Polynomial Chaos and Scaling Limits of Disordered Systems

1. Introduction and overview

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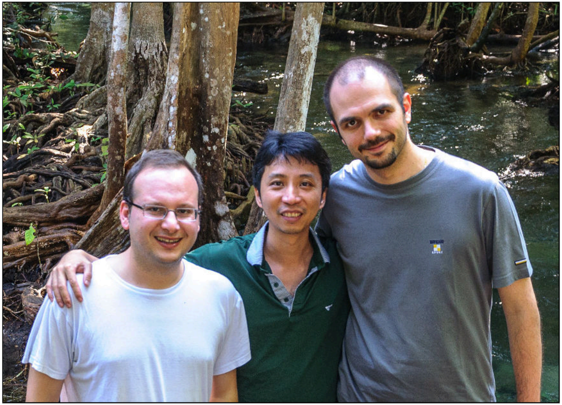
Coworkers



Joint work with Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

Overview

Consider a homogeneous system, described by a probability measure $\mathbf{P}^{\mathrm{ref}}$ on some configuration space (*with "interesting" large scale properties*)

Perturb it in a inhomogeneous way, defining a disordered system $\mathbf{P}^{\omega}$

$$\mathbf{P}^{\omega}(\mathrm{d}\sigma) \propto e^{\boldsymbol{H}^{\omega}(\sigma)}\,\mathbf{P}^{\mathrm{ref}}(\mathrm{d}\sigma) \qquad \text{disorder } \omega = \text{“random landscape”}$$

Are large scale properties affected by (a small amount of) disorder?
Is the law $\mathbf{P}^{\omega}$ radically different from $\mathbf{P}^{\mathrm{ref}}$?

Disorder relevance vs. irrelevance

We are going to look at this problem in the weak disorder regime

General framework ("model independent") $\rightsquigarrow$ Universality

Overview

General framework $\leftrightsquigarrow$ concrete examples

1. Directed polymer in random environment (perturb. of random walk)
2. Disordered pinning models (perturb. of renewal process)
3. Random-field Ising model

1'. Stochastic Heat Equation

(Inspired by [Alberts, Khanin, Quastel 2014] on directed polymers)

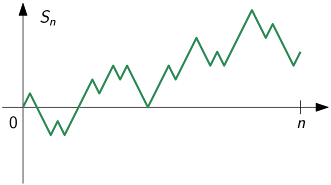
- This lecture: general introduction and overview
- Next lectures: more specific issues

Outline

1. Homogeneous systems
2. Disordered systems
3. Main results
4. Sketch of the proof

Outline

1. Random walk



$\mathbf{P}^{\mathrm{ref}}$ = law of symm. random walk on $\mathbb{Z}^d$

$$S = (S_n)_{n \geq 0}$$

with i.i.d. increments $S_n - S_{n-1}$

S attracted to α-stable Lévy process
Brownian motion

$$\begin{cases} \mathbf{E}^{\mathrm{ref}}[|S_1|^2] < \infty & \text{if } \alpha = 2 \\ \mathbf{P}^{\mathrm{ref}}(|S_1| > x) \sim \frac{C}{x^\alpha} & \text{if } 0 < \alpha < 2 \end{cases}$$

Alternative "language"

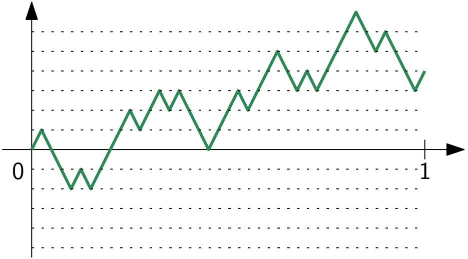
Define "spin" variable $\sigma_{n,x}$ in each space-time point

$$\sigma_{n,x} := \mathbb{1}_{\{S_n = x\}} \in \{0, 1\}$$

The random field $(\sigma_{n,x})_{(n,x) \in \mathbb{N}_0 \times \mathbb{Z}^d}$ is far from independent!

