A Polymer in a Multi-Interface Medium

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Outline

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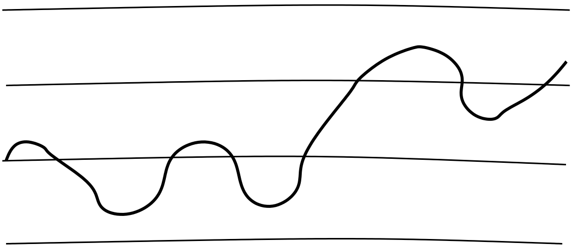
1. Introduction and motivations
2. Definition of the model
3. The free energy
4. Path results
5. Techniques from the proof

A polymer in a multi-interface medium



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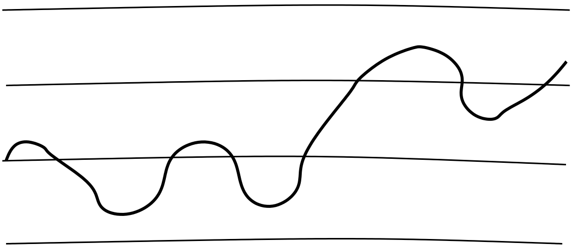

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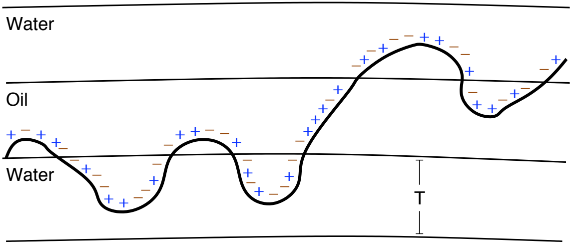

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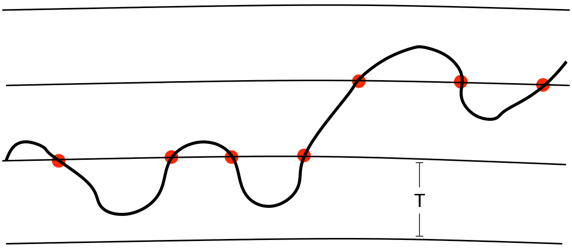

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Focus on (homogeneous, attractive/repulsive) pinning interaction.

Path behavior? Interplay between N and $T = T_N$?

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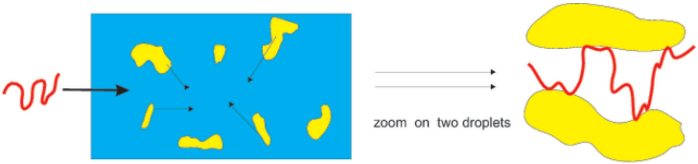

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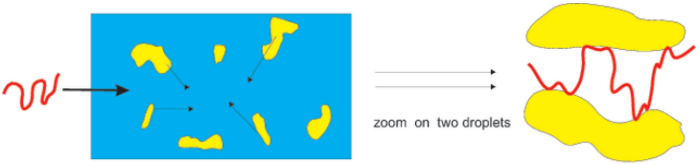
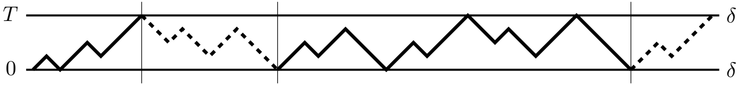
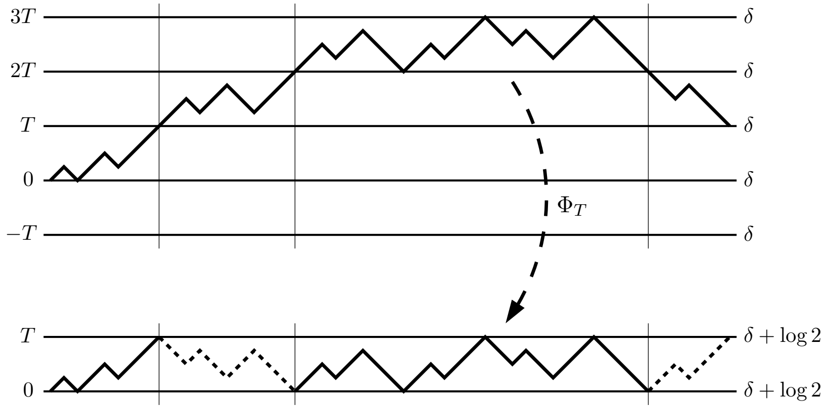


