Pinning Model, Universality and Rough Paths

FRANCESCO CARAVENNA
University of Milano-Bicocca

RHEIN-MAIN-KOLLOQUIUM STOCHASTIK

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- Very preliminary, few "new" results
- Massimiliano Gubinelli, Giulia Comi
- DISORDER RELEVANCE & SINGULAR PDEs

(Rongfeng Sun, Nikos Zygouras)

1. Introduction and Results

OUR EQUATION

- Fix $H \in (0, \frac{1}{2})$. Time interval $[0,1]$.
- Singular Kernel $q_t = \frac{1}{t^{\frac{1}{2}-H}} \mathbb{1}_{[0,1]}(t)$ $\quad (q \in L^2)$
- Our equation for $z : [0,1] \to \mathbb{R}$

$$\text{✪} \qquad z_t = 1 + \int_0^1 z_r \, q_{t-r} \, dw_r \qquad t \in [0,1]$$

$w = (w_r)_{r \in [0,1]}$ Brownian motion

- Affine equation $\rightsquigarrow$ Explicit solution (Picard iteration)

$$z_t = 1 + \underbrace{\int_0^1 q_{t-r} \, dw_r}_{X_t^{(1)}} + \underbrace{\int_0^1 \int_0^1 q_{s-r} \, q_{t-s} \, dw_r \, dw_s}_{X_t^{(2)}} + \dots$$

Our Goal

- Give a robust ("pathwise") meaning to ✪

$$✪ \qquad z_t = 1 + \underbrace{\int_0^1 z_r \, q_{t-r} \, dw_r} = 1 + J_t(z)$$

for a given path $w \in C^{\frac{1}{2}-}$ (deterministic!)

- $x_t^{(1)} = J_t(1) = \int_0^\infty q_{t-r} \, dw_r$ is canonically defined

- **PROBLEM:** higher order terms $x_t^{(n)}$ with $n \geq 2$ are NOT CANONICAL

- Need to "enrich" the noise w : more information needed

MAIN RESULT

$$z_t = 1 + J_t(z) \quad (\star)$$

- Recall that $X_t^{(1)} = \int_0^1 \frac{1}{(t-r)^{\frac{1}{2}-H}} \, dW_r$ is canonically defined

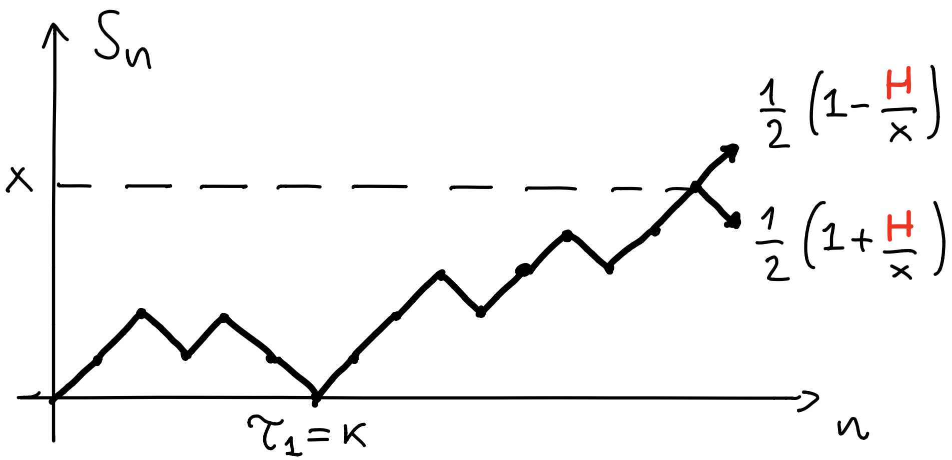
THEOREM Fix $H > \frac{1}{4}$. "Define" $I_t = \int_0^t X_r^{(1)} \, dW_r$ (non canonical!)