1. Random walk - large scale properties

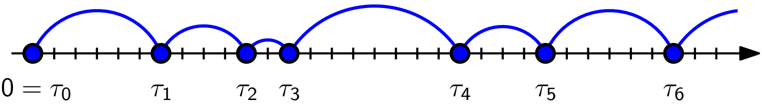
Diffusive rescaling $S^\delta = (\sqrt{\delta}\, S_{\frac{t}{\delta}})_{t \geq 0}$ $\mathbb{T}_\delta := \delta\mathbb{N}_0 \times (\sqrt{\delta}\mathbb{Z})^d$



$\delta = \frac{1}{N}$

S^δ converges in law to BM as $\delta \to 0$ (Donsker)

2. Renewal process



$\mathbf{P}^{\mathrm{ref}}$ = law of a renewal process (= RW with positive increments)

$$\mathbf{P}^{\mathrm{ref}}\big((\tau_{i+1} - \tau_i) = n\big) \sim \frac{c}{n^{1+\alpha}} \qquad \text{tail exponent } \alpha \in (0,1)$$

$\tau = \{0 = \tau_0 < \tau_1 < \tau_2 < \ldots\} \subseteq \mathbb{N}_0$ viewed as a random subset

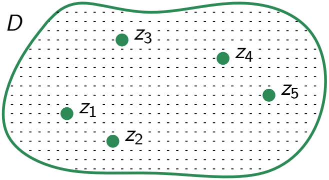
"spins" $\sigma_n := \mathbb{1}_{\{n \in \tau\}} \in \{0,1\}$

$\mathbb{T}_\delta = \delta\mathbb{N}_0$ $\quad \delta\tau \xrightarrow{d} \alpha$-stable regenerative set $\quad$ (as $\delta \to 0$)

The general setup

Lattice $\mathbb{T}_\delta \subseteq D \subseteq \mathbb{R}^d$ (mesh $\approx \delta$)

$z \longmapsto$ two-valued field $\sigma_z \in \{0,1\}$



- $\mathcal{S} = \{0,1\}^{\mathbb{T}_\delta}$ space of spin configurations $\sigma = (\sigma_z)_{z \in \mathbb{T}_\delta}$
- $\mathbf{P}_\delta^{\mathrm{ref}}$ "interesting" probability on $\mathcal{S}$

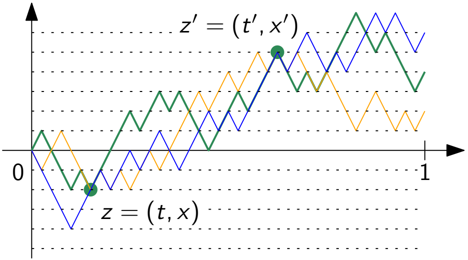
Typically $\mathbf{P}_\delta^{\mathrm{ref}}$ has a non-trivial continuum limit as $\delta \to 0$

Assumption: non-trivial correlations

$$\exists \gamma > 0 : \qquad \frac{\mathbf{P}_\delta^{\mathrm{ref}}\left(\sigma_{\{z_1, z_2, \ldots, z_k\}} = 1\right)}{(\delta^\gamma)^k} \xrightarrow[\delta \to 0]{} \psi_k(z_1, \ldots, z_k)$$

Example 1. Random walk

Large-scale correlations on $\mathbb{T}_\delta := \delta\mathbb{N}_0 \times (\sqrt{\delta}\mathbb{Z})^d$

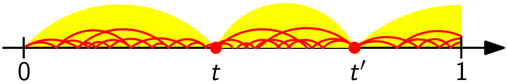


$$\delta = \frac{1}{N}$$

$$\frac{\mathbf{P}_\delta^{\mathrm{ref}}(\sigma_z = 1,\, \sigma_{z'} = 1)}{(\delta^{\frac{d}{2}})^2} \xrightarrow[\delta \to 0]{} \psi(z, z') = \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{\frac{d}{2}}} \, \frac{e^{-\frac{|x'-x|^2}{2(t'-t)}}}{(2\pi(t'-t))^{\frac{d}{2}}}$$

Example 2. Renewal process

Large-scale correlations on $\mathbb{T}_\delta := \delta\mathbb{N}_0$



$$\frac{\mathbf{E}_\delta^{\mathrm{ref}}[\sigma_t\,\sigma_{t'}]\mathbf{P}_\delta^{\mathrm{ref}}(t\in\tau,\ t'\in\tau)}{(\delta^{1-\alpha})^2} \xrightarrow[\delta\to 0]{} \psi(t,t') = \frac{c'}{t^{1-\alpha}}\,\frac{c'}{(t'-t)^{1-\alpha}}$$

