Polynomial Chaos and Scaling Limits of Disordered Systems

3. Marginally relevant models

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Overview

In the previous lectures we focused on systems that are disorder relevant (in particular DPRE with $d = 1$ and Pinning model with $\alpha > \frac{1}{2}$)

- We constructed continuum partition functions $\mathcal{Z}^W$
- We used $\mathcal{Z}^W$ to build continuum disordered models $\mathcal{P}^W$
- We used $\mathcal{Z}^W$ to get estimates on the free energy $\mathbf{F}(\beta, h)$

In this last lecture we consider the subtle marginally relevant regime (in particular DPRE with $d = 2$, Pinning model with $\alpha = \frac{1}{2}$, 2d SHE)

We present some results on the the continuum partition function

The marginal case

We consider simultaneously different models that are marginally relevant:

- Pinning Models with $\alpha = \frac{1}{2}$
- DPRE with $d = 2$ (RW attracted to BM)
- Stochastic Heat Equation in $d = 2$
- DPRE with $d = 1$ (RW with Cauchy tails: $\mathrm{P}(|S_1| > n) \sim \frac{c}{n}$

All these different models share a crucial feature: logarithmic overlap

$$\mathbf{R}_N = \begin{cases} \displaystyle\sum_{1 \le n \le N} \mathbf{P}^{\mathrm{ref}}(n \in \tau)^2 \\ \displaystyle\sum_{1 \le n \le N} \sum_{x \in \mathbb{Z}^d} \mathbf{P}^{\mathrm{ref}}(S_n = x)^2 \end{cases} \sim C \log N$$

For simplicity, we will perform our computations on the pinning model

The $2d$ Stochastic Heat Equation

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x) + \beta\, W(t,x)\, u(t,x) \\ u(0,x) \equiv 1 \end{cases} \qquad (t,x) \in [0,\infty) \times \mathbb{R}^2$$

where $W(t,x)$ is (space-time) white noise on $[0,\infty) \times \mathbb{R}^2$

Mollification in space: fix $j \in C_0^\infty(\mathbb{R}^d)$ with $\|j\|_{L^2} = 1$

$$W_\delta(t,x) := \int_{\mathbb{R}^2} \delta\, j\left(\frac{x-y}{\sqrt{\delta}}\right) W(t,y)\, \mathrm{d}y$$

Then $$u_\delta(t,x) \overset{d}{=} \mathrm{E}_{\frac{x}{\sqrt{\delta}}} \left[\exp\left\{ \int_0^{\frac{t}{\delta}} \left(\beta\, W_1(s, B_s) \;-\; \tfrac{1}{2}\beta^2\right) \mathrm{d}s \right\} \right]$$

By soft arguments $u_\delta(1,x) \overset{d}{\approx} \boldsymbol{Z}_N^\omega$ (partition function of 2d DPRE)

Pinning in the relevant regime $\alpha > \frac{1}{2}$

Recall what we did for $\alpha > \frac{1}{2}$ (for simplicity $h = 0$)

$$
\begin{aligned}
Z_N^{\omega} &= \mathbf{E}^{\mathrm{ref}}\left[e^{H_N^{\omega}}\right] = \mathbf{E}^{\mathrm{ref}}\left[e^{\sum_{n=1}^{N}(\beta\omega_n - \lambda(\beta))\mathbb{1}_{\{n\in\tau\}}}\right] \\
&= \mathbf{E}^{\mathrm{ref}}\left[\prod_{n=1}^{N} e^{(\beta\omega_n - \lambda(\beta))\mathbb{1}_{\{n\in\tau\}}}\right] = \mathbf{E}^{\mathrm{ref}}\left[\prod_{n=1}^{N}\left(1 + X_n\,\mathbb{1}_{\{n\in\tau\}}\right)\right] \\
&= 1 + \sum_{n=1}^{N} \mathbf{P}^{\mathrm{ref}}(n\in\tau)\,X_n + \sum_{0<n<m\leq N} \mathbf{P}^{\mathrm{ref}}(n\in\tau,\, m\in\tau)\,X_n\,X_m + \ldots
\end{aligned}
$$

