The Continuum Disordered Pinning Model

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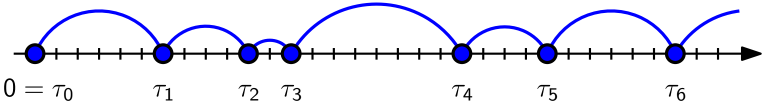
Coworkers

Joint work with Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

Outline

1. Discrete pinning model
2. The continuum limit
3. Main results

α-Renewal Processes



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Gaps $(\tau_{i+1} - \tau_i)_{i \geq 0}$ are i.i.d. integer-valued

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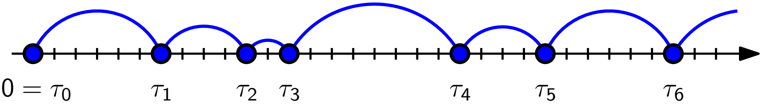

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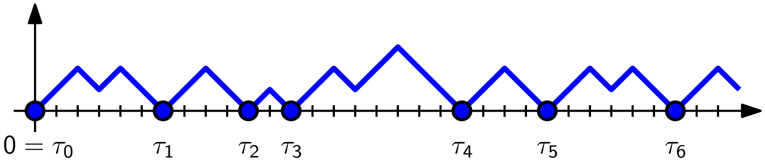

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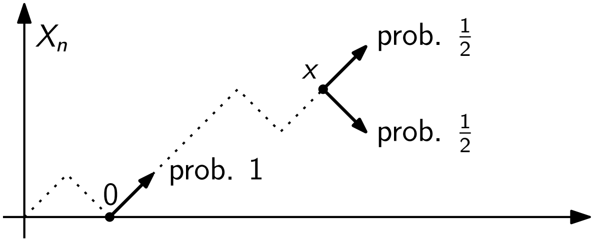
$\tau = \{n \in \mathbb{N}_0 : \ X_n = 0\}$ zero level set of a Markov chain on $\mathbb{N}_0$

Bessel random walks



Case $\alpha = \frac{1}{2}$ $\rightsquigarrow$ (reflected) simple random walk

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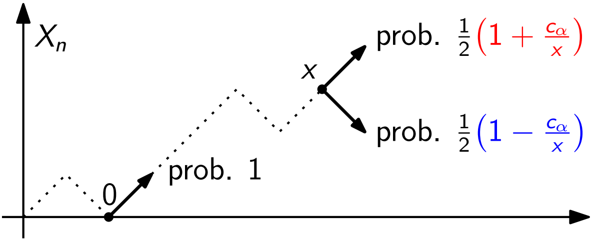

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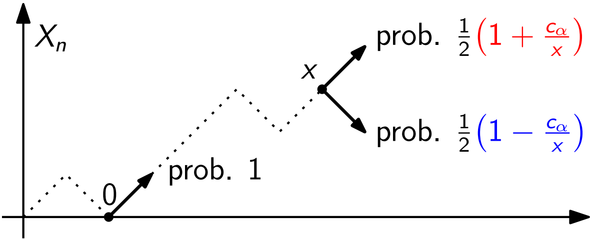

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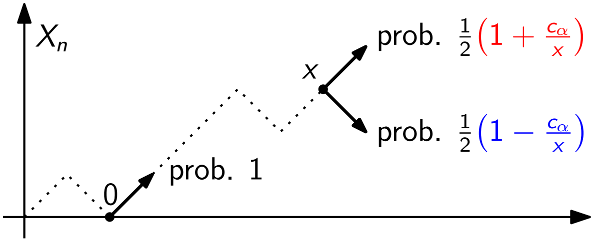

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Case $\alpha = \frac{1}{2}$ $\rightsquigarrow$ (reflected) simple random walk

- $(\alpha < \frac{1}{2})$ drift $\approx \frac{1}{x}$ away from the origin $(c_\alpha > 0)$
- $(\alpha > \frac{1}{2})$ drift $\approx \frac{1}{x}$ toward the origin $(c_\alpha < 0)$

$$\mathrm{P}(\tau_{i+1} - \tau_i = n) \sim \frac{C}{n^{1+\alpha}}$$

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Quenched disorder: i.i.d. random variables $(\widetilde{\omega}_n)_{n\in\mathbb{N}}$, indep. of τ