- Then $J_t(z)$ is CANONICALLY defined for a wide space of z.
- In this space one can solve $(\star)$.
- The solution z is a CONTINUOUS function of $\underbrace{(W, I)}_{\text{"ENRICHED NOISE"}}$

- Toy model for REGULARITY STRUCTURES

2. Pinning Model

BESSEL RANDOM WALK (K. Alexander)



$$E\left[S_{n+1} \mid S_n = x\right] \sim \frac{H}{x}$$

$$P(\tau_1 = k) \sim \frac{c}{k^{\frac{3}{2}+H}} = \frac{c}{k^{1+\alpha}}$$

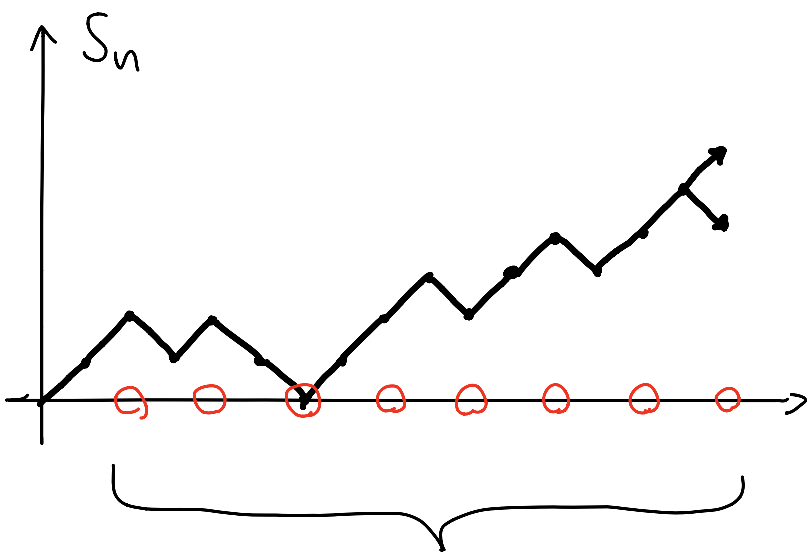
- $H \in (0, \frac{1}{2}) \iff \alpha \in (\frac{1}{2}, 1)$ $\qquad \alpha = \frac{1}{2} + H$

- RENEWAL FUNCTION $u_k = P(S_k = 0) \sim \dfrac{\tilde{c}}{k^{\frac{1}{2}-H}} = \dfrac{\tilde{c}}{k^{1-\alpha}}$

- WE ONLY NEED $\tau = \{\tau_1, \tau_2, \dots\} = \{n \geq 0 : S_n = 0\}$

(RENEWAL PROCESS)

DISORDER



- $\omega = (\omega_n)_{n \in \mathbb{N}}$ I.I.D., law $\mathbb{P}$

- $\mathbb{E}[\omega_n] = 0 \qquad \mathbb{V}\text{AR}[\omega_n] = 1$

- $\lambda(\beta) = \log \mathbb{E}\left[e^{\beta \omega_1}\right]$ FINITE

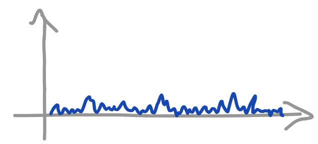
- REWARDS / PENALTIES $\quad e^{\beta \omega_n - \lambda(\beta) + h} \qquad \beta \geq 0, \; h \in \mathbb{R}$

- $\mathbb{E}\left[e^{\beta \omega_n - \lambda(\beta) + h}\right] = e^{h}$

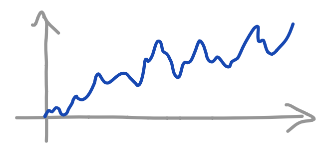
Pinning Model

- Fix $N \in \mathbb{N}$, $\beta \geq 0$, $h \in \mathbb{R}$, a typical realization of ω

$$P_N^{\omega}(dS) = \frac{e^{\sum_{n=1}^{N} \left(\beta\omega_n - \lambda(\beta) + h\right) \mathbb{1}_{\{S_n = 0\}}}}{Z_N^{\omega}} P(dS)$$



- Behavior of P_N^{ω} as $N \to \infty$ $\begin{cases} \text{LOCALIZED at zero} & (h < h_c(\beta)) \\ \text{DELOCALIZED away from zero} & (h > h_c(\beta)) \end{cases}$



- DISORDER IS RELEVANT: any fixed $\beta > 0$ radically different from $\beta = 0$

