Polynomial Chaos and Scaling Limits of Disordered Systems

2. Continuum model and free energy estimates

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Overview

In the previous lecture we saw how to build continuum partition functions $\boldsymbol{Z}_\delta^\omega \xrightarrow[\delta\to 0]{d} \mathcal{Z}^W$ (scaling limits of discrete partition functions)

In this lecture we present two interesting applications of $\mathcal{Z}^W$

- Scaling limit of the full probability measure $\mathbf{P}_\delta^\omega \xrightarrow[\delta\to 0]{d} \mathcal{P}^W$ constructing a continuum version of the disordered system

  We will focus on the DPRE [Alberts, Khanin, Quastel 2014b] drawing inspiration from the Pinning [C., Sun, Zygouras 2016]

- Sharp asymptotics on the discrete model, in terms of free energy and critical curve

  For this we will focus on Pinning models (rather than DPRE)

Outline

1. White noise and Wiener chaos

2. Continuum partition functions

3. The continuum DPRE

4. Pinning models

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1. White noise and Wiener chaos

2. Continuum partition functions

3. The continuum DPRE

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White noise (1 dim.)

We are familiar with (1-dim.) Brownian motion $B = (B(t))_{t \geq 0}$

We are interested in its derivative "$W(t) := \frac{\mathrm{d}}{\mathrm{d}t} B(t)$" called white noise

Think of W as a stochastic process $W = (W(\cdot))$ indexed by

$$\text{Intervals } I = [a, b] \quad \longmapsto \quad W(I) = B(b) - B(a) \sim \mathcal{N}(0, b - a)$$

$$\text{Borel sets } A \in \mathcal{B}(\mathbb{R}) \quad \longmapsto \quad W(A) = \int_{\mathbb{R}} \mathbb{1}_A(t) \, \mathrm{d}B(t) \sim \mathcal{N}(0, |A|)$$

W is a Gaussian process with

$$\mathbb{E}[W(A)] = 0 \qquad \mathbb{C}\mathrm{ov}[W(A), W(B)] = |A \cap B|$$

White noise

White noise on $\mathbb{R}^d$

It is a Gaussian process $W = (W(A))_{A \in \mathcal{B}(\mathbb{R}^d)}$ with

$$\mathbb{E}[W(A)] = 0 \qquad \mathbb{C}\mathrm{ov}[W(A), W(B)] = |A \cap B|$$

- $\forall\, (A_n)_{n\in\mathbb{N}}$ disjoint $\implies$ $W\left(\bigcup_{n\in\mathbb{N}} A_n\right) \overset{\text{a.s.}}{=} \sum_{n\in\mathbb{N}} W(A_n)$

  Almost a random signed measure on $\mathbb{R}^d$... (but not quite!)

We can define single stochastic integrals $W(f) := \int f(x)\, W(\mathrm{d}x)$

$$\mathbb{E}[W(f)] = 0 \qquad \mathbb{E}[W(f)^2] = \|f\|^2_{L^2(\mathbb{R}^d)}$$

Multiple stochastic integrals

We can define

$$W^{\otimes k}(g) = \int_{(\mathbb{R}^d)^k} g(x_1, \ldots, x_k)\, W(\mathrm{d}x_1) \cdots W(\mathrm{d}x_k)$$

For $d = 1$ we can restrict $x_1 < x_2 < \ldots < x_k$ $\rightsquigarrow$ iterated Ito integrals

For symmetric functions we have

$$\mathbb{E}[W^{\otimes k}(g)] = 0 \qquad \mathbb{E}[W^{\otimes k}(g)^2] = k!\, \|g\|^2_{L^2((\mathbb{R}^d)^k)}$$

$$\mathbb{C}\mathrm{ov}[W^{\otimes k}(f), W^{\otimes k'}(g)] = 0 \quad \forall k \neq k'$$

Wiener chaos expansion

Any r.v. $X \in L^2(\Omega_W)$ measurable w.r.t. $\sigma(W)$ can be written as

$$X = \sum_{k=0}^{\infty} \frac{1}{k!} W^{\otimes k}(f_k) \qquad \text{with} \qquad f_k \in L^2_{\mathrm{sym}}((\mathbb{R}^d)^k)$$

Discrete sums and stochastic integrals

Consider a lattice $\mathbb{T}_\delta \subseteq \mathbb{R}^d$ whose cells have volume $v_\delta \to 0$

Take i.i.d. random variables $(X_z)_{z \in \mathbb{T}_\delta}$ with zero mean and unit variance

Consider the "stochastic Riemann sum" (multi-linear polynomial)

$$\Psi_\delta := \sum_{\substack{(z_1,\ldots,z_k)\in(\mathbb{T}_\delta)^k \\ z_i \neq z_j \ \forall i \neq j}} f(z_1,\ldots,z_k)\, X_{z_1} X_{z_2} \cdots X_{z_k}$$

where $f \in L^2(\mathbb{R}^d)$ is (say) continuous.

$$(\sqrt{v_\delta})^k \, \Psi_\delta \ \xrightarrow[\delta \to 0]{d} \ \int_{(\mathbb{R}^d)^k} g(z_1,\ldots,z_k)\, W(\mathrm{d}z_1) \cdots W(\mathrm{d}z_k)$$

(Check the variance!)