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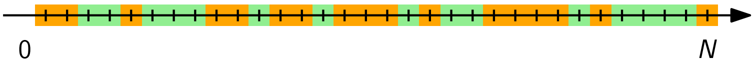
rewards $\widetilde{\omega}_n > 0$ penalties $\widetilde{\omega}_n < 0$



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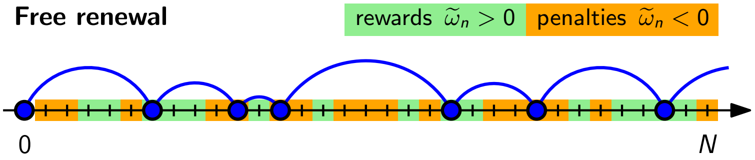

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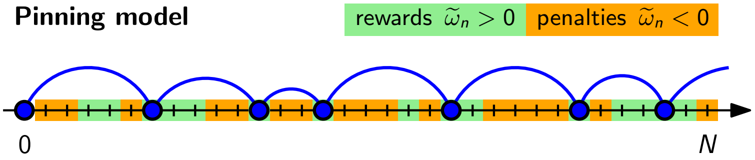

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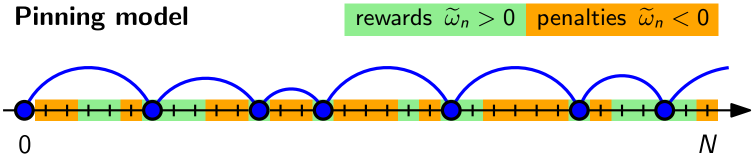

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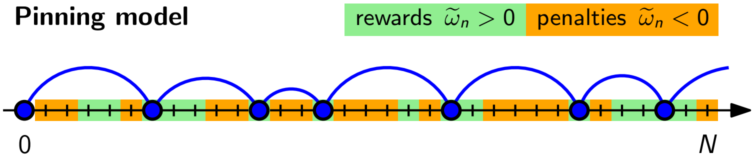

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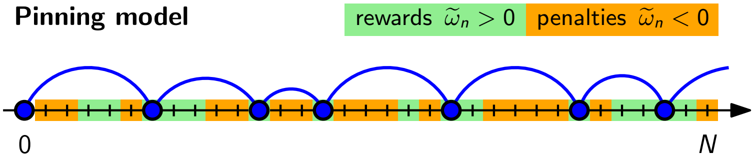

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normalization constant (partition function)

– Hamiltonian: sum of rewards/penalties visited by τ

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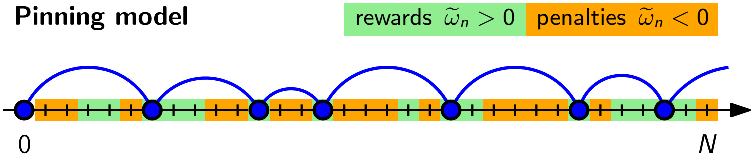

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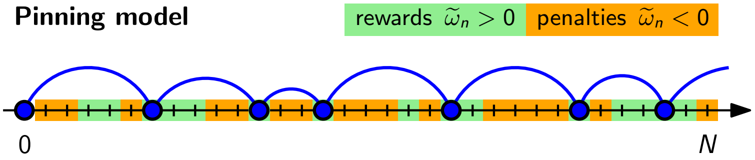

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reward/penalty to visit site n

The effect of disorder

- Write $\widetilde{\omega}_n = \beta\omega_n + h$ with $\mathbb{E}[\omega_1] = 0$, $\mathbb{V}\mathrm{ar}[\omega_1] = 1$

  Parameters $\beta \ge 0$, $h \in \mathbb{R}$ tune disorder strength, bias

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Rescale lattice $\frac{1}{N}\mathbb{N}_0$ and coupling constants $\beta = \beta_N$, $h = h_N$: does $\mathrm{P}_N^{\widetilde{\omega}}$ converge to a "continuum model" as $N \to \infty$?