  $H > 0$ $\left(\alpha > \frac{1}{2}\right)$

WEAK DISORDER AND CONTINUUM LIMIT

- DISORDER RELEVANCE is linked to the existence of CONTINUUM LIMITS as $N \to \infty$, in a suitable WEAK DISORDER regime $\beta = \beta_N \to 0$, $h = h_N \to 0$

- PARTITION FUNCTION $Z_N^\omega = \mathrm{E}\left[e^{\sum_{n=1}^{N} (\beta\omega_n - \lambda(\beta) + h) \mathbb{1}_{\{S_n = 0\}}} \right]$

THEOREM (C., Sun, Zygouras)

Fix $h \equiv 0$ (for simpl.), $\beta = \beta_N = \frac{1}{N^H}$. Reverse $\tilde{\omega}_i := \omega_{N-i}$. Then

$$\left(Z_{Nt}^{\tilde{\omega}} \right)_{t \in [0,1]} \xrightarrow[N \to \infty]{d} (z_t)_{t \in [0,1]} \quad \text{in } C([0,1])$$

Polynomial Chaos

$$Z_N^{\omega} = E\left[e^{\sum_{n=1}^{N} (\beta\omega_n - \lambda(\beta))\, \mathbb{1}\{S_n=0\}} \right]$$

$$\boxed{\eta_n = \frac{e^{\beta\omega_n - \lambda(\beta)} - 1}{\beta} \qquad \mathbb{E}[\eta_n] = 0, \quad \mathbb{V}\mathrm{AR}[\eta_n] \simeq 1}$$

$$= E\left[\prod_{n=1}^{N} \left\{ 1 + \beta \cdot \eta_n \cdot \mathbb{1}\{S_n = 0\} \right\} \right]$$

$$= 1 + \sum_{n=1}^{N} \beta\, \eta_n \underbrace{P(S_n = 0)}_{\frac{1}{n^{\frac{1}{2}-H}}} + \sum_{1 \le m < n \le N} \beta^2\, \eta_m \eta_n \underbrace{P(S_m = 0, S_n = 0)}_{\frac{1}{m^{\frac{1}{2}-H}\,(n-m)^{\frac{1}{2}-H}}} + \dots$$

$$\simeq 1 + \int_0^1 \frac{1}{r^{\frac{1}{2}-H}}\, dW_r + \iint_{0<r<s<1} \frac{1}{r^{\frac{1}{2}-H}\,(s-r)^{\frac{1}{2}-H}}\, dW_r\, dW_s + \dots$$

$$= Z_1$$

GENERAL CONSIDERATIONS

- The last step approximates MULTI-LINEAR POLYNOMIALS of ANY order with MULTIPLE WIENER INTEGRALS $\rightsquigarrow$ INDEPENDENCE of the η_n is used

- Z^{ω}_{Nt} solves approximately our DIFFERENTIAL EQUATION (★)

$$Z^{\omega}_{Nt} \simeq 1 + \int_0^{\infty} Z^{\omega}_{Nr}\, q^{(N)}_{t-r}\, dW^{(N)}_r$$

where
$$W^{(N)}_t = \frac{1}{\sqrt{N}} \{ \eta_1 + \eta_2 + \dots + \eta_{Nt} \}$$

- We expect that Z^{ω}_{Nt} is approx. a CONTINUOUS FUNCTION of $W^{(N)}$ and

$$I^{(N)}_t = \frac{1}{N^{\frac{1}{2}+H}} \sum_{1 \le m < n \le Nt} \frac{\eta_m \eta_n}{(n-m)^{\frac{1}{2}-H}}$$

Results for Pinning Model

<u>Conjecture</u>. Assume that $H \in (\frac{1}{4}, \frac{1}{2}) \iff \alpha \in (\frac{3}{4}, 1)$

If $(W^{(N)}, I^{(N)}) \xrightarrow[N\to\infty]{d\ (\dots)} (W, I)$, then $(Z^{\omega}_{Nt}) \xrightarrow[N\to\infty]{d} (\mathcal{Z}_t)$

- Convergence of $(W^{(N)}, I^{(N)})$ holds much beyond independent η_n

  $\rightsquigarrow$ UNIVERSALITY

- For $H \in (0, \frac{1}{4}] \iff \alpha \in (\frac{1}{2}, \frac{3}{4}]$ a FINITE NUMBER of statistics beyond $W^{(N)}$ and $I^{(N)}$ should be relevant ($\rightsquigarrow$ SINGULAR PDEs)

- The case $H = 0 \iff \alpha = \frac{1}{2}$ is MARGINAL / CRITICAL ...

