On the 2d KPZ and Stochastic Heat Equation

Francesco Caravenna

Università degli Studi di Milano-Bicocca

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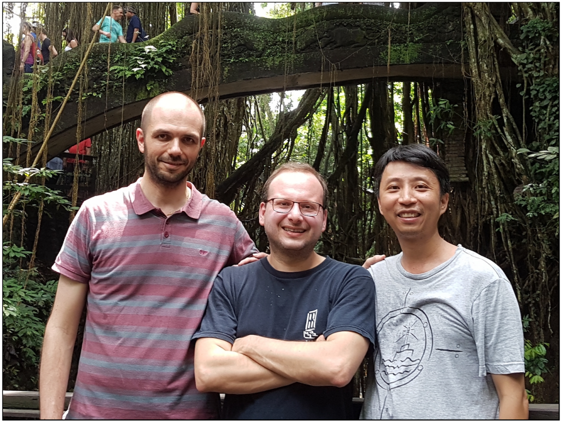
Collaborators



Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

Overview

Two stochastic PDEs on $\mathbb{R}^d$ (mainly $d = 2$)

- Kardar-Parisi-Zhang Equation (KPZ)
- Stochastic Heat Equation (SHE) with multiplicative noise

These are very interesting, yet ill-defined equations

Plan:

1. Consider a regularized version of these equations
2. Study the limit of the solution, when regularisation is removed

Stochastic Analysis $\leftrightsquigarrow$ Statistical Mechanics

White noise

Space-time white noise $\xi = \xi(t,x)$ on $\mathbb{R}^{1+d}$

Generalized random field: random distribution (Schwartz)

- Centered Gaussian with

$$\mathrm{Cov}[\,\xi(t,x), \xi(t',x')\,] = \delta(t-t')\,\delta(x-x')$$

$$\langle \phi, \xi \rangle \;=\; \int_{\mathbb{R}^{1+d}} \phi(t,x)\,\xi(t,x)\,\mathrm{d}t\,\mathrm{d}x \;\sim\; N(0, \|\phi\|_{L^2}^2)$$

- ξ = scaling limit of i.i.d. RVs indexed by $\mathbb{Z}^{1+d}$
  (approximation of PDEs by discrete models)

The KPZ equation

KPZ [Kardar Parisi Zhang PRL'86]

$$\partial_t h = \frac{1}{2}\Delta_x h + \frac{1}{2}|\nabla_x h|^2 + \beta\,\xi \qquad \text{(KPZ)}$$

Model for random interface growth

$h = h(t,x) =$ interface height at time $t \geq 0$, space $x \in \mathbb{R}^d$

$\xi = \xi(t,x) =$ space-time white noise $\qquad \beta > 0$ noise strength

$|\nabla_x h|^2$ ill-defined

For smooth ξ

$$u(t,x) := e^{h(t,x)} \qquad \text{(Cole-Hopf)}$$

The multiplicative Stochastic Heat Equation (SHE)

SHE $(t > 0, \quad x \in \mathbb{R}^d)$

$$\partial_t u = \frac{1}{2} \Delta_x u + \beta\, u\, \xi \qquad \text{(SHE)}$$

Product $u\,\xi$ ill-defined

- $(d = 1)$ Well-posed by Ito integration [Walsh 80's]
- $(d = 1)$ SHE and KPZ are well-understood in a robust sense

  Regularity Structures (Hairer), Paracontrolled Distributions (GIP), Energy Solutions (Goncalves-Jara), Renormalization (Kupiainen)
- $(d \geq 2)$ No general theory

(KPZ) and (SHE) in dimensions $d \geq 2$

Mollify the white noise $\xi(t, x)$ in space on scale $\varepsilon > 0$

$$\xi^{\varepsilon}(t, \cdot) := \xi(t, \cdot) * \varrho_{\varepsilon}$$

Solutions $h^{\varepsilon}(t, x)$, $u^{\varepsilon}(t, x)$ well-defined. Convergence as $\varepsilon \downarrow 0$?