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3. Main results

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$\mathcal{C} := \{\text{closed subsets of } [0, \infty)\}$ compact Polish space

(Hausdorff distance $\rightsquigarrow$ Fell-Matheron topology)

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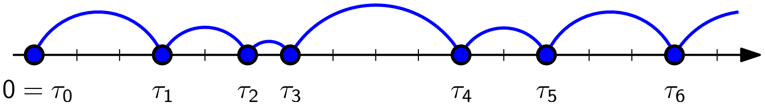

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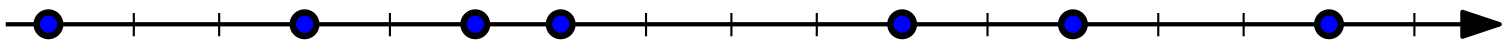

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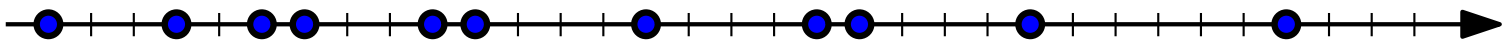

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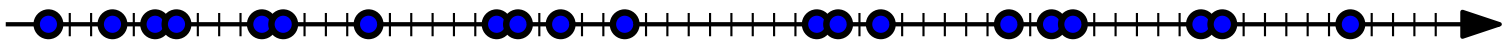

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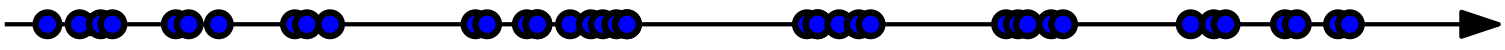

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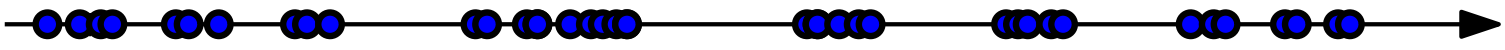

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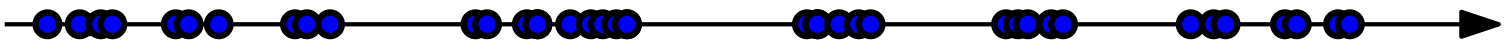

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Consider any α-renewal processes: $\mathrm{P}(\tau_{i+1} - \tau_i = n) \sim \frac{C}{n^{1+\alpha}}$

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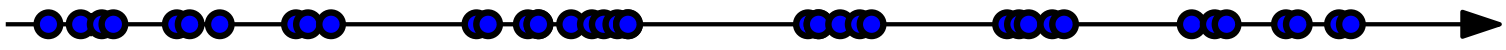

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The law $\mathrm{P}(\frac{d\tau}{N})$ of the rescaled renewal $\frac{\tau}{N}$ converges weakly on $\mathcal{C}$ to a universal limit: the law $\mathscr{P}$ of α-stable regenerative set $\boldsymbol{\tau}$

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- $\alpha \in (0, 1)$: rescaled Bessel RWs converge to Bessel(δ) process

  $$\mathrm{d}X_t = \mathrm{d}B_t + \frac{c_\alpha}{X_t}\mathrm{d}t \qquad c_\alpha = \frac{1}{2} - \alpha, \quad \delta = 2(1 - \alpha)$$

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Rescaled renewal law $\mathrm{P}(\frac{\mathrm{d}\tau}{N})$ is a probability on $\mathcal{C}$, converges to $\mathscr{P}$

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$$H_N^{\widetilde{\omega}}(\tau) = \sum_{n=1}^{N} \mathbb{1}_{\{n\in\tau\}}\, \widetilde{\omega}_n$$

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$\mathrm{Leb}(\boldsymbol{\tau}) = 0$ a.s. $\implies$ integral is ill-defined!