Outline

Continuum partition function for DPRE

1d rescaled RW $S_t^\delta := \sqrt{\delta} S_{t/\delta}$ lives on $\mathbb{T}_\delta = \big([0,1] \cap \delta\mathbb{N}_0\big) \times \sqrt{\delta}\mathbb{Z}$

$$Z_\delta^\omega = \mathbf{E}^{\mathrm{ref}}\Big[\exp\Big(\mathcal{H}^\omega\Big)\Big] = \mathbf{E}^{\mathrm{ref}}\left[\exp\left(\sum_{n=1}^{N}\big(\beta\omega_{(n,S_n)} - \lambda(\beta)\big)\right)\right]$$

$$= 1 + \sum_{(t,x)\in\mathbb{T}_\delta} \mathbf{P}^{\mathrm{ref}}(S_t^\delta = x)\, X_{t,x}$$

$$+ \frac{1}{2} \sum_{(t,x)\neq(t',x')\in\mathbb{T}_\delta} \mathbf{P}^{\mathrm{ref}}(S_t^\delta = x, S_{t'}^\delta = x')\, X_{t,x}\, X_{t',x'} + \ldots$$

Recall the LLT: $\mathbf{P}^{\mathrm{ref}}(S_n = x) \sim \frac{1}{\sqrt{n}}\, g\big(\frac{x}{\sqrt{n}}\big)$ with $g(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$

$$\mathbf{P}^{\mathrm{ref}}(S_t^\delta = x) = \mathbf{P}^{\mathrm{ref}}(S_{\frac{t}{\delta}} = \tfrac{x}{\sqrt{\delta}}) \sim \sqrt{\delta}\, g_t(x) \qquad g_t(x) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}$$

Replacing $X_{t,x} = e^{(\beta\omega_{(t,x)} - \lambda(\beta))} - 1 \approx \beta Y_{t,x}$ with $Y_{t,x}$ i.i.d. $\mathcal{N}(0,1)$

Continuum partition function for DPRE

$$Z_N^\omega = 1 + \beta\sqrt{\delta} \sum_{(t,x)\in\mathbb{T}_\delta} g_t(x)\, Y_{t,x} + \frac{1}{2}(\beta\sqrt{\delta})^2 \sum_{(t,x)\neq(t',x')\in\mathbb{T}_\delta} g_t(x)\, g_{t'-t}(x'-x)\, Y_{t,x}\, Y_{t',x'} + \ldots$$

Cells in $\mathbb{T}_\delta$ have volume $v_\delta = \delta\sqrt{\delta} = \delta^{\frac{3}{2}}$ $\rightsquigarrow$ "Stochastic Riemann sums" converge to stochastic integrals if $\beta\sqrt{\delta} \approx \sqrt{v_\delta}$

$$\boxed{\beta \sim \hat\beta\, \delta^{\frac{1}{4}} = \frac{\hat\beta}{N^{\frac{1}{4}}}}$$

$$Z_N^\omega \xrightarrow[\delta\to 0]{d} \mathcal{Z}^W = 1 + \hat\beta \int_{[0,1]\times\mathbb{R}} g_t(x)\, W(\mathrm{d}t\mathrm{d}x) + \frac{\hat\beta^2}{2} \int_{([0,1]\times\mathbb{R})^2} g_t(x)\, g_{t'-t}(x'-x)\, W(\mathrm{d}t\mathrm{d}x)\, W(\mathrm{d}t'\mathrm{d}x') + \ldots$$

Constrained partition functions

We have constructed $\mathcal{Z}^W$ = "free" partition function on $[0,1] \times \mathbb{R}$
RW paths starting at $(0,0)$ with no constraint on right endpoint

$$\mathcal{Z}^W = \mathcal{Z}^W((0,0),(1,\star)) \qquad \mathbb{E}\left[\mathcal{Z}^W\right] = 1$$

Consider now constrained partition functions: for $(s,y),(t,x) \in [0,1] \times \mathbb{R}$

$$\text{Discrete:} \qquad \boldsymbol{Z}_\delta^\omega((s,y),(t,x)) = \mathbf{E}^{\mathrm{ref}}\left[\exp\left(\mathcal{H}^\omega\right) \mathbb{1}_{\{S_t^\delta = x\}} \,\middle|\, S_s^\delta = y\right]$$

Divided by $\sqrt{\delta}$, they converge to a continuum limit:

$$\mathcal{Z}^W((s,y),(t,x)) \qquad \mathbb{E}\left[\mathcal{Z}^W((s,y),(t,x))\right] = g_{t-s}(x-y)$$

This is a function of white noise in the stripe $W([s,t] \times \mathbb{R})$

Four-parameter random process $\mathcal{Z}^W((s,y),(t,x)) \rightsquigarrow$ regularity?

Key properties

Key properties

For a.e. realization of W the following properties hold:

- Continuity: $\mathcal{Z}^W((s,y),(t,x))$ is jointly continuous in (s,y,t,x) (on the domain $s < t$)
- Positivity: $\mathcal{Z}^W((s,y),(t,x)) > 0$ for all (s,y,t,x) satisfying $s < t$
- Semigroup (Chapman-Kolmogorov): for all $s < r < t$ and $x, y \in \mathbb{R}$

$$\mathcal{Z}^W((s,y),(t,x)) = \int_{\mathbb{R}} \mathcal{Z}^W((s,y),(r,z))\, \mathcal{Z}^W((r,z),(t,x))\, \mathrm{d}z$$

(Inherited from discrete partition functions: drawing!)

How to prove these properties?