Polymer confined between two walls and interacting with them: (polymer in a slit)



Physics literature: [Brak et al. 2005], [Owkzarek et al. 2008], ...

Multi-interface medium vs. slit



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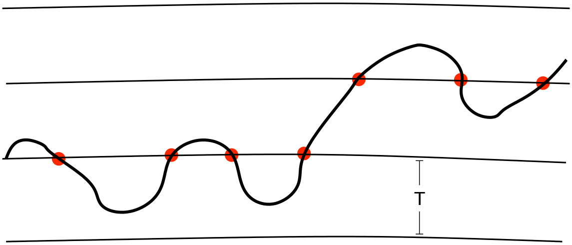

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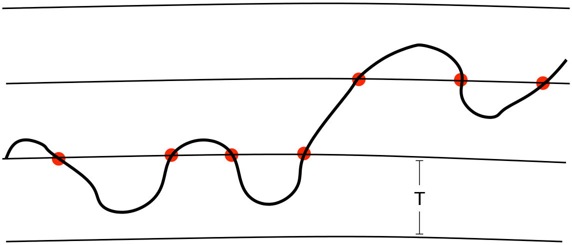

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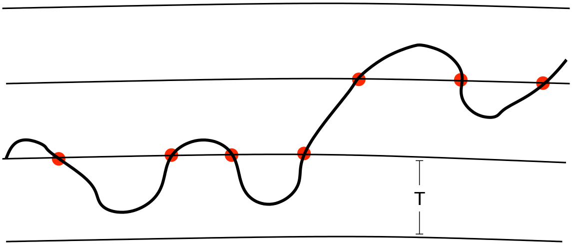

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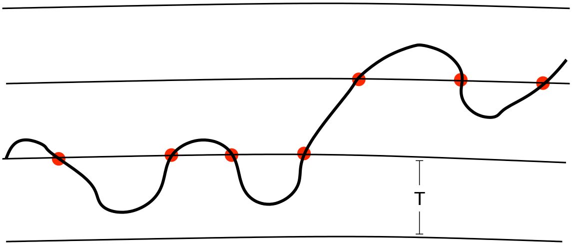

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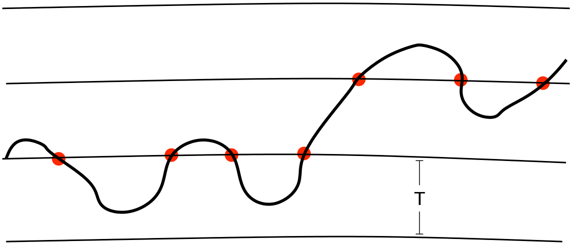

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- $\mathbf{P}_{N,\delta}^{T}$ absolutely continuous w.r.t. SRW $\{S_i\}_{i \geq 0}$

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Ingredients of $\mathbf{P}_{N,\delta}^{T}$:

- Simple symmetric random walk $S = \{S_n\}_{n \geq 0}$ on $\mathbb{Z}$:

$$S_0 := 0 \,, \qquad S_n := X_1 + \ldots + X_n \,,$$

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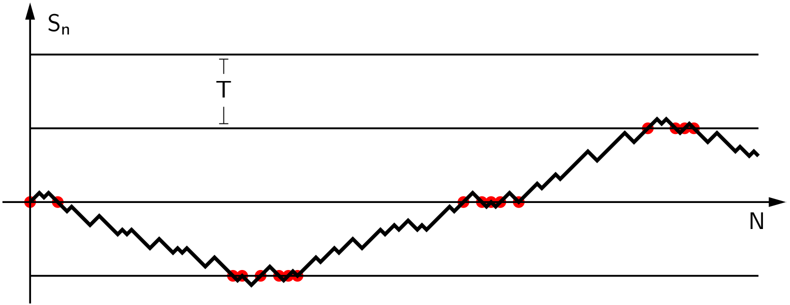

