On the phase diagram of random copolymers at selective interfaces

Francesco Caravenna

caravenna@math.unipd.it

Università degli Studi di Padova

Eindhoven, June 22, 2007

References

- [CGG] C., Giacomin and Gubinelli, *A numerical approach to copolymers at selective interfaces*, J. Stat. Phys. (2006)
- [CG] C. and Giacomin, *On constrained annealed bounds for linear chain pinning models*, Electron. Comm. in Probab. (2005),

Outline of the talk

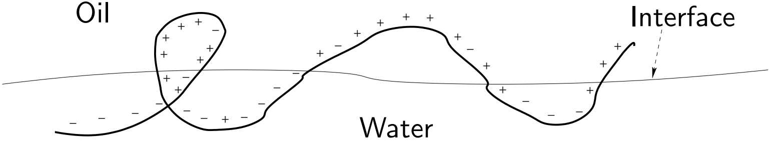
1. Introduction
   - Motivations
   - Definition of the model
   - The phase diagram: UB and LB on the critical line
2. Numerical investigation
   - The transfer matrix approach
   - Beating the LB: a statistical test
   - Beating the UB: numerical observations
   - A conjecture (?) on the critical line
3. Theoretical analysis
   - Improving the UB: the constrained annealing technique

Outline of the talk

Qualitative introduction

Copolymer (= inhomogeneous polymer) near a selective interface

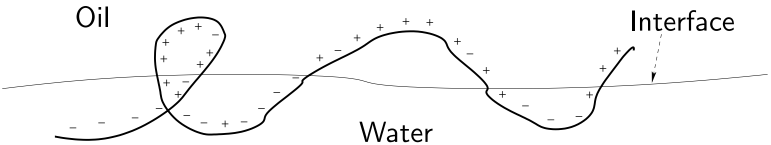
Monomers: (+) → hydrophobic (−) → hydrophilic



Qualitative introduction

Copolymer (= inhomogeneous polymer) near a selective interface

Monomers: (+) → hydrophobic (−) → hydrophilic



Phenomenon:

localization at the interface vs. delocalization in one solvent

Energy–entropy competition

Definition of the model

Free process: Simple Symmetric Random Walk $\{S_n\}_n$ on $\mathbb{Z}$

$$S_0 = 0 \qquad\qquad S_n = \sum_{i=1}^n X_i$$

where $\{X_i\}_i$ are i.i.d. with $\mathbf{P}(X_1 = \pm 1) = 1/2$.

Definition of the model

Free process: Simple Symmetric Random Walk $\{S_n\}_n$ on $\mathbb{Z}$

$$S_0 = 0 \qquad\qquad S_n = \sum_{i=1}^n X_i$$

where $\{X_i\}_i$ are i.i.d. with $\mathbf{P}(X_1 = \pm 1) = 1/2$.

Parameters:

- N (system size) $\lambda, h \geq 0$ (inverse temperature, asymmetry)
- $\omega = \{\omega_n\}_n \in \{-1, +1\}^{\mathbb{N}}$ (charges: hydrophobicity-hydrophily)

Definition of the model

Free process: Simple Symmetric Random Walk $\{S_n\}_n$ on $\mathbb{Z}$

$$S_0 = 0 \qquad\qquad S_n = \sum_{i=1}^n X_i$$

where $\{X_i\}_i$ are i.i.d. with $\mathbf{P}(X_1 = \pm 1) = 1/2$.

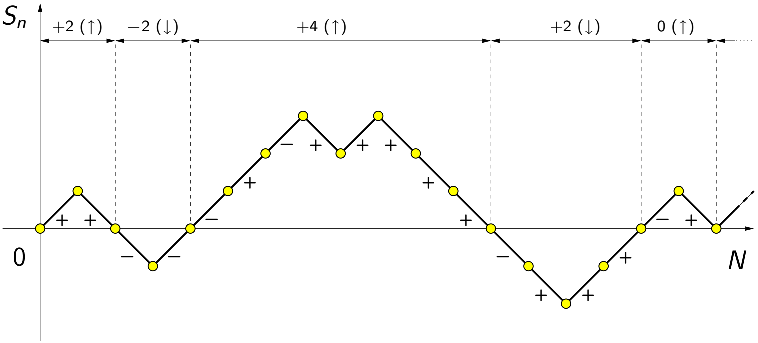
Parameters:

- N (system size) $\quad \lambda, h \geq 0$ (inverse temperature, asymmetry)
- $\omega = \{\omega_n\}_n \in \{-1, +1\}^{\mathbb{N}}$ (charges: hydrophobicity-hydrophily)

Polymer measure $\mathbf{P}_{N,\omega}^{\lambda,h}$ [Bolthausen and den Hollander 97]

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}^{\lambda,h}}{\mathrm{d}\mathbf{P}}(S) := \frac{1}{Z_{N,\omega}^{\lambda,h}} \cdot \exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h) \operatorname{sign}(S_n) \right)$$

A sample path



Energy: $$\mathcal{H}_{N,\omega}^{\lambda,h}(S) := \lambda \sum_{n=1}^{N} (\omega_n + h)\, \mathrm{sign}(S_n) \; = \; \lambda\big(6 + 6h\big)$$

$$\Big(\text{if } S_n = 0 \;\rightarrow\; \mathrm{sign}(S_n) := \mathrm{sign}(S_{n-1}) \Big)$$

The choice of the charges

Quenched randomness: $\omega = \{\omega_n\}_n$ is a typical sample from a centered i.i.d. sequence (law $\mathbb{P}$):

$$\mathbb{E}[\omega_1] = 0 \qquad \mathbb{E}[{\omega_1}^2] = 1$$
$$\mathrm{M}(\alpha) := \mathbb{E}[\exp(\alpha\omega_1)] < \infty \quad \forall\alpha \in \mathbb{R}$$

(generalization: $\omega_n \in \{-1, +1\} \ \rightarrow \ \omega_n \in \mathbb{R}$)

The choice of the charges

Quenched randomness: $\omega = \{\omega_n\}_n$ is a typical sample from a centered i.i.d. sequence (law $\mathbb{P}$):

$$\mathbb{E}[\omega_1] = 0 \qquad \mathbb{E}[{\omega_1}^2] = 1$$
$$\mathrm{M}(\alpha) := \mathbb{E}[\exp(\alpha\omega_1)] < \infty \quad \forall \alpha \in \mathbb{R}$$

(generalization: $\omega_n \in \{-1, +1\} \;\rightarrow\; \omega_n \in \mathbb{R}$)

Typical examples:

- Binary: $\mathbb{P}(\omega_1 = \pm 1) = \frac{1}{2} \quad \rightarrow \quad \log \mathrm{M}(\alpha) = \log \cosh(\alpha)$
- Gaussian: $\omega_1 \sim N(0,1) \quad \rightarrow \quad \log \mathrm{M}(\alpha) = \frac{1}{2}\alpha^2$

The free energy

Interested in the asymptotic properties of the model as $N \to \infty$ (localization-delocalization)

The free energy

Interested in the asymptotic properties of the model as $N \to \infty$ (localization-delocalization)

Partition function: $Z_{N,\omega}^{\lambda,h} := \mathbf{E}\big(\exp(\mathcal{H}_{N,\omega}^{\lambda h})\big)$

Free energy: rate of exponential growth of Z_N:

$$f_\omega(\lambda, h) \;:=\; \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\omega}^{\lambda,h}$$

The free energy

Interested in the asymptotic properties of the model as $N \to \infty$ (localization-delocalization)

Partition function: $Z_{N,\omega}^{\lambda,h} := \mathbf{E}\big(\exp(\mathcal{H}_{N,\omega}^{\lambda h})\big)$

Free energy: rate of exponential growth of Z_N:

$$f_\omega(\lambda, h) \;:=\; \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\omega}^{\lambda,h}$$

- The limit exists $\mathbb{P}$–a.s. (and in $L_1(\mathrm{d}\mathbb{P})$) by superadditivity
- Self-averaging property: $f_\omega(\lambda, h) = f(\lambda, h)$ for $\mathbb{P}$–a.e. ω

Localization vs. Delocalization

How to read the (de)localized character of the model from $f(\lambda, h)$?

Localization vs. Delocalization

How to read the (de)localized character of the model from $f(\lambda, h)$?

Basic observation: $\boxed{f(\lambda, h) \geq \lambda h}$

Localization vs. Delocalization

How to read the (de)localized character of the model from $f(\lambda, h)$?