The 1d Stochastic Heat Equation

The four-parameter field $\mathcal{Z}^{W}((s,y),(t,x))$ solves the 1d SHE

$$\begin{cases} \partial_t \mathcal{Z}^{W} = \frac{1}{2}\Delta_x \mathcal{Z}^{W} + \hat{\beta}\, W\, \mathcal{Z}^{W} \\ \lim_{t\downarrow s} \mathcal{Z}^{W}((s,y),(t,x)) = \delta(y-x) \end{cases}$$

Checked directly from Wiener chaos expansion (mild solution)

It is known that solutions to the SHE satisfy the properties above

Alternative approach (to check, OK for pinning [C., Sun, Zygouras 2016])

- Prove continuity by Kolmogorov criterion, showing that

$$\frac{\mathcal{Z}^{W}((s,y),(t,x))}{g_{t-s}(x-y)} \quad \text{is continuous also for } t = s$$

- Use continuity to prove semigroup for all times
- Use continuity to deduce positivity for close times, then bootstrap to arbitrary times using semigroup

Outline

A naive approach

Consider DPRE in $d = 1$ (random walk + disorder)

$$\mathbf{P}^{\omega}(S) \propto e^{\sum_{n=1}^{N} \beta\, \omega(n,S_n)}\, \mathbf{P}^{\mathrm{ref}}(S)$$

Can we define its continuum analogue (BM + disorder)? Naively

$$\mathcal{P}^{W}(\mathrm{d}B) \propto e^{\int_0^1 \hat\beta\, W(t,B_t)\, \mathrm{d}t}\, \mathcal{P}^{\mathrm{ref}}(\mathrm{d}B)$$

$\mathcal{P}^{\mathrm{ref}}$ = law of BM $\quad W(t,x)$ = white noise on $\mathbb{R}^2$ (space-time)

- $\int_0^1 W(t,B_t)\, \mathrm{d}t$ ill-defined. Regularization?

NO! The problem is more subtle (and interesting!)

Partition functions and f.d.d.

Start from discrete: distribution of DPRE at two times $0 < t < t' < 1$

$$\mathbf{P}_\delta^\omega(S_t^\delta = x, S_{t'}^\delta = x') = \frac{\boldsymbol{Z}_\delta^\omega\big((0,0),(t,x)\big)\,\boldsymbol{Z}_\delta^\omega\big((t,x),(t',x')\big)\,\boldsymbol{Z}_\delta^\omega\big((t',x'),(1,\star)\big)}{\boldsymbol{Z}_\delta^\omega\big((0,0),(1,\star)\big)}$$

(drawing!) Analogous formula for any finite number of times

Idea: Replace $\boldsymbol{Z}_\delta^\omega \rightsquigarrow \mathcal{Z}^W$ to *define* the law of continuum DPRE

Recall: to define a process $(X_t)_{t\in[0,1]}$ it is enough (Kolmogorov) to assign finite-dimensional distributions (f.d.d.)

$$\mu_{t_1,\ldots,t_k}(A_1,\ldots,A_k) \text{ "} = \mathrm{P}(X_{t_1} \in A_1, \ldots, X_{t_k} \in A_k) \text{ "}$$

that are consistent

$$\mu_{t_1,\ldots,t_j,\ldots,t_k}(A_1,\ldots,\mathbb{R},\ldots,A_k) = \mu_{t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_k}(A_1,\ldots,A_{j-1},A_{j+1},\ldots,A_k)$$

The continuum 1d DPRE

- Fix $\hat{\beta} \in (0, \infty)$ (on which $\mathcal{Z}^W$ depend) [recall that $\beta \sim \hat{\beta}\delta^{\frac{1}{4}}$]
- Fix space-time white noise W on $[0,1] \times \mathbb{R}$ and a realization of continuum partition functions $\mathcal{Z}^W$ satisfying the key properties (continuity, strict positivity, semigroup)

The Continuum DPRE is the process $(X_t)_{t\in[0,1]}$ with f.d.d.

$$\frac{\mathcal{P}^W(X_t \in \mathrm{d}x, X_{t'} \in \mathrm{d}x')}{\mathrm{d}x\,\mathrm{d}x'} := \frac{\mathcal{Z}^W((0,0),(t,x))\,\mathcal{Z}^W((t,x),(t',x'))\,\mathcal{Z}^W((t',x'),(1,\star))}{\mathcal{Z}^W((0,0),(1,\star))}$$

- Well-defined by strict positivity of $\mathcal{Z}^W$
- Consistent by semigroup property

Relation with Wiener measure

The law of the continuum DPRE is a random probability

$$\mathcal{P}^{W}(X \in \cdot\,) \qquad\qquad \text{(quenched law)}$$

for the process $X = (X_t)_{t\in[0,1]}$ [Probab. kernel $\mathcal{S}'(\mathbb{R}) \to \mathbb{R}^{[0,1]}$]

Define a new law $\tilde{\mathbb{P}}$ (mutually absolutely continuous) for disorder W by

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(W) = \mathcal{Z}^{W}\big((0,0),(1,\star)\big)$$

Key Lemma

$$\mathcal{P}^{\mathrm{ann}}(X \in \cdot\,) := \int_{\mathcal{S}'(\mathbb{R})} \mathcal{P}^{W}(X \in \cdot\,)\, \tilde{\mathbb{P}}(\mathrm{d}W) = \mathrm{P}(BM \in \cdot\,)$$

Proof. The factor $\mathcal{Z}^{W}$ in $\tilde{\mathbb{P}}$ cancels the denominator in the f.d.d. for $\mathcal{P}^{W}$

Since $\mathbb{E}\big[\mathcal{Z}^{W}\big((s,y),(t,x)\big)\big] = g_{t-s}(x-y)$ one gets f.d.d. of BM $\square$