Renormalize disorder strength $\beta = \beta_{\varepsilon}$

$$\beta_{\varepsilon} = \begin{cases} \hat{\beta} \ \text{(fixed)} & d = 1 \\ \hat{\beta} \, \dfrac{1}{\sqrt{\log \varepsilon^{-1}}} & d = 2 \\ \hat{\beta} \, \varepsilon^{\frac{d-2}{2}} & d \geq 3 \end{cases} \qquad \hat{\beta} \in (0, \infty)$$

Mollified equations

Mollified SHE

$$\begin{cases} \partial_t u^\varepsilon = \frac{1}{2}\Delta u^\varepsilon + \beta_\varepsilon \, u^\varepsilon \, \xi^\varepsilon \\ u^\varepsilon(0,\cdot) \equiv 1 \end{cases} \qquad (\varepsilon\text{-SHE})$$

Then $u^\varepsilon(t,x) \geq 0$ with $\mathbb{E}[u^\varepsilon(t,x)] \equiv 1$

Set $h^\varepsilon(t,x) := \log u^\varepsilon(t,x)$ $\rightsquigarrow$ Ito's formula

Mollified KPZ

$$\begin{cases} \partial_t h^\varepsilon = \frac{1}{2}\Delta h^\varepsilon + \frac{1}{2}|\nabla h^\varepsilon|^2 + \beta_\varepsilon \, \xi^\varepsilon - c\, \beta_\varepsilon^2 \, \varepsilon^{-d} \\ h^\varepsilon(0,\cdot) \equiv 0 \end{cases} \qquad (\varepsilon\text{-KPZ})$$

Main results

Space dimension $d = 2$ $\qquad \beta_\varepsilon = \frac{\hat{\beta}}{\sqrt{\log \varepsilon^{-1}}} \qquad \hat{\beta} \in (0, \infty)$

I. Phase transition for SHE and KPZ [CSZ 17]

Solutions $u^\varepsilon(t,x)$ and $h^\varepsilon(t,x)$ undergo phase transition at $\hat{\beta}_c = \sqrt{2\pi}$

II. Sub-critical regime of SHE and KPZ [CSZ 17] [CSZ 18b]

LLN + fluctuations of solutions $u^\varepsilon(t,x)$ and $h^\varepsilon(t,x)$ for all $\hat{\beta} < \hat{\beta}_c$

III. Critical regime of SHE [CSZ 18a]

Moment estimates of SHE solution $u^\varepsilon(t,x)$ for $\hat{\beta} = \hat{\beta}_c$

References

With Rongfeng Sun and Nikos Zygouras:

- [CSZ 17] *Universality in marginally relevant disordered systems* Ann. Appl. Probab. 2017
- [CSZ 18a] *On the moments of the (2+1)-dimensional directed polymer and Stochastic Heat Equation in the critical window* Commun. Math. Phys. (to appear)
- [CSZ 18b] *The two-dimensional KPZ equation in the entire subcritical regime* Ann. Probab. (to appear)

$(d = 2)$ [Bertini Cancrini 98]
[Chatterjee Dunlap 18] [Gu 18] [Gu Quastel Tsai 19]

$(d \geq 3)$ [Magnen Unterberger 18] [Gu Ryzhik Zeitouni 18]
[Dunlap Gu Ryzhik Zeitouni 19] [Comets Cosco Mukherjee 18 19a 19b]

Main result I. Phase transition

Space dimension $d = 2$ $\qquad \beta_\varepsilon = \dfrac{\hat{\beta}}{\sqrt{\log \varepsilon^{-1}}}$ $\qquad \hat{\beta} \in (0, \infty)$

Theorem (Phase transition for SHE) [CSZ 17]

- For $\hat{\beta} < \sqrt{2\pi}$ $\qquad u^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} \exp\left(\sigma Z - \frac{1}{2}\sigma^2\right)$

$$Z \sim N(0,1) \qquad \sigma^2 := \log \frac{2\pi}{2\pi - \hat{\beta}^2}$$

$u^\varepsilon(t,x_i) \xrightarrow[\varepsilon \downarrow 0]{d}$ asympt. independent (for distinct points x_i's)

- For $\hat{\beta} \geq \sqrt{2\pi}$ $\qquad u^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} 0$

Main result I. Phase transition

Space dimension $d = 2$ $\qquad \beta_\varepsilon = \dfrac{\hat\beta}{\sqrt{\log \varepsilon^{-1}}} \qquad \hat\beta \in (0, \infty)$

Theorem (Phase transition for KPZ) [CSZ 17]

- For $\hat\beta < \sqrt{2\pi}$ $\qquad h^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} \sigma\, Z - \frac{1}{2}\,\sigma^2$

$$Z \sim N(0,1) \qquad \sigma^2 := \log \frac{2\pi}{2\pi - \hat\beta^2}$$

$h^\varepsilon(t, x_i) \xrightarrow[\varepsilon \downarrow 0]{d}$ asympt. independent (for distinct points x_i's)

- For $\hat\beta \geq \sqrt{2\pi}$ $\qquad h^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} -\infty$