Basic observation: $f(\lambda, h) \geq \lambda h$

Proof: restrict Z_N on positive trajectories

$$Z_{N,\omega}^{\lambda,h} \geq \mathbf{E}\left[\exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h)\, \mathrm{sign}(S_n) \right) ; \, S_1 > 0, \ldots, S_N > 0 \right]$$

Localization vs. Delocalization

How to read the (de)localized character of the model from $f(\lambda, h)$?

Basic observation: $f(\lambda, h) \geq \lambda h$

Proof: restrict Z_N on positive trajectories

$$Z_{N,\omega}^{\lambda,h} \geq \mathbf{E}\left[\exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h)\, \mathrm{sign}(S_n) \right) ;\, S_1 > 0, \ldots, S_N > 0 \right]$$

$$= \exp\left(N \lambda h + \lambda \sum_{n=1}^{N} \omega_n \right) \cdot \mathbf{P}\left[S_1 > 0 \,, \, \ldots \,, \, S_N > 0 \right]$$

Localization vs. Delocalization

How to read the (de)localized character of the model from $f(\lambda, h)$?

Basic observation: $f(\lambda, h) \geq \lambda h$

Proof: restrict Z_N on positive trajectories

$$Z_{N,\omega}^{\lambda,h} \geq \mathbf{E}\left[\exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h)\, \mathrm{sign}(S_n) \right) ;\, S_1 > 0, \ldots, S_N > 0 \right]$$

$$= \exp\left(N\lambda h + \lambda \sum_{n=1}^{N} \omega_n \right) \cdot \mathbf{P}\left[S_1 > 0, \ldots, S_N > 0 \right]$$

$$= \exp\Big(N\lambda h + o(N) \Big) \cdot \frac{\text{const.} + o(1)}{\sqrt{N}}$$

Localization vs. Delocalization

How to read the (de)localized character of the model from $f(\lambda, h)$?

Basic observation: $$\boxed{f(\lambda, h) \geq \lambda h}$$

Therefore we split the phase diagram $\{(\lambda, h) : \lambda, h \geq 0\}$ into

- Localized region $\mathcal{L} = \{(\lambda, h) : f(\lambda, h) > \lambda h\}$
- Delocalized region $\mathcal{D} = \{(\lambda, h) : f(\lambda, h) = \lambda h\}$

Localization vs. Delocalization

How to read the (de)localized character of the model from $f(\lambda, h)$?

Basic observation: $$f(\lambda, h) \geq \lambda h$$

Therefore we split the phase diagram $\{(\lambda, h) : \lambda, h \geq 0\}$ into

- Localized region $\mathcal{L} = \{(\lambda, h) : f(\lambda, h) > \lambda\, h\}$
- Delocalized region $\mathcal{D} = \{(\lambda, h) : f(\lambda, h) = \lambda\, h\}$

Two main questions:

1. Study of the phase diagram

Localization vs. Delocalization

How to read the (de)localized character of the model from $f(\lambda, h)$?

Basic observation: $f(\lambda, h) \geq \lambda h$

Therefore we split the phase diagram $\{(\lambda, h) : \lambda, h \geq 0\}$ into

- Localized region $\mathcal{L} = \{(\lambda, h) : f(\lambda, h) > \lambda h\}$
- Delocalized region $\mathcal{D} = \{(\lambda, h) : f(\lambda, h) = \lambda h\}$

Two main questions:

1. Study of the phase diagram
2. Free energy definition of $\mathcal{L}$ and $\mathcal{D}$ $\rightarrow$ path properties?
   - $\mathcal{L}$: strong path localization [Sinai 93] [Biskup and den Hollander 99]
   - $\mathcal{D}$: many open questions [Giacomin and Toninelli 05]

The critical line

Theorem ([BdH 97])

There exists a continuous, increasing curve $h_c : [0, \infty) \to [0, \infty)$, *with* $h_c(0) = 0$ *and* $0 < h_c'(0) < \infty$, *such that*