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Continuum Disordered Pinning Model (CDPM)

Fix $\alpha \in (\frac{1}{2}, 1)$ [Rescale $\beta_N = \hat{\beta} N^{\frac{1}{2}-\alpha}$, $h_N = \hat{h} N^{-\alpha} - \frac{1}{2}\beta_N^2$]

Theorem (existence and universality of CDPM)

$\mathrm{P}_N^{\widetilde{\omega}}(\frac{\mathrm{d}\tau}{N})$ converges in distrib. to a random law $\mathscr{P}^{\widetilde{W}}$, called CDPM

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Fix $\alpha \in (\frac{1}{2}, 1)$ [Rescale $\beta_N = \hat{\beta} N^{\frac{1}{2}-\alpha}$, $h_N = \hat{h} N^{-\alpha} - \frac{1}{2}\beta_N^2$]

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The CDPM has any a.s. property of α-stable regenerative set $\mathscr{P}$

$$A \subseteq \mathcal{C}, \quad \mathscr{P}(A) = 1 \qquad \Longrightarrow \qquad \mathscr{P}^{\widetilde{W}}(A) = 1 \,, \quad \mathbb{P}(\mathrm{d}\widetilde{W})\text{-a.s.}$$

Example: $A = \big\{ C \subseteq [0, \infty) : \; \dim_H(C) = \alpha \big\}$

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Theorem (singularity)

The CDPM $\mathscr{P}^{\widetilde{W}}$ is singular w.r.t. regenerative set $\mathscr{P}$ for $\mathbb{P}$-a.e. $\widetilde{W}$

Sketch of the proof

$Z_N^{\widetilde{\omega}} := \mathrm{E}\left[e^{H_N^{\widetilde{\omega}}(\tau)}\right]$ (free) $\qquad \widehat{Z}_N^{\widetilde{\omega}} := \mathrm{E}\left[e^{H_N^{\widetilde{\omega}}(\tau)} \mathbb{1}_{\{N\in\tau\}}\right]$ (constrained)

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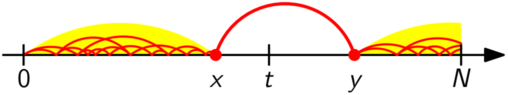
$A :=$ {  }

Discrete pinning model

$$\mathrm{P}_N^{\widetilde{\omega}}(A) =$$

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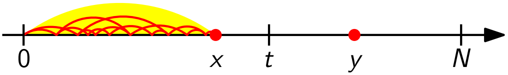

}

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$$\mathrm{P}_N^{\widetilde{\omega}}(A) = \widehat{Z}_{[0,x]}^{\widetilde{\omega}}$$

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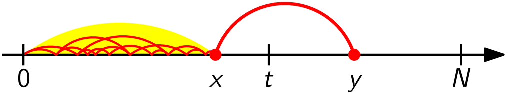
$A := \Big\{$ 

$\Big\}$

Discrete pinning model

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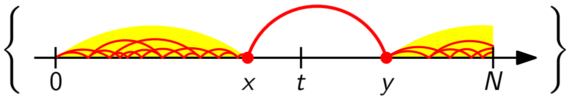

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$$\mathrm{P}_N^{\widetilde{\omega}}(A) = \widehat{Z}_{[0,x]}^{\widetilde{\omega}} \frac{C}{(y-x)^{1+\alpha}} Z_{[y,N]}^{\widetilde{\omega}}$$

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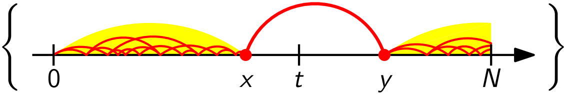
$A :=$ 

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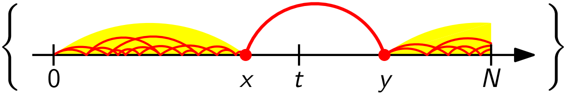
$A :=$ 

Discrete pinning model

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Probability determined by partition functions $\widehat{Z}$ and Z

(that have continuum limits $\widehat{\mathscr{Z}}$ and $\mathscr{Z}\ldots$)

Further observations

The CDPM yields sharp asymptotic predictions on free energy and critical curve, for $\alpha \in (\frac{1}{2}, 1)$, in the weak coupling regime $\beta, h \to 0$

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Marginal case $\alpha = \frac{1}{2}$ is under investigation...

Thanks