Outline

Enters disorder

$(\omega_z)_{z \in \mathbb{T}_\delta}$ i.i.d. random variables (e.g. $\mathcal{N}(0,1)$)

$\mathbb{E}[\omega_z] = 0 \qquad \mathbb{E}[\omega_z^2] = 1 \qquad \lambda(\beta) = \log \mathbb{E}[e^{\beta \omega_z}] < \infty$

Each site $z \in \mathbb{T}_\delta$ carries a charge ω_z that can be $\begin{cases} > 0 & \text{reward} \\ < 0 & \text{penalty} \end{cases}$

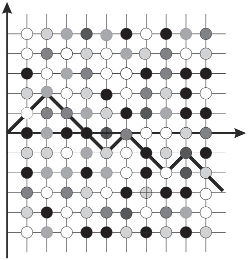
Spatial inhomogeneities in $\mathbf{P}_\delta^{\mathrm{ref}}(\mathrm{d}\sigma) \quad \rightsquigarrow \quad$ new probability law $\mathbf{P}_\delta^\omega(\mathrm{d}\sigma)$

Gibbs measure: $\mathbf{P}_\delta^\omega(\mathrm{d}\sigma) := \dfrac{1}{\boldsymbol{Z}_\delta^\omega} \, e^{\boldsymbol{H}^\omega(\sigma)} \, \mathbf{P}_\delta^{\mathrm{ref}}(\mathrm{d}\sigma)$

(−) Energy: $\sigma \mapsto \boldsymbol{H}^\omega(\sigma) := \sum_{z \in \mathbb{T}_\delta} \big(\beta\, \omega_z \;+\; h \;-\; \lambda(\beta) + h\big)\, \sigma_z$

$\beta \geq 0$ disorder strength $\qquad h \in \mathbb{R}$ disorder bias

1. Directed Polymer in Random Environment (random walk)



- Symmetric random walk $S = (S_n)_{n \geq 0}$ on $\mathbb{Z}^d$ attracted to BM (finite variance)
- $\omega_{n,x} > 0$ reward $\omega_{n,x} < 0$ penalty
- "spin" $\sigma_{n,x} := \mathbb{1}_{\{S_n = x\}} \in \{0,1\}$

Directed polymer in random environment ($N = 1/\delta$ steps)

$$\mathbf{P}^{\omega}(S) = \frac{1}{\mathbf{Z}_{\delta}^{\omega}} \, e^{\sum_{n=1}^{N} (\beta \omega_{n,S_n} - \lambda(\beta) + h)} \, \mathbf{P}^{\mathrm{ref}}(S)$$

RW paths in corridors of large $\omega > 0$ have high probability (energy gain)

...but such paths are few! (entropy loss) $\rightsquigarrow$ Who wins?

1. Directed Polymer in Random Environment (random walk)

- $[d \geq 3, \;\; \beta > 0$ small$]$ $\quad \mathbf{P}^\omega$ "similar" to $\mathbf{P}^{\mathrm{ref}}$ (entropy wins)

$$\frac{S_N}{\sqrt{N}} \text{ under } \mathbf{P}^\omega \xrightarrow[N\to\infty]{d} \mathcal{N}(0,1) \qquad (\mathbb{P}(\mathrm{d}\omega)\text{-a.s.})$$

i.e. the same under $\mathbf{P}^{\mathrm{ref}}$ [Imbrie, Spencer 1988] [Bolthausen 1989]

- $[d \leq 2, \;\;$ any $\beta > 0]$ $\quad \mathbf{P}^\omega$ "different" from $\mathbf{P}^{\mathrm{ref}}$ (energy wins)

$$\max_{x \in \mathbb{Z}^d} \mathbf{P}^\omega\big(S_N = x\big) \geq c > 0 \qquad (\mathbb{P}(\mathrm{d}\omega)\text{-a.s.})$$

unlike $\mathbf{P}^{\mathrm{ref}}(S_N = x) = O\Big(\frac{1}{\sqrt{N}}\Big) = o(1)$ [Carmona, Hu 2002]

[Comets, Shiga, Yoshida 2003]

[Vargas 2007]

For DPRE disorder is irrelevant for $d \geq 3$ and relevant for $d \leq 2$

($d = 2$ is actually marginally relevant, cf. below)

Disorder Relevance vs. Irrelevance

Does arbitrarily small (but fixed!) disorder affect large scale properties?