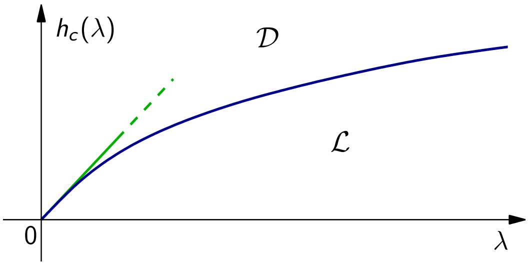
$$\mathcal{L} = \{(\lambda, h) : h < h_c(\lambda)\} \qquad \mathcal{D} = \{(\lambda, h) : h \geq h_c(\lambda)\}$$

The critical line

Theorem ([BdH 97])

There exists a continuous, increasing curve $h_c : [0,\infty) \to [0,\infty)$, *with* $h_c(0) = 0$ *and* $0 < h_c'(0) < \infty$, *such that*

$$\mathcal{L} = \{(\lambda, h) : h < h_c(\lambda)\} \qquad \mathcal{D} = \{(\lambda, h) : h \geq h_c(\lambda)\}$$

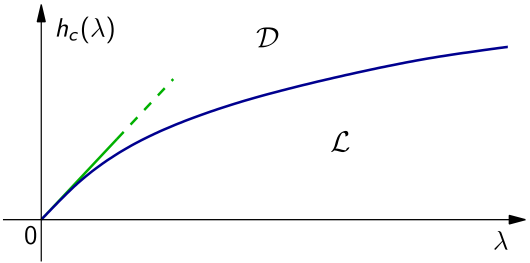


The critical line

Theorem ([BdH 97])

There exists a continuous, increasing curve $h_c : [0,\infty) \to [0,\infty)$, *with* $h_c(0) = 0$ *and* $0 < h_c'(0) < \infty$, *such that*

$$\mathcal{L} = \{(\lambda,h) : h < h_c(\lambda)\} \qquad \mathcal{D} = \{(\lambda,h) : h \geq h_c(\lambda)\}$$



Slope at the origin:

- Brownian scaling
- Universality

Upper and Lower Bound on the critical line

Family of increasing curves indexed by $m > 0$:

$$h^{(m)}(\lambda) \; := \; \frac{1}{2m\lambda} \log \mathrm{M}(-2m\lambda) \qquad \left(\frac{\mathrm{d}h^{(m)}}{\mathrm{d}\lambda}(0) \; = \; m \right)$$

Binary: $h^{(m)}(\lambda) = \frac{\log\cosh(2m\lambda)}{2m\lambda}$ Gaussian: $h^{(m)}(\lambda) = m\lambda$

Upper and Lower Bound on the critical line

Family of increasing curves indexed by $m > 0$:

$$h^{(m)}(\lambda) := \frac{1}{2m\lambda} \log \mathrm{M}(-2m\lambda) \qquad \left(\frac{\mathrm{d}h^{(m)}}{\mathrm{d}\lambda}(0) = m \right)$$

Binary: $h^{(m)}(\lambda) = \frac{\log\cosh(2m\lambda)}{2m\lambda}$ Gaussian: $h^{(m)}(\lambda) = m\lambda$

Physical literature:

- $h_c(\cdot) = h^{(1)}(\cdot)$ [Garel et al. '89, Maritan and Trovato '99]
- $h_c(\cdot) = h^{(2/3)}(\cdot)$ [Monthus '00, Stepanov et al. '98]

Upper and Lower Bound on the critical line

Family of increasing curves indexed by $m > 0$:

$$h^{(m)}(\lambda) \;:=\; \frac{1}{2m\lambda} \log \mathrm{M}(-2m\lambda) \qquad \left(\frac{\mathrm{d}h^{(m)}}{\mathrm{d}\lambda}(0) \;=\; m \right)$$

Binary: $h^{(m)}(\lambda) = \frac{\log\cosh(2m\lambda)}{2m\lambda}$ Gaussian: $h^{(m)}(\lambda) = m\lambda$

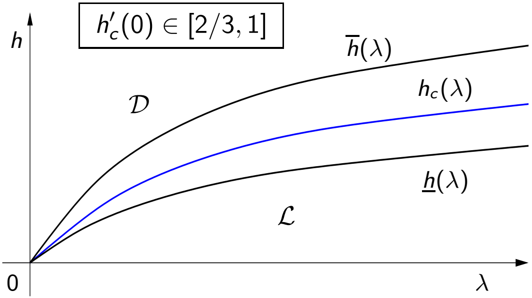
Physical literature:

- $h_c(\cdot) = h^{(1)}(\cdot)$ [Garel et al. '89, Maritan and Trovato '99]
- $h_c(\cdot) = h^{(2/3)}(\cdot)$ [Monthus '00, Stepanov et al. '98]

Theorem ([BdH 97], [Bodineau and Giacomin 04])

$$\underline{h}(\cdot) \;:=\; h^{(2/3)}(\cdot) \;\le\; h_c(\cdot) \;\le\; h^{(1)}(\cdot) \;=:\; \overline{h}(\cdot)$$

Upper and Lower Bound on the critical line



Theorem ([BdH 97], [Bodineau and Giacomin 04])

$$\underline{h}(\cdot) \;:=\; h^{(2/3)}(\cdot) \;\leq\; h_c(\cdot) \;\leq\; h^{(1)}(\cdot) \;=:\; \overline{h}(\cdot)$$

Outline of the talk

A preliminary transformation

Reduced free energy: $\mathtt{F}(\lambda, h) := f(\lambda, h) - \lambda\, h$

$$\mathcal{L} = \big\{(\lambda, h) \,:\, \mathtt{F}(\lambda, h) > 0\big\} \qquad \mathcal{D} = \big\{(\lambda, h) \,:\, \mathtt{F}(\lambda, h) = 0\big\}$$

A preliminary transformation

Reduced free energy: $\mathtt{F}(\lambda, h) := f(\lambda, h) - \lambda\, h$

$$\mathcal{L} = \big\{(\lambda, h) \,:\, \mathtt{F}(\lambda, h) > 0\big\} \qquad \mathcal{D} = \big\{(\lambda, h) \,:\, \mathtt{F}(\lambda, h) = 0\big\}$$

$\mathtt{F}(\lambda, h)$ corresponds to the new partition function

$$\mathcal{Z}_N \;=\; Z_N \cdot \exp\Bigg(-\lambda h N \Bigg)$$

A preliminary transformation

Reduced free energy: $\mathtt{F}(\lambda, h) := f(\lambda, h) - \lambda h$

$$\mathcal{L} = \{(\lambda, h) : \mathtt{F}(\lambda, h) > 0\} \qquad \mathcal{D} = \{(\lambda, h) : \mathtt{F}(\lambda, h) = 0\}$$

$\mathtt{F}(\lambda, h)$ corresponds to the new partition function

$$\mathcal{Z}_N = Z_N \cdot \exp\left(- \lambda h N - \lambda \sum_{i=1}^{N} \omega_i \right)$$

A preliminary transformation

Reduced free energy: $\mathtt{F}(\lambda, h) := f(\lambda, h) - \lambda\, h$

$$\mathcal{L} = \big\{(\lambda, h) \,:\, \mathtt{F}(\lambda, h) > 0\big\} \qquad \mathcal{D} = \big\{(\lambda, h) \,:\, \mathtt{F}(\lambda, h) = 0\big\}$$

$\mathtt{F}(\lambda, h)$ corresponds to the new partition function

$$\begin{aligned} \mathcal{Z}_N &= Z_N \cdot \exp\Bigg(- \lambda h N - \lambda \sum_{i=1}^N \omega_i \Bigg) \\ &= \mathbf{E}\Bigg[\exp\Bigg(- 2\lambda \sum_{n=1}^N (\omega_n + h)\, \mathbf{1}_{\{\mathrm{sign}(S_n) = -1\}} \Bigg) \Bigg] \end{aligned}$$

A preliminary transformation

Reduced free energy: $\mathtt{F}(\lambda, h) := f(\lambda, h) - \lambda\, h$

$$\mathcal{L} = \big\{(\lambda, h)\,:\, \mathtt{F}(\lambda, h) > 0\big\} \qquad \mathcal{D} = \big\{(\lambda, h)\,:\, \mathtt{F}(\lambda, h) = 0\big\}$$

$\mathtt{F}(\lambda, h)$ corresponds to the new partition function

$$\begin{aligned} \mathcal{Z}_N &= Z_N \cdot \exp\left(-\lambda h N - \lambda \sum_{i=1}^{N} \omega_i \right) \\ &= \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h)\, \mathbf{1}_{\{\mathrm{sign}(S_n) = -1\}} \right) \right] \end{aligned}$$

$(\lambda, h) \in \mathcal{L}$ iff $\mathcal{Z}_{N,\omega}^{\lambda,h}$ grows exponentially in N.

