesssim \lambda^{\alpha}$ FOR A SINGLE $\varphi \in \mathcal{D}$ WITH $\int \varphi \neq 0$.

THEOREM. IF $F = (F_x)$ IS γ-COHERENT WITH HOMOGENEITY $\beta_K \equiv \beta \leq 0$, THEN THE RECONSTRUCTION $f := \mathcal{R}F \in \mathcal{C}^{\beta}$.

7 - YOUNG PRODUCT

WE CAN MULTIPLY DISTRIBUTIONS $g \in \mathcal{D}'$ WITH SMOOTH FUNCTIONS $f \in C^\infty$:

$$(g \cdot f)(\varphi) := g(f\varphi)$$

IF $f \in \mathcal{C}^\alpha$ WITH $\alpha > 0$ THIS NO LONGER MAKES SENSE $(f\varphi \notin \mathcal{D})$

BUT WE CAN STILL GIVE A LOCAL DESCRIPTION OF THE PRODUCT:

$$(g \cdot F_x)(\varphi) := g(F_x \varphi),$$

F_x = TAYLOR POLYNOMIAL OF f AT x OF MAXIMAL DEGREE

THEOREM. IF $f \in \mathcal{C}^\alpha$ AND $g \in \mathcal{C}^\beta$ WITH $\beta \leq 0$, THE GERM $(g \cdot F_x)_x$ IS $(\alpha+\beta)$-COHERENT. IF $\alpha+\beta > 0$, ITS RECONSTRUCTION IS A CANONICAL EXTENSION OF THE PRODUCT FOR $(f,g) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$.

(IF $\alpha+\beta \leq 0$, WE CAN STILL DEFINE A NON-CANONICAL "PRODUCT")

8 - SKETCH OF THE PROOF OF THE RT ($\gamma > 0$)

FIX ANY TEST FUNCTION $\rho \in \mathcal{D}$ WITH $\int \rho = 1$ AND ANY SEQUENCE $\varepsilon_n \downarrow 0$.

THEN $\rho^{\varepsilon_n}(z) := \frac{1}{\varepsilon_n^d} \rho\left(\frac{z}{\varepsilon_n}\right)$ ARE MOLLIFIERS: $\rho^{\varepsilon_n} * \psi \xrightarrow[n \to \infty]{} \psi$

IT FOLLOWS THAT FOR ANY DISTRIBUTION $f \in \mathcal{D}'$

$$f(\psi) = \lim_{n \to \infty} f(\rho^{\varepsilon_n} * \psi) = \lim_{n \to \infty} \int_{\mathbb{R}^d} f(\rho_z^{\varepsilon_n}) \, \psi(z) \, dz$$

GIVEN A GERM $F = (F_x)$ WE REPLACE $f(\rho_z^{\varepsilon_n})$ BY $F_z(\rho_z^{\varepsilon_n})$:

$\rightsquigarrow$ DEFINE $f_n(\psi) := \int_{\mathbb{R}^d} F_z(\rho_z^{\varepsilon_n}) \, \psi(z) \, dz$

GOAL: SHOW THAT $f_n \to f = \mathcal{R}F$.

WE CHOOSE $\varepsilon_n = \frac{1}{2^n}$ AND $\rho := \varphi^{2} * \overset{\substack{\text{COHERENCE ⊛}\\ \uparrow}}{\varphi}$ SO THAT

$$\rho^{\varepsilon_{n+1}} - \rho^{\varepsilon_n} = \left(\rho^{1/2} - \rho\right)^{\varepsilon_n} = \left(\varphi * \check{\varphi}\right)^{\varepsilon_n} = \varphi^{\varepsilon_n} * \check{\varphi}^{\varepsilon_n}$$

WITH $\check{\varphi} := \varphi^{1/2} - \varphi^2$. THE DIFFERENCE $\rho^{\varepsilon_{n+1}} - \rho^{\varepsilon_n}$ IS A CONVOLUTION!

[CRUCIAL TO SHOW THAT $f_{n+1} - f_n$ IS SMALL $\Rightarrow$ f_n CONVERGES]

ASSUME THAT $\int z^k \varphi(z)\,dz = 0 \quad \forall\ 1 \leq |k| \leq r-1$ [RECALL THAT $\int \varphi \neq 0$]

THEN $\int z^k \check{\varphi}(z)\,dz = 0 \quad \forall\ 0 \leq |k| \leq r-1$

LEMMA $\displaystyle\int_{\mathbb{R}^d} \left|\left(\check{\varphi}^{\varepsilon_n} * \psi\right)(z)\right| dz = \left\| \check{\varphi}^{\varepsilon_n} * \psi \right\|_{L^1} \lesssim \varepsilon_n^{r} \|\psi\|_{C^r}$

THE CHOICE $\varrho = \varphi^2 * \varphi$ ALLOWS US TO COMPARE EFFICIENTLY DIFFERENT DYADIC SCALES, PROVIDED φ ANNIHILATES MONOMIALS

THE TEST FUNCTION $\varphi \in \mathcal{D}$ (WITH $\int \varphi \neq 0$) IN ✪ ↗ COHERENCE WAS ARBITRARY.

WE "TWEAK φ" TO MAKE IT ANNIHILATE MONOMIALS (FROM DEGREE 1) UP TO A FIXED DEGREE $r-1$ (WITHOUT DISTROYNG COHERENCE ✪ !)

LEMMA (TWEAKING). FIX ANY DISTINCT $\lambda_0, \lambda_1, \dots, \lambda_{r-1}$ AND DEFINE

$$c_i := \prod_{k \in \{0,\dots,r-1\} \setminus \{i\}} \frac{\lambda_k}{\lambda_k - \lambda_i}$$

THEN $\hat{\varphi} := \sum_{k=0}^{r-1} c_i \varphi^{\lambda_i}$ SATISFIES $\int_{\mathbb{R}^d} z^k \hat{\varphi}(z)\, dz = 0 \quad \forall\, 1 \leq |k| \leq r-1$

DANKE!