Law of large numbers

Consider the sub-critical regime $\hat{\beta} < \sqrt{2\pi}$

- as $\varepsilon \downarrow 0$: $\quad \mathbb{E}[u^\varepsilon(t,x)] \equiv 1 \qquad \mathbb{E}[h^\varepsilon(t,x)] \equiv -\frac{1}{2}\sigma^2 + o(1)$
- $u^\varepsilon(t,x)$ and $h^\varepsilon(t,x)$ asymptotically independent for distinct x's

Corollary: LLN $\quad (\hat{\beta} < \sqrt{2\pi})$

$$u^\varepsilon(t,\cdot) \xrightarrow[\varepsilon\downarrow 0]{d} 1 \qquad h^\varepsilon(t,\cdot) \xrightarrow[\varepsilon\downarrow 0]{d} -\frac{1}{2}\,\sigma^2 \qquad \text{as distributions on } \mathbb{R}^2$$

$$\text{e.g.} \qquad \int_{\mathbb{R}^2} h^\varepsilon(t,x)\,\phi(x)\,\mathrm{d}x \xrightarrow[\varepsilon\downarrow 0]{d} -\frac{1}{2}\,\sigma^2 \int_{\mathbb{R}^2} \phi(x)\,\mathrm{d}x$$

Main result II. Fluctuations for SHE

Rescaled SHE solution $\quad \mathcal{U}^\varepsilon(t,x) := \big(u^\varepsilon(t,x) - 1\big)/\beta_\varepsilon$

Theorem (Fluctuations for SHE) [CSZ 17]

for $\hat\beta < \sqrt{2\pi}$ $\qquad \mathcal{U}^\varepsilon(t,\cdot) \xrightarrow[\varepsilon\downarrow 0]{d} v(t,\cdot)$ $\qquad$ as a distrib.

$v =$ solution of Edwards-Wilkinson equation

$$\partial_t v = \tfrac{1}{2}\Delta_x v + \gamma\,\xi \qquad \text{where} \qquad \gamma = \sqrt{\frac{2\pi}{2\pi - \hat\beta^2}} > 1$$

$$\partial_t \mathcal{U}^\varepsilon = \tfrac{1}{2}\Delta_x \mathcal{U}^\varepsilon + \xi^\varepsilon + \beta_\varepsilon\,\mathcal{U}^\varepsilon\,\xi^\varepsilon$$

Remarkably $\quad \beta_\varepsilon\,\mathcal{U}^\varepsilon\,\xi^\varepsilon \not\to 0 \qquad$ (while $\beta_\varepsilon \to 0$, $\mathcal{U}^\varepsilon \to v$, $\xi^\varepsilon \to \xi$)

Fluctuations for KPZ ?

Proof based on Wiener Chaos expansions, not available for KPZ

$$h^\varepsilon(t,x) = \log u^\varepsilon(t,x) \qquad \text{(Cole-Hopf)}$$

We might hope that

$$h^\varepsilon(t,\cdot) = \log\big(1 + (u^\varepsilon(t,\cdot) - 1)\big) \approx \big(u^\varepsilon(t,\cdot) - 1\big) \text{ ?}$$

NO, because $u^\varepsilon(t,x)$ is not close to 1 pointwise

Correct comparison (non trivial!)

$$h^\varepsilon(t,\cdot) - \mathbb{E}[h^\varepsilon] \approx \big(u^\varepsilon(t,\cdot) - 1\big)$$

Main result II. Fluctuations for KPZ

Rescaled KPZ solution $\mathcal{H}^\varepsilon(t,x) := \big(h^\varepsilon(t,x) - \mathbb{E}[h^\varepsilon]\big)/\beta_\varepsilon$

Theorem (Fluctuations for KPZ) [CSZ 18b]

for $\hat\beta < \sqrt{2\pi}$ $\qquad \mathcal{H}^\varepsilon(t,\cdot) \xrightarrow[\varepsilon\downarrow 0]{d} v(t,\cdot)$ as a distrib.

$v =$ solution of Edwards-Wilkinson equation

$$\partial_t v = \tfrac{1}{2}\Delta_x v + \gamma\,\xi \qquad \text{where} \qquad \gamma = \sqrt{\frac{2\pi}{2\pi - \hat\beta^2}} > 1$$

$$\partial_t \mathcal{H}^\varepsilon = \tfrac{1}{2}\Delta_x \mathcal{H}^\varepsilon + \xi^\varepsilon + \beta_\varepsilon\big(|\nabla \mathcal{H}^\varepsilon|^2 - c\,\varepsilon^{-2}\big)$$

Term $\beta_\varepsilon\big(|\nabla \mathcal{H}^\varepsilon|^2 - c\,\varepsilon^{-2}\big)$ responsible for "extra" white noise