The transfer matrix approach

Naïve idea: for fixed (λ, h) and typical ω, **compute** numerically $\mathcal{Z}_{N,\omega}^{\lambda,h}$ as a function of N, to decide between $\mathcal{L}$ and $\mathcal{D}$

The transfer matrix approach

Naïve idea: for fixed (λ, h) and typical ω, **compute** numerically $\mathcal{Z}_{N,\omega}^{\lambda,h}$ as a function of N, to decide between $\mathcal{L}$ and $\mathcal{D}$

Problem: how far can we push N? ($\mathcal{Z}_N$ is the sum of 2^N terms. . .)

The transfer matrix approach

Naïve idea: for fixed (λ, h) and typical ω, **compute** numerically $\mathcal{Z}_{N,\omega}^{\lambda,h}$ as a function of N, to decide between $\mathcal{L}$ and $\mathcal{D}$

Problem: how far can we push N? ($\mathcal{Z}_N$ is the sum of 2^N terms. . .)

Transfer Matrix approach: compute $\mathcal{Z}_N$ exactly in $O(N^2)$ steps!

The transfer matrix approach

Naïve idea: for fixed (λ, h) and typical ω, **compute** numerically $\mathcal{Z}_{N,\omega}^{\lambda,h}$ as a function of N, to decide between $\mathcal{L}$ and $\mathcal{D}$

Problem: how far can we push N? ($\mathcal{Z}_N$ is the sum of 2^N terms...)

$\mathcal{Z}_N(x) := \{\mathcal{Z}_N$ restricted to path ending at $x \in \mathbb{Z}\}$

Markov property + Additivity of the Hamiltonian give:

$$\mathcal{Z}_{M+2}(y) = \begin{cases} \frac{1}{4}\mathcal{Z}_M(y+2) \;+\; \frac{1}{2}\mathcal{Z}_M(y) \;+\; \frac{1}{4}\mathcal{Z}_M(y-2) & y > 0 \\ \frac{1}{4}\Big[\mathcal{Z}_M(2) + \mathcal{Z}_M(0)\Big] \;+\; \frac{1}{4}\alpha_M\Big[\mathcal{Z}_M(0) + \mathcal{Z}_M(-2)\Big] & y = 0 \\ \alpha_M\Big[\frac{1}{4}\mathcal{Z}_M(y+2) \;+\; \frac{1}{2}\mathcal{Z}_M(y) \;+\; \frac{1}{4}\mathcal{Z}_M(y-2)\Big] & y < 0 \end{cases},$$

where $\alpha_M := \exp\big(-2\lambda\,(\omega_{2M+1} + \omega_{2M+2} + 2h)\big)$. □

The transfer matrix approach

Naïve idea: for fixed (λ, h) and typical ω, **compute** numerically $\mathcal{Z}_{N,\omega}^{\lambda,h}$ as a function of N, to decide between $\mathcal{L}$ and $\mathcal{D}$

Problem: how far can we push N? ($\mathcal{Z}_N$ is the sum of 2^N terms. . .)

Transfer Matrix approach: compute $\mathcal{Z}_N$ exactly in $O(N^2)$ steps!

The transfer matrix approach

Naïve idea: for fixed (λ, h) and typical ω, **compute** numerically $\mathcal{Z}_{N,\omega}^{\lambda,h}$ as a function of N, to decide between $\mathcal{L}$ and $\mathcal{D}$

Problem: how far can we push N? ($\mathcal{Z}_N$ is the sum of 2^N terms...)

Transfer Matrix approach: compute $\mathcal{Z}_N$ exactly in $O(N^2)$ steps!

$$\Downarrow$$

Accessible values: $N \sim 10^7$

Approximate computation (LB) in $O(N^{3/2})$ steps $\Rightarrow$ $N \sim 10^8$

The transfer matrix approach

Naïve idea: for fixed (λ, h) and typical ω, **compute** numerically $\mathcal{Z}_{N,\omega}^{\lambda,h}$ as a function of N, to decide between $\mathcal{L}$ and $\mathcal{D}$

Problem: how far can we push N? ($\mathcal{Z}_N$ is the sum of 2^N terms...)