3. Rough Paths

Our Equation

- All functions will be defined on $[0,1]$.
- Hölder space $C^\alpha = \{ f : [0,1] \to \mathbb{R} : |f_t - f_s| \leq \underbrace{C}_{\|f\|_\alpha} |t-s|^\alpha \}$
- Fix a path $w \in C^{\frac{1}{2}-}$

$$(\star) \qquad z_t = 1 + \int_0^\infty z_r \, g_{t-r} \, dw_r$$

$$= 1 + \int_0^\infty \underbrace{z_r}_{\text{ROUGH: } z \in C^{H-}} \underbrace{\frac{1}{(t-r)^{\frac{1}{2}-H}}}_{\text{SMOOTH, BUT DIVERGES AS } r \to t} dw_r$$

Controlled Paths (M. Gubinelli)

- Fix $\alpha, \beta \in (0,1)$. Take $f \in C^\alpha$ and $g, h \in C^\beta$.

- Definition h is controlled by g (through f) iff

$$h_t - h_s = f_s (g_t - g_s) + O(|t-s|^{\alpha+\beta})$$

"Increments of h are not worse than those of g"

- Key Example: $h_t = \Phi(g_t)$ is controlled by g_t, for Φ smooth

YOUNG INTEGRAL

- Fix $x \in C^{\alpha}$, $w \in C^{\beta}$ with $\boxed{\alpha + \beta > 1}$
- CANONICAL DEFINITION of the integral

$$I_t = \int_0^t x_z \, dw_z \in C^{\beta}$$

$$= \lim_{|\pi| \to 0} \sum_{t_i \in \pi} x_{t_i} \left(w_{t_{i+1}} - w_{t_i} \right)$$

- CHARACTERIZING PROPERTY: I is controlled by w through x

$$I_t - I_s = x_s \left(w_t - w_s \right) + O\left(|t-s|^{\alpha+\beta} \right)$$

Rough Paths (T. Lyons)

- For $\boxed{\alpha+\beta<1}$, NO CANONICAL DEFINITION of $\int x_r \, dw_r$

Ex: Ito vs. Stratonovich for $x_r = w_r$

- Solution: "choose" $I_t = \int_0^t x_r \, dw_r \in C^\beta$ as any path satisfying

$$I_t - I_s = x_s (w_t - w_s) + \underbrace{O(|t-s|^{\alpha+\beta})}_{\mathbb{X}\mathbb{W}_{s,t}}$$

▸ Existence OK ▸ Non uniqueness OK $\left(I_t - \tilde{I}_t = \Delta_t \in C^{\alpha+\beta}\right)$

- $I \iff$ Remainder $\mathbb{X}\mathbb{W}_{s,t} = I_t - I_s - x_s(w_t - w_s) = \text{"}\int_s^t (x_r - x_s) \, dw_r\text{"}$

$$\begin{cases} \mathbb{X}\mathbb{W}_{s,t} = O(|t-s|^{\alpha+\beta}) \\ \mathbb{X}\mathbb{W}_{s,t} - (\mathbb{X}\mathbb{W}_{s,u} + \mathbb{X}\mathbb{W}_{u,t}) = (x_u - x_s)(w_t - w_u) \qquad \text{(CHEN)} \end{cases}$$

ROUGH INTEGRAL

- Assume now $\boxed{\alpha + \beta < 1}$ but $\boxed{2\alpha + \beta > 1}$ $\qquad \begin{cases} \alpha = H- \\ \beta = \frac{1}{2}- \end{cases} \Rightarrow H > \frac{1}{4}$

- Choose your favourite $I_t = \int_0^t x_r \, dW_r$

- ROUGH INTEGRAL. There is a CANONICAL DEFINITION of $\int z_r \, dW_r$ for a wide class $z \in D_x$ $\qquad \lim_{|\pi| \to 0} \sum_{t_i \in \pi} \left\{ z_{t_i} (W_{t_{i+1}} - W_{t_i}) + z'_{t_i} \mathbb{W}_{t_i t_{i+1}} \right\}$

$$D_x = \text{paths } z \in C^\alpha \text{ controlled by } x \in C^\alpha$$

$$= \left\{ (z, z') \in C^\alpha \times C^\alpha : \; z_t - z_s = z'_s (x_t - x_s) + O(|t-s|^{2\alpha}) \right\}$$