- $X_n = e^{\beta\omega_n - \lambda(\beta)} - 1 \approx \beta\, Y_n$ with $Y_n \sim \mathcal{N}(0,1)$
- $\mathbf{P}^{\mathrm{ref}}(n\in\tau) \sim \dfrac{c}{n^{1-\alpha}}$

Pinning in the relevant regime $\alpha > \frac{1}{2}$

$$\boldsymbol{Z}_N^{\omega} = 1 + \beta \sum_{0<n\le N} \frac{Y_n}{n^{1-\alpha}} + \beta^2 \sum_{0<n<m\le N} \frac{Y_n\, Y_m}{n^{1-\alpha}\,(m-n)^{1-\alpha}} + \ldots$$

$$= 1 + \frac{\beta}{N^{1-\alpha}} \sum_{t\in(0,1]\cap\frac{\mathbb{Z}}{N}} \frac{Y_t}{t^{1-\alpha}} + \left(\frac{\beta}{N^{1-\alpha}}\right)^2 \sum_{s<t\in(0,1]\cap\frac{\mathbb{Z}}{N}} \frac{Y_s\, Y_t}{s^{1-\alpha}\,(t-s)^{1-\alpha}} +$$

Lattice $\frac{\mathbb{Z}}{N}$ has cells with volume $\frac{1}{N}$, hence if $\frac{\beta}{N^{1-\alpha}} \approx \sqrt{\frac{1}{N}}$ that is

$$\beta = \frac{\hat\beta}{N^{\alpha-\frac{1}{2}}}$$

We obtain

$$\boldsymbol{Z}_N^{\omega} \xrightarrow[N\to\infty]{d} 1 + \hat\beta \int_0^1 \frac{\mathrm{d}W_t}{t^{1-\alpha}} + \hat\beta^2 \int_{0<s<t<1} \frac{\mathrm{d}W_s\,\mathrm{d}W_t}{s^{1-\alpha}\,(t-s)^{1-\alpha}} + \ldots$$

What happens for $\alpha = \frac{1}{2}$? Stochastic integrals ill-defined: $\frac{1}{\sqrt{t}} \not\in L^2_{\mathrm{loc}}$ ...

The marginal regime $\alpha = \frac{1}{2}$

$$\boldsymbol{Z}_N^{\omega} \;=\; 1 \;+\; \beta \sum_{0<n\le N} \frac{Y_n}{\sqrt{n}} \;+\; \beta^2 \sum_{0<n<m\le N} \frac{Y_n\, Y_m}{\sqrt{n}\,\sqrt{m-n}} \;+\; \ldots$$

Goal: find the joint limit in distribution of all these sums

Linear term is easy ($Y_n \sim \mathcal{N}(0,1)$ by Lindeberg): asympt. $\mathcal{N}(0,\sigma^2)$

$$\sigma^2 \;=\; \beta^2 \sum_{0<n\le N} \frac{1}{n} \sim \beta^2 \log N$$

We then rescale $\boxed{\beta = \beta_N \sim \frac{\hat\beta}{\sqrt{\log N}}}$ Other terms converge?

Interestingly, every sum gives contribution 1 to the variance!

$$\mathbb{V}\mathrm{ar}\big[\boldsymbol{Z}_N^{\omega}\big] = 1 + \hat\beta^2 + \hat\beta^4 + \ldots = \frac{1}{1-\hat\beta^2} \qquad \text{blows up at} \qquad \hat\beta = 1!$$

Scaling limit of marginal partition function

Theorem 1. [C., Sun, Zygouras '15b]

Consider DPRE $d = 2$ or Pinning $\alpha = \frac{1}{2}$ or 2d SHE
(or long-range DPRE with $d = 1$ and Cauchy tails)