Absolute continuity properties

Theorem

$\forall A \subseteq \mathbb{R}^{[0,1]}: \quad \mathrm{P}(BM \in A) = 1 \quad \Rightarrow \quad \mathcal{P}^W(X \in A) = 1$ for $\mathbb{P}$-a.e. W

Any given a.s. property of BM is an a.s. property of continuum DPRE, for a.e. realization of the disorder W

Corollary

$$\mathcal{P}^W(X \text{ has Hölder paths with exp. } \tfrac{1}{2}-) = 1 \qquad \text{for } \mathbb{P}\text{-a.e. } W$$

We can thus realize $\mathcal{P}^W$ as a law on $C([0,1],\mathbb{R})$, for $\mathbb{P}$-a.e. W

(More precisely: $\mathcal{P}^W$ *admits a modification* with Hölder paths)

One is tempted to conclude that $\mathcal{P}^W$ is absolutely continuous w.r.t. Wiener measure, for $\mathbb{P}$-a.e. W ...

NO ! "$\forall A$" and "for $\mathbb{P}$-a.e. W" cannot be exchanged!

Singularity properties

Theorem

The law $\mathcal{P}^W$ is singular w.r.t. Wiener measure, for $\mathbb{P}$-a.e. W.

$$\text{for } \mathbb{P}\text{-a.e. } W \qquad \exists\, A = A_W \subseteq C([0,1],\mathbb{R}) :$$
$$\mathcal{P}^W(X \in A) = 1 \qquad \text{vs.} \qquad \mathrm{P}(BM \in A) = 0$$

Unlike discrete DPRE, there is no continuum Hamiltonian

$$\mathcal{P}^W(X \in \cdot\,) \not\propto e^{\mathcal{H}^W(\,\cdot\,)} \mathrm{P}(BM \in \cdot\,)$$

Absolute continuity is lost in the scaling limit

In a sense, the laws $\mathcal{P}^W$ are just *barely* not absolutely continuous w.r.t. Wiener measure ("stochastically absolutely continuous")

Proof of singularity

Let $(X_t)_{t\in[0,1]}$ be the canonical process on $C([0,1],\mathbb{R})$ $\quad [\; X_t(f) = f(t) \;]$

Let $\mathcal{F}_n := \sigma\big(X_{t_i^n} :\; t_i^n = \frac{i}{2^n},\; 0 \le i \le 2^n\big)$ be the dyadic filtration

Fix a typical realization of W. Setting $\mathcal{P}^{\mathrm{ref}} =$ Wiener measure

$$R_n^W(X) := \frac{\mathrm{d}\mathcal{P}^W|_{\mathcal{F}_n}}{\mathrm{d}\mathcal{P}^{\mathrm{ref}}|_{\mathcal{F}_n}}(X)$$

The process $(R_n^W)_{n\in\mathbb{N}}$ is a martingale w.r.t. $\mathcal{P}^{\mathrm{ref}}$ (exercise!)

Since $R_n^W \ge 0$, the martingale converges: $\quad R_n^W \xrightarrow[n\to\infty]{\text{a.s.}} R_\infty^W$

- $\mathcal{P}^W \ll \mathcal{P}^{\mathrm{ref}}$ if and only if $\mathcal{E}^{\mathrm{ref}}[R_\infty^W] = 1$ $\qquad$ (the martingale is UI)
- $\mathcal{P}^W$ is singular w.r.t. $\mathcal{P}^{\mathrm{ref}}$ if and only if $R_\infty^W = 0$

Proof of singularity

It suffices to show that $R_n^W(X) \xrightarrow[n\to\infty]{} 0$ in $\mathbb{P} \otimes \mathcal{P}^{\mathrm{ref}}$-probability

Fractional moment

For $\mathcal{P}^{\mathrm{ref}}$-a.e. X $\qquad \mathbb{E}\tilde{\mathbb{E}}\big[\big(R_n^W(X)\big)^\gamma\big] \xrightarrow[n\to\infty]{} 0 \qquad$ for some $\gamma \in (0,1)$

$$R_n^W(X) = \frac{1}{\mathcal{Z}^W((0,0),(1,\star))} \prod_{i=0}^{2^n-1} \frac{\mathcal{Z}^W\big((t_i^n, X_{t_i^n}),(t_{i+1}^n, X_{t_{i+1}^n})\big)}{g_{\frac{1}{2^n}}(X_{t_{i+1}^n} - X_{t_i^n})}$$

- Switch from $\mathbb{E}$ to equivalent law $\tilde{\mathbb{E}}$ to cancel the denominator
- For fixed X, the $\mathcal{Z}^W\big((t_i^n, X_{t_i^n}),(t_{i+1}^n, X_{t_{i+1}^n})\big)$'s are independent

We need to exploit translation and scale invariance of their laws

Proof of singularity

Lemma 1 (Translation and scale invariance)

If we set $\Delta_i^n := \dfrac{X_{t_{i+1}^n} - X_{t_i^n}}{\sqrt{t_{i+1}^n - t_i^n}}$ we have

$$\frac{\mathcal{Z}_{\hat\beta}^{W}\big((t_i^n, X_{t_i^n}), (t_{i+1}^n, X_{t_{i+1}^n})\big)}{g_{\frac{1}{2^n}}(X_{t_{i+1}^n} - X_{t_i^n})} \overset{d}{=} \frac{\mathcal{Z}_{\frac{\hat\beta}{2^{n/4}}}^{W}\big((0,0),(1,\Delta_i^n)\big)}{g_1(\Delta_i^n)}$$

Lemma 2 (Expansion)