Is $\mathbf{P}_\delta^\omega$ qualitatively different from $\mathbf{P}_\delta^{\mathrm{ref}}$?

$[\,\delta \to 0 \;\; (N \to \infty) \;$ with fixed $\beta > 0\,]$

YES: model is disorder relevant   NO: model is disorder irrelevant

2. Disordered Pinning Model (renewal process + disorder)

$$\mathbf{P}^{\mathrm{ref}}(\tau_1 = n) \sim \frac{c}{n^{1+\alpha}}$$

$\left[\, \alpha > \frac{1}{2} \,\right]$ disorder relevant $\left[\, \alpha < \frac{1}{2} \,\right]$ disorder irrelevant

$\left[\, \alpha = \frac{1}{2} \,\right]$ marginal: (ir)relevance depends on finer details

(cf. free energy and critical exponents)   [References: ...]

What are we going to do?

We focus on models $\mathbf{P}_\delta^\omega$ which are disorder relevant

Any fixed disorder strength $\beta > 0$, no matter how small, has dramatic effects in the large scale regime $\delta \to 0$ (i.e. $N \to \infty$)

Weak disorder regime

Can we tune $\beta \to 0$ as $\delta \to 0$ and still see interesting effects on $\mathbf{P}_\delta^\omega$?

(For instance, does $\mathbf{P}_\delta^\omega$ converge to a random limit law $\mathcal{P}^W$?)

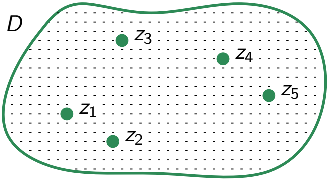
YES! This is the goal of our course

Very robust approach $\leftrightsquigarrow$ Universality

Outline

Key assumption (disorder relevant vs. marginal)

- Lattice $\mathbb{T}_\delta \subseteq D \subseteq \mathbb{R}^d$ (mesh $\approx \delta$)
  Two-valued field $\sigma = (\sigma_z)_{z \in \mathbb{T}_\delta}$
- $\mathbf{P}_\delta^{\mathrm{ref}}$ interesting probability for σ



$$\exists \gamma > 0 : \quad \frac{\mathbf{E}_\delta^{\mathrm{ref}}\left[\sigma_{z_1}\,\sigma_{z_2} \cdots \sigma_{z_k}\right]}{(\delta^\gamma)^k} \xrightarrow[\delta \to 0]{} \xrightarrow[\delta \to 0]{L^2(D)} \psi_k(z_1, \ldots, z_k) \quad (\star)$$

L^2 characterizes disorder relevant regime! (Harris criterion)

1. DPRE. $\psi(t,x) = \dfrac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{\frac{d}{2}}}$ $\qquad L^2([0,1] \times \mathbb{R}^d) \rightsquigarrow d < 2$
2. Pinning. $\psi(t) = \dfrac{c'}{t^{1-\alpha}}$ $\qquad L^2([0,1]) \rightsquigarrow \alpha > \frac{1}{2}$ marginal!

The partition function

Recall the definition of the disordered system

$$\mathbf{P}_\delta^\omega(\mathrm{d}\sigma) := \frac{1}{\boldsymbol{Z}_\delta^\omega}\, e^{\boldsymbol{H}^\omega(\sigma)}\, \mathbf{P}_\delta^{\mathrm{ref}}(\mathrm{d}\sigma)$$

We focus on the normalizing constant $\boldsymbol{Z}_\delta^\omega$ called partition function

$$\boldsymbol{Z}_\delta^\omega = \mathbf{E}^{\mathrm{ref}}\left[e^{\boldsymbol{H}^\omega(\sigma)}\right] = \mathbf{E}^{\mathrm{ref}}\left[\exp\left(\sum_{z\in\mathbb{T}_\delta}(\beta\omega_z - \lambda(\beta) + h)\,\sigma_z \quad \sum_{1\le n\le N}\left(\beta\omega_{(n,S_n)} - \lambda(\beta\right.\right.\right.$$