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Penalization of the simple random walk

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$\phi(\delta, \{T_n\}_n)$ non-analytic at $\delta \longleftrightarrow$ phase transition at δ

The free energy: characterization

We assume that $T_N \to T \in 2\mathbb{N} \cup \{\infty\}$, i.e.,

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Theorem ([CP1]). Let $T_N \to T$.

$$\phi(\delta, \{T_n\}_n) = \phi(\delta, T) = \begin{cases} \big(Q_T\big)^{-1}(e^{-\delta}) & \text{if } T < +\infty \\ \big(Q_\infty\big)^{-1}(e^{-\delta} \wedge 1) & \text{if } T = +\infty \end{cases}$$

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- Same path behavior? NO!

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Free energy says that for fixed T

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- Force exerted by the polymer on the confining walls (slit)
- Path behavior (multi-interface)

Outline

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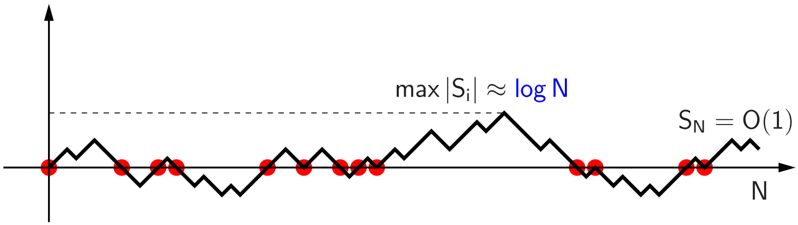

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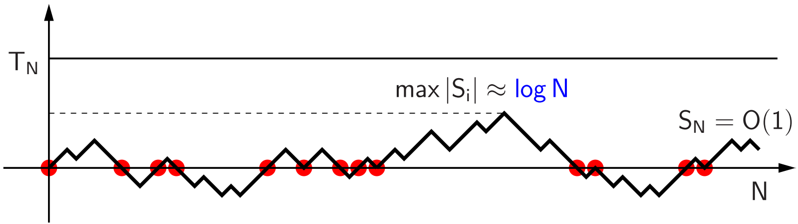

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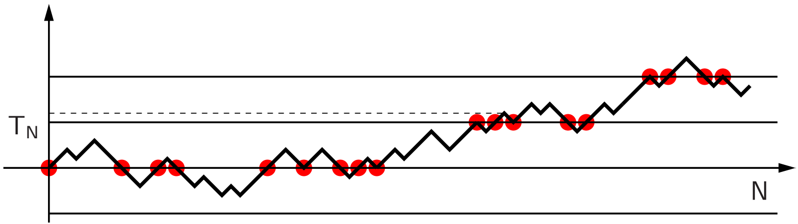

- If $T_N \gg \log N$ nothing changes: polymer localized at zero
- If $T_N \ll \log N$ different interfaces worth visiting

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$S_N \asymp \alpha_N$ means S_N/α_N is tight and $\mathbf{P}_{N,\delta}^{T_N}(|S_N/\alpha_N| \geq \varepsilon) \geq \varepsilon$

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The attractive case $\delta > 0$: path results

Theorem ([CP1])

For every $\delta > 0$ *there exists* $c_\delta > 0$ *such that under* $\mathbf{P}_{N,\delta}^{T_N}$*:*