Rescaling $\beta := \dfrac{\hat{\beta}}{\sqrt{\log N}}$ (and $h \equiv 0$) the partition function converges in law to an explicit limit: $\boldsymbol{Z}_N^\omega \xrightarrow[N\to\infty]{d} \mathcal{Z}^W = \begin{cases} \text{log-normal} & \text{if } \hat{\beta} < 1 \\ 0 & \text{if } \hat{\beta} \geq 1 \end{cases}$

$$\mathcal{Z}^W \stackrel{d}{=} \exp\left\{\sigma_{\hat{\beta}} W_1 - \frac{1}{2}\sigma_{\hat{\beta}}^2\right\} \qquad \text{with} \quad \sigma_{\hat{\beta}} = \log \frac{1}{1-\hat{\beta}^2}$$

Multi-scale correlations for $\hat{\beta} < 1$

Define $Z_N^{\omega}(t,x)$ as partition function for rescaled RW starting at (t,x)

$$Z_N^{\omega}(t,x) = \mathbf{E}^{\mathrm{ref}}\left[e^{H^{\omega}(S)} \,\middle|\, S_t^{\delta} = x\right]$$

where $\{S_t^{\delta} = x\} = \{S_{Nt} = \sqrt{N}x\}$ $\qquad [\delta = \frac{1}{N}]$

Multi-scale correlations for $\hat\beta < 1$

Theorem 2. [C., Sun, Zygouras '15b]

Consider DPRE with $d = 2$ or 2d SHE (fix $\hat\beta < 1$)

$Z_N^\omega(\mathrm{X})$ and $Z_N^\omega(\mathrm{X}')$ are asymptotically independent for fixed $\mathrm{X} \neq \mathrm{X}'$

More generally, if $\mathrm{X} = (t_N, x_N)$ and $\mathrm{X}' = (t_N', x_N')$ are such that

$$d(X, X') := |t_N - t_N'| + |x_N - x_N'|^2 \sim \frac{1}{N^{1-\zeta}} \qquad \zeta \in [0,1]$$

then
$$\left(Z_N^\omega(\mathrm{X}),\, Z_N^\omega(\mathrm{X}')\right) \xrightarrow[N\to\infty]{d} \left(e^{Y - \frac{1}{2}\mathbb{V}\mathrm{ar}[Y]},\, e^{Y' - \frac{1}{2}\mathbb{V}\mathrm{ar}[Y']}\right)$$

Y, Y' joint $\mathcal{N}(0, \sigma^2_{\hat\beta})$ with
$$\mathbb{C}\mathrm{ov}\left[Y, Y'\right] = \log \frac{1 - \zeta\hat\beta^2}{1 - \hat\beta^2}$$

Multi-scale correlations for $\hat{\beta} < 1$

We can integrate $\boldsymbol{Z}_N^{\omega}$ against a test function $\phi \in C_0([0,1] \times \mathbb{R}^2)$

$$\begin{aligned}\langle \boldsymbol{Z}_N^{\omega}, \phi \rangle &:= \int_{[0,1]\times\mathbb{R}^2} \phi(t,x)\ \boldsymbol{Z}_N^{\omega}(t,x)\, \mathrm{d}t\, \mathrm{d}x \\ &\simeq \frac{1}{N^2} \sum_{t \in [0,1]\cap\frac{\mathbb{Z}}{N}\,,\ x \in (\frac{\mathbb{Z}}{\sqrt{N}})^2} \phi(t,x)\ \boldsymbol{Z}_N^{\omega}(t,x)\end{aligned}$$

Corollary

$$\langle \boldsymbol{Z}_N^{\omega}, \phi \rangle \to \langle 1, \phi \rangle \quad \text{in probability} \quad \text{as } N \to \infty$$

Fluctuations for $\hat{\beta} < 1$

Theorem 3. [C., Sun, Zygouras '15b]