DPRE: sample ω's along a RW path $(S_n)_{n\ge 0}$, then average their exp

The partition function $\boldsymbol{Z}_\delta^\omega$ encodes the key properties of $\mathbf{P}_\delta^\omega$

- $\boldsymbol{Z}_\delta^\omega$ is simpler than $\mathbf{P}_\delta^\omega(\mathrm{d}\sigma)$ (random number vs. random measure)
- It is still a complicated function of i.i.d. random variables $(\omega_x)_{x\in\mathbb{T}_\delta}$

Plan of the course

Key Result (scaling limit of $\boldsymbol{Z}_{\delta}^{\omega}$)

The partition function $\boldsymbol{Z}_{\delta}^{\omega}$ has a non-trivial limit in distribution $\mathcal{Z}^{W}$ (continuum partition function) when $\beta, h \to 0$ at suitable rates as $\delta \downarrow 0$

A. disorder relevant systems B. marginal systems

- Lecture I. Key Result A
  - Sketch of the proof
  - Lindeberg principle for polynomial chaos
- Lecture II. Some consequences of Key Result A
  - Disordered continuum model
  - Free energy estimates
- Lecture III. Key Result B
  - DPRE $d = 2$, Pinning $\alpha = \frac{1}{2}$, 2d Stochastic Heat Equation

Key Result A (disorder relevant systems)

Theorem A [C., Sun, Zygouras '15+]

Let $\mathbf{P}_\delta^{\mathrm{ref}}$ satisfy $(\star)$ with exponent γ and dimension d.

If we scale $\beta, h \to 0$ appropriately:

$$\beta := \hat{\beta}\,\delta^{d/2-\gamma} \qquad h := \hat{h}\,\delta^{d-\gamma} \qquad (\hat{\beta},\ \hat{h} \text{ fixed})$$

the partition function has a non-trivial limit in law: $\boldsymbol{Z}_\delta^\omega \xrightarrow[\delta\downarrow 0]{d} \mathcal{Z}^W$

The limit $\mathcal{Z}^W$ is explicit function of $W(\mathrm{d}x) :=$ white noise on $\mathbb{R}^d$

$$\mathcal{Z}^W := \sum_{k=0}^\infty \frac{1}{k!} \int \cdots \int_{\Omega^k} \psi_k(z_1, \ldots, z_k) \prod_{i=1}^k \left(\hat{\beta}\, W(\mathrm{d}z_i) + \hat{h}\, \mathrm{d}z_i\right)$$

(Wiener chaos expansion)

Outline

The partition function

Let us take a breath... Forget about $\mathbf{P}_\delta^\omega(\mathrm{d}\sigma)$

Just look at the partition function Z_δ^ω from a probabilistic viewpoint:

$$Z_\delta^\omega = \mathbf{E}^{\mathrm{ref}}\left[\exp\left(\sum_{z\in\mathbb{T}_\delta} (\beta\omega_z - \lambda(\beta) + h)\,\sigma_z \sum_{1\le n\le N} \left(\beta\omega_{(n,S_n)} - \lambda(\beta) + h\right)\right)\right]$$

DPRE: sample ω's along a RW path $(S_n)_{n\ge 0}$, then average their exp

Problem

Z_δ^ω is a complicated function of i.i.d. random variables $(\omega_x)_{x\in\mathbb{T}_\delta}$

How to study its convergence in law as $\delta \to 0$?

Solution

Z_δ^ω is a simpler function of other i.i.d. random variables $(X_x)_{x\in\mathbb{T}_\delta}$