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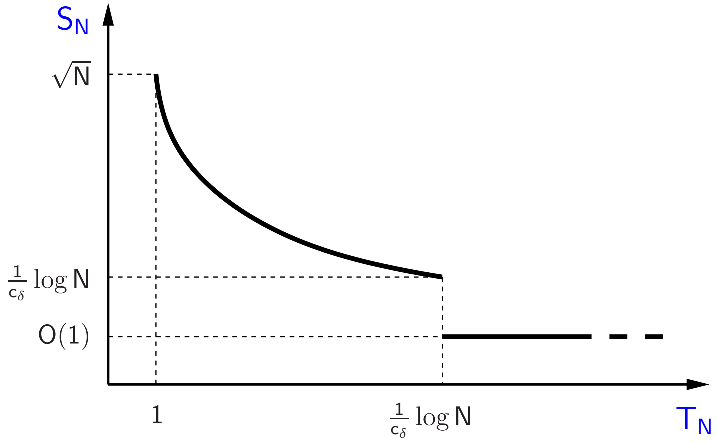

- Sub-diffusive scaling ($T_N \to \infty$)
- Transition at $T_N \approx \log N$

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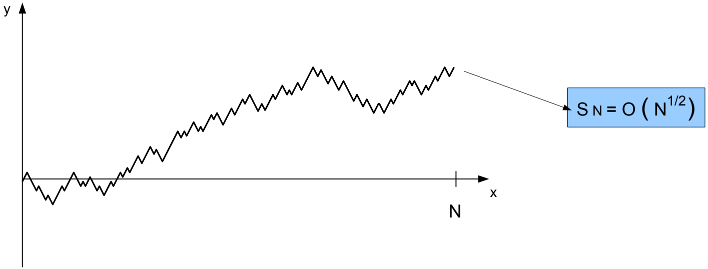

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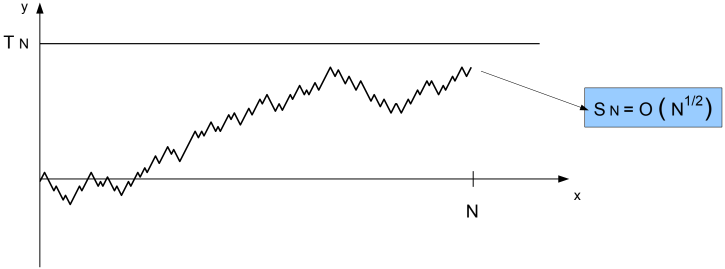

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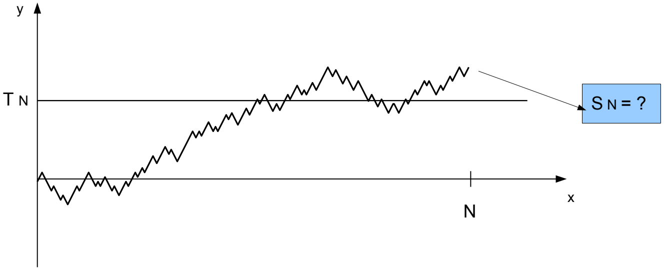

- If $T_N \gg \sqrt{N}$ nothing changes: $S_N \asymp \sqrt{N}$
- If $T_N \ll \sqrt{N}$ does the polymer visit other interfaces?

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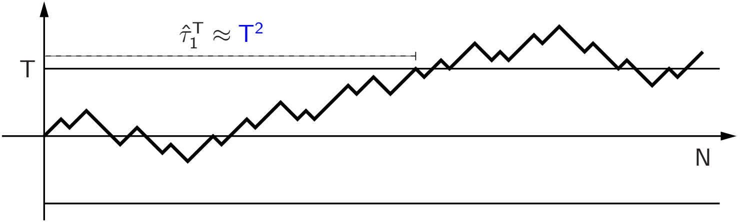