Consider DPRE with $d = 2$ or 2d SHE (fix $\hat{\beta} < 1$)

$$\boldsymbol{Z}_N^{\omega}(t,x) \approx 1 + \frac{1}{\sqrt{\log N}}\, G(t,x) \qquad \text{(in } \mathcal{S}'\text{)}$$

where $G(t,x)$ is a generalized Gaussian field on $[0,1] \times \mathbb{R}^2$ with

$$\mathbb{C}\mathrm{ov}\left[G(\mathrm{X}), G(\mathrm{X}')\right] \sim C \log \frac{1}{\|\mathrm{X} - \mathrm{X}'\|}$$

The regime $\hat{\beta} = 1$ (in progress)

For $\hat{\beta} = 1$: $\qquad Z_N^\omega(t,x) \to 0$ in law $\qquad \mathbb{V}\mathrm{ar}[Z_N^\omega(t,x)] \to \infty$

However, covariances are finite: cf. [Bertini, Cancrini 95]

$$\mathbb{C}\mathrm{ov}[Z_N^\omega(t,x)\,,\, Z_N^\omega(t',x')] \underset{N\to\infty}{\sim} K\big((t,x)\,,\,(t',x')\big) < \infty$$

where

$$K\big((t,x)\,,\,(t',x')\big) \approx \frac{1}{\log|(t,x)-(t',x')|}$$

Then

$$\mathbb{V}\mathrm{ar}\big[\langle Z_N^\omega\,,\,\phi\rangle\big] \to (\phi, K\phi) < \infty$$

Conjecture

For $\hat{\beta} = 1$ the partition function $Z_N^\omega(t,x)$ has a non-trivial limit in law, viewed as a random Schwartz distribution in (t,x)

Proof of Theorem 1. for pinning

$$\boldsymbol{Z}_N^\omega = \sum_{k=0}^{N} \beta^k \sum_{0<n_1<\ldots<n_k\le N} \frac{Y_{n_1} Y_{n_2} \cdots Y_{n_k}}{\sqrt{n_1}\sqrt{n_2-n_1}\cdots\sqrt{n_k-n_{k-1}}}$$

$$= 1 + \frac{\hat\beta}{\sqrt{\log N}} \sum_{0<n\le N} \frac{Y_n}{\sqrt{n}} + \left(\frac{\hat\beta}{\sqrt{\log N}}\right)^2 \sum_{0<n<n'\le N} \frac{Y_n Y_{n'}}{\sqrt{n}\sqrt{n'-n}} + \ldots$$

Goal: find the joint limit in distribution of all these sums

$\rightsquigarrow$ blackboard!

Fourth moment theorem

4th Moment Theorem [de Jong 90] [Nualart, Peccati, Reinert 10]

Consider homogeneous (deg. ℓ) polynomial chaos $Y_N = \sum_{|I|=\ell} \psi_N(I) \prod_{i \in I} Y_i$

- $\max_i \psi_N(i) \xrightarrow[N\to\infty]{} 0$ (in case $\ell = 1$) [Small influences!]
- $\mathbb{E}[(Y_N)^2] \xrightarrow[N\to\infty]{} \sigma^2$
- $\mathbb{E}[(Y_N)^4] \xrightarrow[N\to\infty]{} 3\,\sigma^4$

Then $$Y_N \xrightarrow[N\to\infty]{d} \mathcal{N}(0, \sigma^2)$$

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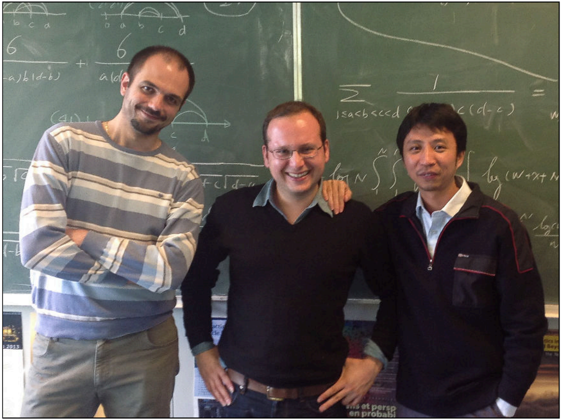
Collaborators



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