For $z \in \mathbb{R}$ and $\varepsilon \in [0,1]$ (say)

$$\frac{\mathcal{Z}_{\varepsilon}^{W}\big((0,0),(1,z)\big)}{g_1(z)} = 1 + \varepsilon\, X_z + \varepsilon^2\, Y_{\varepsilon,z}$$

$\mathbb{E}[X_z] = 0 \quad \mathbb{E}[X_{\varepsilon,z}] = 0 \quad \mathbb{E}\big[X_z^2\big] \le C \quad \mathbb{E}\big[Y_{\varepsilon,z}^2\big] \le C$ unif. in ε, z

Proof of singularity

By Taylor expansion, for fixed $\gamma \in (0,1)$

$$\mathbb{E}\left[\left(\frac{\mathcal{Z}_\varepsilon^W((0,0),(1,z))}{g_1(z)}\right)^\gamma\right] = \mathbb{E}\left[\left(1+\varepsilon X_z + \varepsilon^2 Y_{\varepsilon,z}\right)^\gamma\right]$$

$$= 1 + \gamma\left\{\varepsilon\mathbb{E}[X_z] + \varepsilon^2\mathbb{E}[Y_{\varepsilon,z}]\right\} + \frac{\gamma(\gamma-1)}{2}\left\{\varepsilon^2\mathbb{E}[(X_x)^2] + \ldots\right\} + \ldots$$

$$= 1 - c\,\varepsilon^2 \le e^{-c\,\varepsilon^2}$$

($\star$) First order terms vanish ($\star$) $\gamma(\gamma-1) < 0$ ($\star$) For some $c > 0$

Estimate is uniform over $z \in \mathbb{R}$ $\rightsquigarrow$ We can set $z = \Delta_i^n$ and $\varepsilon = \frac{1}{2^{n/4}}$

$$\tilde{\mathbb{E}}\left[\left(R_n^W(X)\right)^\gamma\right] = \prod_{i=0}^{2^n-1} \mathbb{E}\left[\left(\frac{\mathcal{Z}_\varepsilon^W((0,0),(1,\Delta_i^n))}{g_1(\Delta_i^n)}\right)^\gamma\right] \le e^{-c\,\varepsilon^2\,2^n} = e^{-c\,2^{n/2}}$$

which vanishes as $n \to \infty$ □

Proof of Lemma 1

Introducing the dependence on $\hat\beta$

$$\mathcal{Z}^{W}_{\hat\beta}((s,y),(t,x)) \overset{d}{=} \mathcal{Z}^{W}_{\hat\beta}((0,0),(t-s,x-y))$$

$$\mathcal{Z}^{W}_{\hat\beta}((0,0),(t,x)) \overset{d}{=} \frac{1}{\sqrt{t}}\, \mathcal{Z}^{W}_{\hat\beta t^{\frac14}} \left((0,0),\left(1,\frac{x}{\sqrt{t}}\right)\right)$$

transl. invariance $+$ diffusive rescaling (prefactor, new $\hat\beta$) (drawing!)

$$\mathcal{Z}^{W}((0,0),(t,x)) = g_t(x) + \hat\beta \int_{[0,t]\times\mathbb{R}} g_s(z)\, g_{t-s}(x-z)\, W(\mathrm{d}s\mathrm{d}z) + \ldots$$

$$= \frac{1}{\sqrt{t}}\, g_1\big(\tfrac{x}{\sqrt{t}}\big) + \frac{1}{\sqrt{t}} \left(\frac{\hat\beta\, t^{\frac34}}{\sqrt{t}}\right) \int_{[0,t]\times\mathbb{R}} g_{\frac{s}{t}}\big(\tfrac{z}{\sqrt{t}}\big)\, g_{1-\frac{s}{t}}\big(\tfrac{x-z}{\sqrt{t}}\big)\, \frac{W(\mathrm{d}s\mathrm{d}z)}{t^{\frac34}} + \ldots$$

$= \text{OK}!$ $\square$

Convergence of discrete DPRE

- $\mathbf{P}_\delta^\omega$ = law of discrete DPRE (recall that $S_t^\delta := \sqrt{\delta} S_{t/\delta}$)
  "Rescaled RW S^δ moving in an i.i.d. environment ω"
- $\mathcal{P}^W$ = law of continuum DPRE
  "BM moving in a white noise environment W"

Both $\mathbf{P}_\delta^\omega$ and $\mathcal{P}^W$ are random probability laws on $E := C([0,1],\mathbb{R})$
i.e. RVs (defined on different probab. spaces) taking values in $\mathcal{M}_1(E)$

Does $\mathbf{P}_\delta^\omega$ converge in distribution toward $\mathcal{P}^W$ as $\delta \to 0$?

$$\forall\, \psi \in C_b(\mathcal{M}_1(E) \to \mathbb{R}) : \qquad \mathbb{E}\big[\psi(\mathbf{P}_\delta^\omega)\big] \xrightarrow[\delta\to 0]{} \mathbb{E}\big[\psi(\mathcal{P}^W)\big]$$

The answer is positive… almost surely ;-)

Statement for Pinning model proved in [C., Sun, Zygouras 2016]

Details need to be checked for DPRE (stronger assumptions on RW ?)