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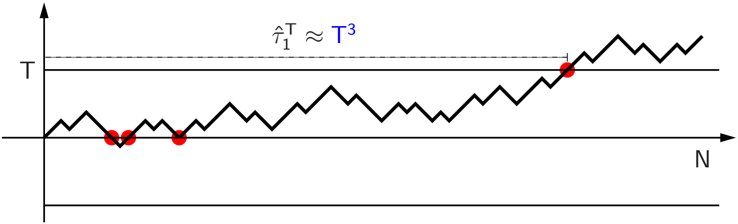

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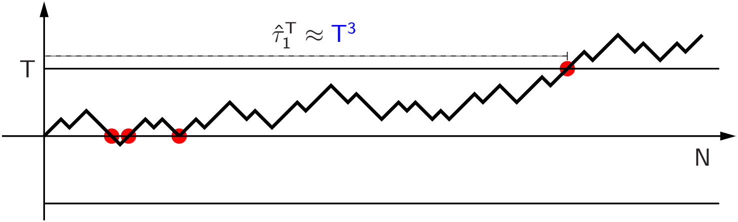

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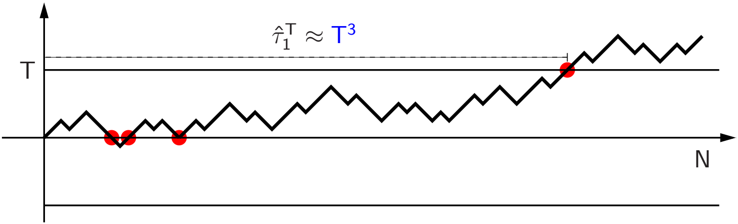

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Theorem ([CP2])

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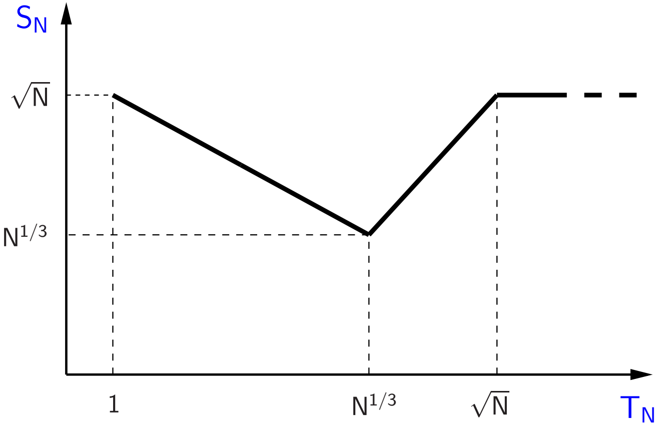

• Sub-diffusive if $1 \ll T_N \ll \sqrt{N}$ • Transitions $T_N \approx N^{1/3}, \sqrt{N}$

Outline

1. Introduction and motivations

2. Definition of the model

3. The free energy

4. Path results

5. Techniques from the proof

A renewal theory approach

Let $\tau_1^T, \tau_2^T, \tau_3^T \ldots$ be the points at which S_n visits an interface

$$\tau_{k+1}^T \; := \; \inf\left\{ n > \tau_k^T \, : \; S_n - S_{\tau_k^T} \in \{-T, 0, T\} \right\} \qquad (T \text{ is fixed})$$

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Under the simple random walk law $\{\tau_n^T\}_{n\in\mathbb{N}}$ is a classical renewal process with explicit law

$$q_T(n) \ := \ P(\tau_1^T = n) \ \approx \ \frac{1}{(\min\{n, T^2\})^{3/2}} \, e^{-\frac{\pi^2}{2T^2} n}$$

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Note that under P

$$\tau_1^T \approx \begin{cases} O(1) \text{ with probab. } 1 - \frac{1}{T} & \left[q_T(n) \approx \frac{1}{n^{3/2}}\right] \\ & \end{cases}$$

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Strategy of the proof

For fixed T, the law of $\tau^T \cap [0,N] = \{\tau_1^T, \ldots, \tau_{L_N}^T\}$ is the same

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- Uniform renewal theorems

Thanks.

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