Sketch of the approach: Polynomial Chaos

1. Linearization. Since $\sigma_x \in \{0,1\}$, every function of σ_x is linear

$$Z_\delta^\omega = \mathbf{E}_\delta^{\mathrm{ref}}\left[e^{\sum_{x\in\mathbb{T}_\delta}(\beta\omega_x - \lambda(\beta)+h)\sigma_x}\right] = \mathbf{E}_\delta^{\mathrm{ref}}\left[\prod_{x\in\mathbb{T}_\delta} e^{(\beta\omega_x-\lambda(\beta)+h)\sigma_x}\right] = \mathbf{E}_\delta^{\mathrm{ref}}\left[\prod_{x\in\mathbb{T}_\delta} (1$$

where $X_x := e^{\beta\omega_x - \lambda(\beta)+h} - 1$. New random variables (X_x) with

$$\mathbb{E}[X_x] \simeq h \qquad \mathbb{V}\mathrm{ar}[X_x] \simeq \beta^2$$

2. High-temperature expansion. By a binomial expansion of the product

$$Z_\delta^\omega = \sum_{k=0}^{|\mathbb{T}_\delta|} \frac{1}{k!} \sum_{(x_1,\ldots,x_k)\in(\mathbb{T}_\delta)^k} \mathbf{E}_\delta^{\mathrm{ref}}\left[\sigma_{x_1}\cdots\sigma_{x_k}\right] X_{x_1}\cdots X_{x_k}$$

Multilinear polynomial of random variables X_x's $\rightsquigarrow$ Decoupling !

Formally replace $\sum \rightsquigarrow \int$ and $X_{x_i} \rightsquigarrow W(\mathrm{d}x_i)$. Justification ?

A concrete example: Disordered Pinning Model

Pinning Models with $\alpha > \frac{1}{2}$ (disorder relevant) $[\,\delta = \frac{1}{N}\,]$

$$Z_\delta^\omega \approx 1 + \sum_{0<n\le N} \frac{X_n}{n^{1-\alpha}} + \sum_{0<m<n\le N} \frac{X_m \quad X_n}{m^{1-\alpha}(n-m)^{1-\alpha}} + \dots$$

Rescaling $\beta \sim \delta^{\alpha-\frac{1}{2}}$ ($h \equiv 0$ for simplicity)

$$\xrightarrow[\delta\to 0]{d} 1 + \int_{0<t<1} \frac{\mathrm{d}W_t}{t^{1-\alpha}} + \int_{0<s<t<1} \frac{\mathrm{d}W_s \quad \mathrm{d}W_t}{s^{1-\alpha}(t-s)^{1-\alpha}} + \dots$$

Intriguing question: what happens for $\alpha = \frac{1}{2}$?

This is marginal! Like 2d DPRE and 2d Stochastc Heat Equation

Justification

General problem: convergence in law for random variables of the form

Polynomial chaos

$$Z = \Psi(X) = \psi(\emptyset) + \sum_{i \in \mathbb{T}} \psi(i)\, X_i + \frac{1}{2} \sum_{i \neq j \in \mathbb{T}} \psi(i,j)\, X_i\, X_j + \ldots$$

$$= \sum_{I \subseteq \mathbb{T}} \psi(I) \prod_{i \in I} X_i$$

$X = (X_i)_{i \in \mathbb{T}}$ independent (possibly non i.d.) random variables in L^2

- Can we pretend that X_i's are i.i.d. Gaussians?
  YES, thanks to a Lindeberg principle that we now discuss
- Can we replace Gaussian X_i's by white noise $W(\mathrm{d}x_i)$?
  YES, by coupling + L^2 estimates

Variance and influences

Fix a multi-linear polynomial

$$\Psi(x) = \sum_{I \subseteq \mathbb{T}} \psi(I)\, x^I \qquad \text{with} \quad x^I := \prod_{i \in I} x_i$$

$$\mathrm{C}_\Psi := \sum_{I \subseteq \mathbb{T},\ I \neq \emptyset} \psi(I)^2 = \mathbb{V}\mathrm{ar}[\Psi(X)]$$

$$\mathrm{Inf}_i[\Psi] := \sum_{I \subseteq \mathbb{T},\, I \ni i} \psi(I)^2 = \mathbb{E}\Big[\mathbb{V}\mathrm{ar}\big[\, \Psi(X) \,\big|\, X_{\mathbb{T} \setminus \{i\}} \,\big]\Big]$$

For any family of r.v.'s $X = (X_i)_{i \in \mathbb{T}}$ with $\mathbb{E}[X_i] = 0$ $\mathbb{V}\mathrm{ar}[X_i] = 1$

$\mathrm{Inf}_i[\Psi]$ quantifies how much $\Psi(x)$ depends on the variable x_i

Noise sensitivity [Benjaimini, Kalai, Schramm 2001] [Garban, Steif 2012]