Universality

The convergence of $\mathbf{P}_\delta^\omega$ toward $\mathcal{P}^W$ is an instance of universality

There are many discrete DPRE:

- any RW S (zero mean, finite variance + technical assumptions)
- any (i.i.d.) disorder ω (finite exponential moments)

In the continuum ($\delta \to 0$) and weak disorder ($\beta \to 0$) regime, all these microscopic models $\mathbf{P}_\delta^\omega$ give rise to a unique macroscopic model $\mathcal{P}^W$

Tomorrow we will see how the continuum model $\mathcal{P}^W$ can tell quantitative information on discrete models $\mathbf{P}_\delta^\omega$ (free energy estimates)

Convergence

How to prove convergence in distribution $\mathbf{P}_\delta^\omega \xrightarrow[\delta\to 0]{d} \mathcal{P}^W$?

Prove a.s. convergence through a suitable coupling of (ω, W)

Assume we have convergence in distribution of discrete partition functions to continuum ones, in the space of continuum functions of $(s, y), (t, x)$

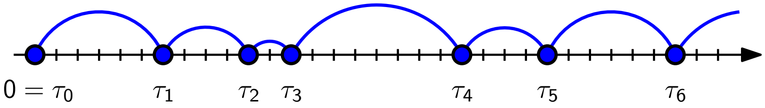
$$\boldsymbol{Z}_\delta^\omega\big((s,y),(t,x)\big) \xrightarrow[\delta\to 0]{d} \mathcal{Z}^W\big((s,y),(t,x)\big)$$

By Skorokhod representation theorem, there is a coupling of (ω, W) under which this convergence holds a.s.

Fix such a coupling: for a.e. (ω, W) the f.d.d. of $\mathbf{P}_\delta^\omega$ converge weakly to those of $\mathcal{P}^W$. It only remains to prove tightness of $\mathbf{P}_\delta^\omega(\cdot)$.

Outline

Ingredients: renewal process & disorder



Discrete renewal process $\tau = \{0 = \tau_0 < \tau_1 < \tau_2 < \ldots\} \subseteq \mathbb{N}_0$

Gaps $(\tau_{i+1} - \tau_i)_{i \ge 0}$ are i.i.d. with polynomial-tail distribution:

$$\mathbf{P}^{\mathrm{ref}}(\tau_1 = n) \sim \frac{c_K}{n^{1+\alpha}}\,, \qquad c_K > 0, \quad \alpha \in (0,1)$$

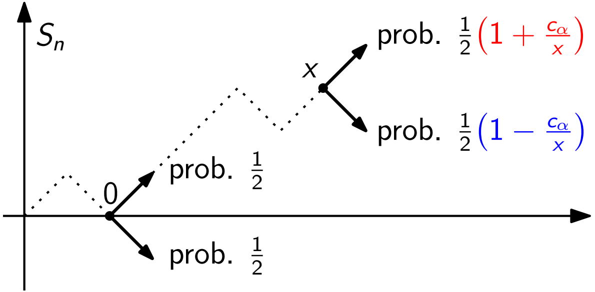
$\tau = \{n \in \mathbb{N}_0 : \; S_n = 0\}$ zero level set of a Markov chain $S = (S_n)_{n \ge 0}$

Disorder $\omega = (\omega_i)_{i \in \mathbb{N}}$: i.i.d. real random variables with law $\mathbb{P}$

$$\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_1}] < \infty \qquad \mathbb{E}[\omega_1] = 0 \qquad \mathbb{V}\mathrm{ar}[\omega_1] = 1$$

Bessel random walks

For $\alpha \in (0,1)$ the α-Bessel random walk is defined as follows:

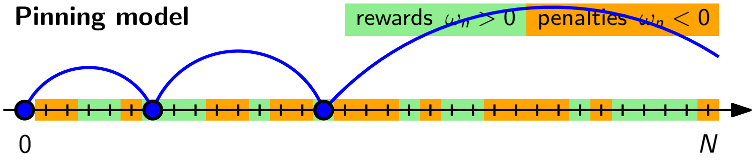


$$c_\alpha := \frac{1}{2} - \alpha$$

- $(\alpha = \frac{1}{2})$ no drift $(c_\alpha = 0)$ $\rightsquigarrow$ simple random walk
- $(\alpha < \frac{1}{2})$ drift away from the origin $(c_\alpha > 0)$
- $(\alpha > \frac{1}{2})$ drift toward the origin $(c_\alpha < 0)$

Disordered pinning model

Pinning model



$N \in \mathbb{N}$ (system size) $\beta \geq 0$, $h \in \mathbb{R}$ (disorder strength, bias)

The pinning model

Gibbs change of measure $\mathbf{P}_N^\omega = \mathbf{P}_{N,\beta,h}^\omega$ of the renewal distribution $\mathbf{P}^{\mathrm{ref}}$

$$\frac{\mathrm{d}\mathbf{P}_N^\omega}{\mathrm{d}\mathbf{P}^{\mathrm{ref}}}(\tau) := \frac{1}{Z_N^\omega} \exp\left(\sum_{n=1}^{N} (\beta\omega_n + h - \lambda(\beta)) \mathbb{1}_{\{n\in\tau\}} \; \mathbb{1}_{\{S_n=0\}} \right)$$

The phase transition

How are the typical paths τ of the pinning model $\mathbf{P}_N^\omega$?