(This is the space where we will solve our equation ⍟)

Singular Kernel (Hairer)

- It remains to introduce a SINGULAR KERNEL

$$q_r \approx \frac{1}{|r|^\gamma} = \sum_{n \geq 0} (2^n)^\gamma \cdot \underbrace{\varphi_n(r)}_{\text{SMOOTH}} \cdot \mathbb{1}_{\{|r| \leq \frac{1}{2^n}\}}$$

THEOREM. Assume that $\gamma < \beta$, where $w \in C^\beta$.

Then there is a canonical definition of

$$J_t(z) = \int_0^\infty z_r \, q_{t-r} \, dw_r \qquad \text{for all } z \in D_x$$

such that $J_t(z) \in C^{\beta - \gamma}$

LOSS OF REGULARITY!

4. Robust Analysis of our Equation

BACK TO OUR EQUATION

$$\circledast \qquad z_t = 1 + \underbrace{\int_0^\infty z_r \, q_{t-r} \, dw_r}$$

$$= 1 + J_t(z)$$

$$\begin{cases} q_s = \dfrac{1}{s^{\frac{1}{2}-H}} \mathbb{1}_{\{s>0\}} \\ w \in C^{\frac{1}{2}-} = C^\beta \end{cases}$$

1. Define $x \in C^{H-} = C^\alpha$ canonically by

$$x_t = J_t(1) = \int_0^\infty q_{t-r} \, dw_r$$

2. Choose $I_t = \text{"}\int_0^t x_r \, dw_r\text{"}$ (non canonical), equivalently

$$\mathbb{XW}_{s,t} = \text{"}\int_s^t (x_r - x_s) \, dw_r\text{"} = O\left(|t-s|^{H+\frac{1}{2}-}\right) \quad + \quad \text{(CHEN)}$$

TOPOLOGIES

- ENRICHED NOISE $(W, \mathbb{W})$

$$\|W\|_{\frac{1}{2}-} := \sup_{0 \le s < t \le 1} \frac{|W_t - W_s|}{|t-s|^{\frac{1}{2}-}}$$

$$\|\mathbb{W}\|_{H+\frac{1}{2}-} := \sup_{0 \le s < t \le 1} \frac{|\mathbb{W}_{s,t}|}{|t-s|^{H+\frac{1}{2}-}}$$

- CONTROLLED PATHS

$$\mathcal{D}_X = \left\{ (Z, Z') : \ Z_t - Z_s = Z'_s (X_t - X_s) + \underbrace{O(|t-s|^{2H-})}_{Z^R_{s,t}} \right\}$$

$$\|Z\|_{\mathcal{D}_X} := \|Z'\|_{H-} + \|Z^R\|_{2H-}$$

Main Result

$$(\star) \qquad Z_t = 1 + \underbrace{\int_0^\infty Z_r \, q_{t-r} \, dW_r}_{} = 1 + J_t(Z)$$

- **Theorem** Assume $H \in (\frac{1}{4}, 1)$. Fix $(W, \mathbb{W})$ and x as above.

  ▶ $J_t(Z)$ is canonically defined for all $Z \in \mathcal{D}_x$.
  Moreover it defines a controlled path with $J_t'(Z) = Z_t$.

  ▶ The map $Z \mapsto 1 + J(Z)$ is a contraction on $\mathcal{D}_x$ for short time.

  ▶ There exists a unique solution $Z = 1 + J(Z)$ of $(\star)$
  which depends continuously on the enriched noise $(W, \mathbb{W})$.

CONCLUSIONS

- 1-dimensional stochastic equation, motivated by PINNING
- Robust analysis for $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$ ($\rightsquigarrow$ extend to $H \in \left(0, \frac{1}{2}\right)$)
- Rough Paths techniques + ideas from Regularity Structures
- Ultimate goal (dream?): push the analysis to $H = 0$

Danke