Lindeberg Principle

If influences $\mathrm{Inf}_i(\Psi)$ are small, the law of $\Psi(X)$ is insensitive to the details of the laws of the individual X_i's

- Fix a multi-linear polynomial $\Psi(x) = \sum_{I \subseteq \mathbb{T},\, |I| \le \ell} \psi(I)\, x^I$ of degree ℓ
- $X = (X_i)_{i\in\mathbb{T}}$, $X' = (X'_i)_{i\in\mathbb{T}}$ indep. with zero mean, unit variance

$$m_3 := \max_{i\in\mathbb{T}} \left(\mathbb{E}[|X_i|^3] \vee \mathbb{E}[|X'_i|^3]\right) < \infty$$

Theorem [Mossel, O'Donnel, Oleszkiewicz 2010]

$$\mathrm{dist}\big(\Psi(X), \Psi(X')\big) := \sup_{f\in C^3:\ \|f'\|_\infty, \|f''\|_\infty, \|f'''\|_\infty \le 1} \Big|\mathbb{E}\big[f(\Psi(X))\big] - \mathbb{E}\big[f(\Psi(X'))\big]\Big|$$

$$\le\ 30^\ell\, C_\Psi\, {m_3}^\ell \sqrt{\max_{i\in\mathbb{T}} \left(\mathrm{Inf}_i[\Psi]\right)}$$

Lindeberg Principle

We can go beyond finite 3rd moment. Define the truncated moments

$$m_2^{>M} := \sup_{X \in \{X_i, X_i'\}} \mathbb{E}\left[X^2 \, \mathbb{1}_{\{|X|>M\}}\right] \qquad m_3^{\leq M} := \sup_{X \in \{X_i, X_i'\}} \mathbb{E}\left[|X|^3 \, \mathbb{1}_{\{|X|\leq M\}}\right]$$

Theorem [C., Sun, Zygouras 2015+]

$$\mathrm{dist}\big(\Psi(X), \Psi(X')\big)$$

$$\leq 70^{\ell+1} \, C_\Psi \left\{ m_2^{>M} + \left(m_3^{\leq M}\right)^{\ell} \sqrt{\max_{i \in \mathbb{T}} \left(\mathrm{Inf}_i[\Psi]\right)} \right\} \qquad \leq e^{\frac{2}{\varepsilon}} \sum$$

- Explicit, non-asymptotic estimate!
- Extension to the case $\mathbb{E}[X_i] = \mathbb{E}[X_i'] = \mu_i \neq 0$

$$\Psi^\varepsilon(x) = \sum_{I \subseteq \mathbb{T}} (1+\varepsilon)^{|I|} \, \psi(I) \, x^I$$

Lindeberg Principle

$$\mathrm{dist}\big(\Psi(X), \Psi(X')\big) \le 70^{\ell+1}\, C_\Psi \left\{ m_2^{>M} + \left(m_3^{\le M}\right)^{\ell} \sqrt{\max_{i \in \mathbb{T}} \left(\mathrm{Inf}_i[\Psi]\right)} \right\}$$

Corollary

Consider a family $(\Psi_\delta)_{\delta>0}$ of multi-linear polynomials

- Assume $\sup_{\delta>0} C_{\Psi_\delta} < \infty$ $\max_{i \in \mathbb{T}_\delta} \left(\mathrm{Inf}_i[\Psi_\delta]\right) \xrightarrow[\delta \to 0]{} 0$
- Take $(X_{\delta,i})$, $(X'_{\delta,i})$ with zero mean, unit variance and u.i. squares

$$\lim_{M\to\infty} m_2^{>M} := \sup_{X \in \{X_{\delta,i}, X'_{\delta,i}\}} \mathbb{E}\left[X^2 \, \mathbb{1}_{\{|X|>M\}}\right] = 0$$

Then $$\boxed{\mathrm{dist}\big(\Psi_\delta(X_\delta), \Psi_\delta(X'_\delta)\big) \xrightarrow[\delta \to 0]{} 0}$$

Does $\Psi_N(X_\delta)$ have a limit in law as $\delta \to 0$? Check it for Gaussian X_δ's !

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