Contact number $\quad \mathcal{C}_N := \big|\tau \cap (0,N]\big| = \sum_{n=1}^N \mathbb{1}_{\{n\in\tau\}} = \sum_{n=1}^N \mathbb{1}_{\{S_n=0\}}$

Theorem (phase transition)

$\exists$ *continuous, non decreasing, deterministic critical curve* $h_c(\beta)$:

- *Localized regime: for* $h > h_c(\beta)$ *one has* $\mathcal{C}_N \approx N$

$$\exists \mu = \mu_{\beta,h} > 0 : \quad \mathbf{P}_N^\omega\left(\left|\frac{\mathcal{C}_N}{N} - \mu\right| > \varepsilon\right) \xrightarrow[N\to\infty]{} 0 \qquad \omega\text{–a.s.}$$

- *Deocalized regime: for* $h < h_c(\beta)$ *one has* $\mathcal{C}_N = O(\log N)$

$$\exists A = A_{\beta,h} > 0 : \quad \mathbf{P}_N^\omega\left(\frac{\mathcal{C}_N}{\log N} > A\right) \xrightarrow[N\to\infty]{} 0 \qquad \omega\text{–a.s.}$$

Estimates on the critical curve

For $\beta = 0$ (homogeneous pinning, no disorder) one has $h_c(0) = 0$

What is the behavior of $h_c(\beta)$ for $\beta > 0$ small ?

Theorem ($\mathrm{P}(\tau_1 = n) \sim \frac{c_K}{n^{1+\alpha}}$)

- $(\alpha < \frac{1}{2})$ *disorder is irrelevant:* $h_c(\beta) = 0$ *for* $\beta > 0$ *small*
  [Alexander] [Toninelli] [Lacoin] [Cheliotis, den Hollander]
- $(\alpha \geq \frac{1}{2})$ *disorder is relevant:* $h_c(\beta) > 0$ *for all* $\beta > 0$
  - $(\alpha > 1)$ $h_c(\beta) \sim C\,\beta^2$ *with explicit* $C = \frac{\alpha}{1+\alpha}\frac{1}{2\mathrm{E}(\tau_1)}$
    [Berger, C., Poisat, Sun, Zygouras]
  - $(\frac{1}{2} < \alpha < 1)$ $C_1\,\beta^{\frac{2\alpha}{2\alpha-1}} \leq h_c(\beta) \leq C_2\,\beta^{\frac{2\alpha}{2\alpha-1}}$ $h_c(\beta) \sim \hat{C}\,\beta^{\frac{2\alpha}{2\alpha-1}}$
    using continuum part. funct.!
    [Derrida, Giacomin, Lacoin, Toninelli] [Alexander, Zygouras] [C., Torri, Toninelli]
  - $(\alpha = \frac{1}{2})$ $h_c(\beta) = e^{-\frac{c+o(1)}{\beta^2}}$ [Giacomin, Lacoin, Toninelli] [Berger, Lacoin]

Discrete free energy and critical curve

Partition function $\quad Z_N^{\omega} := \mathrm{E}\left[e^{H_N(\tau)}\right] = \mathrm{E}\left[e^{\sum_{n=1}^{N}(h+\beta\omega_n-\Lambda(\beta))\mathbb{1}_{\{n\in\tau\}}}\right]$

Consider first the regime of $N \to \infty$ with fixed β, h

- Free energy $\quad \mathrm{F}(\beta, h) := \lim_{N\to\infty} \frac{1}{N} \log Z_N^{\omega} \geq 0 \qquad \mathbb{P}(\mathrm{d}\omega)$-a.s.

$$Z_N^{\omega} \geq \mathrm{E}\left[e^{H_N(\tau)}\,\mathbb{1}_{\{\tau\cap(0,N]=\emptyset\}}\right] = \mathrm{P}(\tau \cap (0, N] = \emptyset) \sim \frac{(const.)}{N^{\alpha}}$$

- Critical curve $\quad h_c(\beta) = \sup\{h \in \mathbb{R} : \ \mathrm{F}(\beta, h) = 0\} \qquad$ non analiticity!

$$\text{(convexity)} \quad \frac{\partial \mathrm{F}(\beta, h)}{\partial h} = \lim_{N\to\infty} \mathbf{E}_N^{\omega}\left[\frac{\mathcal{C}_N}{N}\right] \begin{cases} > 0 & \text{if } h > h_c(\beta) \\ = 0 & \text{if } h < h_c(\beta) \end{cases}$$

$\mathrm{F}(\beta, h)$ and $h_c(\beta)$ depend on the law of τ and ω

Universality as $\beta, h \to 0$? YES, connected to continuum model

Continuum partition functions

Build continuum partition functions for Pinning Model with $\alpha \in (\frac{1}{2}, 1)$ (disorder relevant) following "usual" approach [C, Sun, Zygouras 2015+]

We need to rescale

$$\beta = \beta_N = \frac{\hat{\beta}}{N^{\alpha - 1/2}} \qquad h = h_N = \frac{\hat{h}}{N^{\alpha}}$$

One has $\boldsymbol{Z}_N^{\omega} \xrightarrow[N\to\infty]{d} \mathcal{Z}^{W}$ with

$$\mathcal{Z}^{W} := 1 + C \int_{0<t<1} \frac{\mathrm{d}W_t^{\hat{\beta},\hat{h}}}{t^{1-\alpha}} + C^2 \int_{0<t<t'<1} \frac{\mathrm{d}W_t^{\hat{\beta},\hat{h}} \, \mathrm{d}W_{t'}^{\hat{\beta},\hat{h}}}{t^{1-\alpha}(t'-t)^{1-\alpha}} + \ldots$$

where $W_t^{\hat{\beta},\hat{h}} := \hat{\beta} W_t + \hat{h}\, t$ and $C = \frac{\alpha \sin(\alpha\pi)}{\pi\, c_K}$

Continuum free energy

In analogy with the discrete model, define

Continuum free energy $\mathcal{F}(\hat\beta, \hat h) := \lim_{t\to\infty} \frac{1}{t} \log \mathcal{Z}^{W}_{\hat\beta,\hat h}(0,t)$ a.s.