Sketch of the proof

We approximate $u^\varepsilon(t,x)$ by "local version" $\tilde{u}^\varepsilon(t,x)$ which samples noise ξ in a tiny region around (t,x)

Then we approximate KPZ solution $h^\varepsilon(t,x)$ by Taylor expansion

$$h^\varepsilon \;=\; \log u^\varepsilon \;=\; \log \tilde{u}^\varepsilon + \log\left(1 + \frac{u^\varepsilon - \tilde{u}^\varepsilon}{\tilde{u}^\varepsilon}\right) \approx\; \log \tilde{u}^\varepsilon + \frac{u^\varepsilon - \tilde{u}^\varepsilon}{\tilde{u}^\varepsilon} + R^\varepsilon$$

- Remainder is small $\quad \big(R^\varepsilon(t,\cdot) - \mathbb{E}[R^\varepsilon]\big)/\beta_\varepsilon \xrightarrow{d} 0$
- Local dependence $\quad \big(\log \tilde{u}^\varepsilon(t,\cdot) - \mathbb{E}[\log \tilde{u}^\varepsilon]\big)/\beta_\varepsilon \xrightarrow{d} 0$
- Crucial approximation $\quad \dfrac{u^\varepsilon(t,\cdot) - \tilde{u}^\varepsilon(t,\cdot)}{\tilde{u}^\varepsilon(t,\cdot)} \approx u^\varepsilon(t,\cdot) - 1$

Some Comments

Key tools in our approach are

- Wiener chaos + Renewal Theory arguments
- 4th Moment Theorems to prove Gaussianity
- Hypercontractivity + Concentration of Measure

Alternative proof by [Gu 18] via Malliavin calculus (only for small $\hat{\beta}$)

[Chatterjee and Dunlap 18] first considered fluctuations for KPZ
They proved tightness of $\mathcal{H}^{\varepsilon}$ (only for small $\hat{\beta}$)

We identify the limit (EW) in the entire sub-critical regime $\hat{\beta} < \sqrt{2\pi}$

Analogous results in dimensions $d \geq 3$ by many authors

A variation on KPZ

Chatterjee and Dunlap [CD 18] looked at a different KPZ

$$\partial_t \tilde{h}^\varepsilon = \tfrac{1}{2}\Delta \tilde{h}^\varepsilon + \tfrac{1}{2}\,\beta_\varepsilon\,|\nabla \tilde{h}^\varepsilon|^2 + \xi^\varepsilon$$

where β_ε tunes the strength of the non-linearity

In our setting, β_ε tunes the strength of the noise

$$\partial_t h^\varepsilon = \tfrac{1}{2}\Delta h^\varepsilon + \tfrac{1}{2}|\nabla h^\varepsilon|^2 + \beta_\varepsilon\,\xi^\varepsilon - c\,\beta_\varepsilon^2\,\varepsilon^{-d}$$

The two equations have the same fluctuations

$$\tilde{h}^\varepsilon(t,x) - \mathbb{E}[\tilde{h}^\varepsilon] = \frac{1}{\beta_\varepsilon}\left(h^\varepsilon(t,x) - \mathbb{E}[h^\varepsilon]\right) = \mathcal{H}^\varepsilon(t,x)$$

The critical regime

What about the critical point $\hat{\beta} = \sqrt{2\pi}$?

More generally, critical window [Bertini Cancrini 98]

$$\beta_\varepsilon = \sqrt{\frac{2\pi}{\log \varepsilon^{-1}} \left(1 + \frac{\vartheta}{\log \varepsilon^{-1}}\right)} \qquad \text{with} \quad \vartheta \in \mathbb{R}$$

Key conjecture for critical SHE

$$u^\varepsilon(t,\cdot) \xrightarrow[\varepsilon \downarrow 0]{d} \mathcal{U}(t,\cdot) \qquad \text{(random distribution on } \mathbb{R}^2\text{)}$$

Nothing known for KPZ solution $h^\varepsilon(t,\cdot)$

Second moment

Known results [Bertini Cancrini 98]

$$\mathbb{E}\big[\langle u^\varepsilon(t,\cdot),\phi\rangle\big] \equiv \big\langle 1,\phi\big\rangle \qquad \sup_{\varepsilon>0} \mathbb{E}\big[\langle u^\varepsilon(t,\cdot),\phi\rangle^2\big] < \infty$$

$$\mathbb{E}\big[\langle u^\varepsilon(t,\cdot),\phi\rangle^2\big] \xrightarrow[\varepsilon\downarrow 0]{} \big\langle \phi,\, K\phi\big\rangle \qquad K\big(x,x'\big) \sim C \log \tfrac{1}{|x-x'|}$$

Corollary: tightness

$\exists$ subseq. limits $u^{\varepsilon_k}(t,\cdot) \xrightarrow[k\to\infty]{d} \mathcal{U}(t,\cdot)$ as random distributions

Could the limit be trivial $\mathcal{U}(t,\cdot) \equiv 1$?