(existence and self-averaging need some work)

Again $\mathcal{F}(\hat\beta, \hat h) \geq 0$ and define

Continuum critical curve $\mathcal{H}_c(\hat\beta) := \sup\{\hat h \in \mathbb{R} : \mathcal{F}(\hat\beta, \hat h) = 0\}$

Scaling relations

$\forall c > 0 :$ $\mathcal{Z}^{W}_{\hat\beta,\hat h}(c\, t) \overset{d}{=} \mathcal{Z}^{W}_{c^{\alpha-\frac{1}{2}}\hat\beta,\, c^{\alpha}\hat h}(t)$ (Wiener chaos exp.)

$\mathcal{F}\big(c^{\alpha-\frac{1}{2}}\hat\beta, c^{\alpha}\hat h\big) = c\,\mathcal{F}(\hat\beta, \hat h)$ $\boxed{\mathcal{H}_c(\hat\beta) = \mathcal{H}_c(1)\,\hat\beta^{\frac{2\alpha}{2\alpha-1}}}$

Interchanging the limits

Can we relate continuum free energy to the discrete one?

By construction of continuum partition functions

$$\mathcal{Z}_{\hat{\beta},\hat{h}}^{W}(t) \stackrel{d}{=} \lim_{N\to\infty} \boldsymbol{Z}_{\beta_N,h_N}^{\omega}(Nt)$$

Assuming uniform integrability of $\log \boldsymbol{Z}^{\omega}$ (OK)

$$\mathcal{F}(\hat{\beta},\hat{h}) = \lim_{t\to\infty} \frac{1}{t}\, \mathbb{E}\big[\log \mathcal{Z}_{\hat{\beta},\hat{h}}^{W}(t)\big] = \lim_{t\to\infty} \frac{1}{t} \lim_{N\to\infty} \mathbb{E}\big[\log \boldsymbol{Z}_{\beta_N,h_N}^{\omega}(Nt)\big]$$

Assuming we can interchange the limits $N \to \infty$ and $t \to \infty$

$$\mathcal{F}(\hat{\beta},\hat{h}) = \lim_{N\to\infty} N \lim_{t\to\infty} \frac{1}{Nt\ t}\, \mathbb{E}\big[\log \boldsymbol{Z}_{\beta_N,h_N}^{\omega}(Nt)\big] = \lim_{N\to\infty} N\, \mathrm{F}(\beta_N, h_N)$$

Setting $\delta = \frac{1}{N}$ for clarity, we arrive at...

Interchanging the limits

Conjecture

$$\mathcal{F}(\hat\beta, \hat h) \;=\; \lim_{\delta \to 0} \frac{\mathrm{F}\big(\hat\beta\delta^{\alpha-\frac{1}{2}}, \hat h \delta^{\alpha}\big)}{\delta}$$

Theorem [C., Toninelli, Torri 2015]

For all $\hat\beta > 0$, $\hat h \in \mathbb{R}$ and $\eta > 0$ there is $\delta_0 > 0$ such that $\forall \delta < \delta_0$

$$\mathcal{F}(\hat\beta, \hat h - \eta) \;\le\; \frac{\mathrm{F}\big(\hat\beta\delta^{\alpha-\frac{1}{2}}, \hat h \delta^{\alpha}\big)}{\delta} \;\le\; \mathcal{F}(\hat\beta, \hat h + \eta)$$

This implies Conj. and $\boxed{h_c(\beta) \sim \mathcal{H}_c(\beta) \sim \mathcal{H}_c(1)\,\beta^{\frac{2\alpha}{2\alpha-1}}}$

For any discrete Pinning model with $\alpha \in (\frac{1}{2}, 1)$, the free energy $\boldsymbol{F}(\beta, h)$ and the critical curve $h_c(\beta)$ have a universal shape in the regime $\beta, h \to 0$

Interchanging the limits

Very delicate result. How to prove it?

- Assume that there is a continuum Hamiltonian:

$$\boldsymbol{Z}^{\omega} = \mathbf{E}\Big[e^{\boldsymbol{H}^{\omega}_{Nt}}\Big] \qquad \mathcal{Z}^{W} = \mathcal{E}\Big[e^{\mathcal{H}^{W}_{t}}\Big]$$

- Couple $\boldsymbol{H}^{\omega}_{Nt}$ and $\mathcal{H}^{W}_{t}$ on the same probability space in such a way that the difference $\Delta_{N,t} := \boldsymbol{H}^{\omega}_{Nt} - \mathcal{H}^{W}_{t}$ is "small"
- Deduce that

$$\mathbb{E}\Big[\log \boldsymbol{Z}^{\omega}\Big] \le \mathbb{E}\Big[\log \mathcal{Z}^{W}\Big] + \log \mathbb{E}\mathbf{E}\Big[e^{\Delta_{N,t}}\Big]$$

  and show that the last term is "negligible"

Problem: there is no continuum Hamiltonian!

Solution: perform coarse-graining and define an "effective" Hamiltonian

The DPRE case

What about the DPRE?

We can still define discrete $\mathsf{F}(\beta)$ and continuum $\mathcal{F}(\hat{\beta})$ free energy

Since $\mathcal{F}(\hat{\beta}) \sim \mathcal{F}(1)\,\beta^4$ we can hope that

$$\mathsf{F}(\beta) \sim \mathcal{F}(1)\,\beta^4 \qquad \text{as } \beta \to 0$$

provided the "interchanging of limits" is justified

N. Torri is currently working on this problem. A finer coarse-graining is needed, together with sharper estimates on continuum partition functions

References

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