Main result III. Third moment in the critical window

We computed the sharp asymptotics of third moment

Theorem [CSZ 18a]

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\left[\langle u^\varepsilon(t,\cdot), \phi\rangle^3\right] = C(\phi) < \infty$$

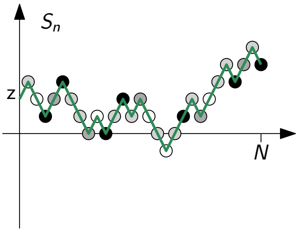
Corollary

Any subseq. limit $u^{\varepsilon_k}(t,\cdot) \xrightarrow{d} \mathcal{U}(t,\cdot)$ has the same covariance $K(x,x')$

$$\rightsquigarrow \quad \mathcal{U}(t,\cdot) \not\equiv 1 \quad \text{is non-trivial}$$

Recently [Gu Quastel Tsai 19] proved convergence of all moments

Directed Polymer in Random Environment



- $(S_n)_{n\geq 0}$ simple random walk on $\mathbb{Z}^d$
- Disorder: i.i.d. random variables $\omega(n,x)$ zero mean, unit variance

$$\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega(n,x)}] < \infty$$

- (-) Hamiltonian $\quad H_{N,\beta}(\omega, S) := \beta \sum_{n=1}^{N} \omega(n, S_n) - \lambda(\beta)\, N$

Partition Functions $\quad (N \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z_{N,\beta}(z) = \mathbf{E}^{\mathrm{rw}}\left[e^{H_{N,\beta}(\omega,S)} \middle| S_0 = z \right] = \frac{1}{(2d)^N} \sum_{(s_0,\ldots,s_N) \text{ n.n.: } s_0=z} e^{H_{N,\beta}(\omega,s)}$$

Directed Polymer and SHE

Partition functions $\boldsymbol{Z}_{N,\beta}(z)$ are discrete analogues of $u^{\varepsilon}(t,x)$

- They solve a lattice SHE

$$\boldsymbol{Z}_{N+1}(z) - \boldsymbol{Z}_N(z) \;=\; \Delta \boldsymbol{Z}_N(z) \;+\; \beta\, \tilde{\omega}(N+1, z)\, \widetilde{\boldsymbol{Z}}_N(z)$$

  Alternative way of regularizing SHE (discretize vs. mollify)

- Quantitative analogy via Feynman-Kac formula for SHE

  We can use partition functions $\boldsymbol{Z}_N(z)$ to approximate $u^{\varepsilon}(t,x)$

Feynman-Kac for SHE

Recall the mollified SHE
$$\begin{cases} \partial_t u^\varepsilon = \frac{1}{2}\Delta u^\varepsilon + \beta_\varepsilon \, u^\varepsilon \, \xi^\varepsilon \\ u^\varepsilon(0,\cdot) \equiv 1 \end{cases}$$

A stochastic Feynman-Kac formula holds

$$u^\varepsilon(t,x) \overset{d}{=} \mathrm{E}_{\varepsilon^{-1}x}\left[\exp\left(\beta_\varepsilon \, \varepsilon^{-\frac{d-2}{2}} \int_0^{\varepsilon^{-2}t} \int_{\mathbb{R}^2} \varrho(B_s - y) \, \xi(\mathrm{d}s, \mathrm{d}y) \; - \; \text{q.v.} \right) \right]$$

where $\varrho \in C_c^\infty(\mathbb{R}^d)$ is the mollifier and $B = (B_s)_{s \ge 0}$ is Brownian motion

We can identify $u^\varepsilon(t,x) \;\approx\; \boldsymbol{Z}_{N,\beta}(z)$ with

$$N = \varepsilon^{-2} t \qquad z = \varepsilon^{-1} x \qquad \beta_\varepsilon = \varepsilon^{\frac{d-2}{2}} \, \beta$$

In conclusion

Directed Polymers provides a friendly framework for SHE

All mentioned tools have "discrete stochastic analysis" analogues

Polynomial Chaos, Concentration Inequalities, Hypercontractivity, ...

Probabilistic arguments (e.g. renewal theory) are often more transparent for Directed Polymers than for SHE

Our results are first proved for Directed Polymer, then for SHE and KPZ

Robustness + Universality

Thanks.