Transfer Matrix approach: compute $\mathcal{Z}_N$ exactly in $O(N^2)$ steps!

$$\Downarrow$$

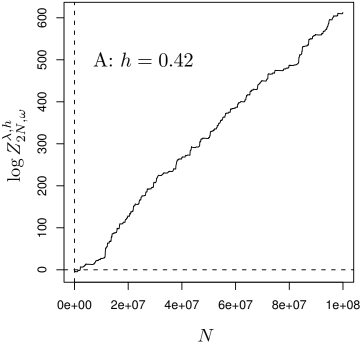
Accessible values: $N \sim 10^7$

Approximate computation (LB) in $O(N^{3/2})$ steps $\Rightarrow$ $N \sim 10^8$

Computations in the case $\omega_n \in \{-1, +1\}$ (tried also Gaussian, ...)

Qualitative results ($\lambda = 0.6,\ \underline{h} = 0.36,\ \overline{h} = 0.49$)

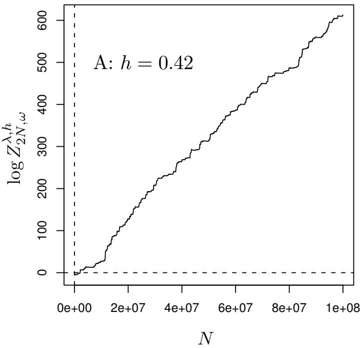
Dichotomy for the asymptotics of $\{\mathcal{Z}_{N,\omega}^{\lambda,h}\}_N$:



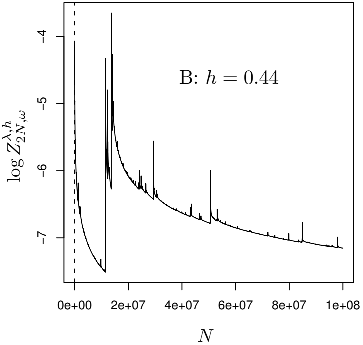
it diverges exponentially. . .

Qualitative results ($\lambda = 0.6$, $\underline{h} = 0.36$, $\overline{h} = 0.49$)

Dichotomy for the asymptotics of $\{\mathcal{Z}_{N,\omega}^{\lambda,h}\}_N$:



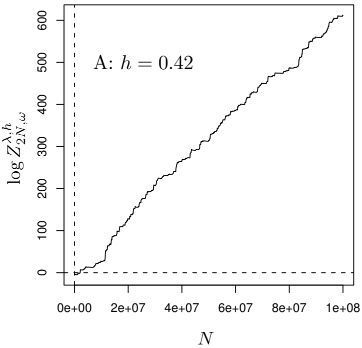
it diverges exponentially. . .

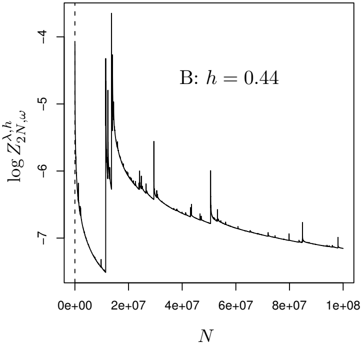


or it vanishes!

Qualitative results ($\lambda = 0.6,\ \underline{h} = 0.36,\ \overline{h} = 0.49$)

Dichotomy for the asymptotics of $\{\mathcal{Z}_{N,\omega}^{\lambda,h}\}_N$:





it diverges exponentially. . .

or it vanishes!

(outside of the "critical" region. . . come back after)

Qualitative results

It is numerically rather evident that the true critical line $h_c(\lambda)$ lies strictly in between $\underline{h}(\lambda)$ and $\overline{h}(\lambda)$, i.e.:

- for some $h > \underline{h}(\lambda)$ we see an exponential growth of $\mathcal{Z}_N$
- for some $h < \overline{h}(\lambda)$ we see a vanishing behavior of $\mathcal{Z}_N$

Qualitative results

It is numerically rather evident that the true critical line $h_c(\lambda)$ lies strictly in between $\underline{h}(\lambda)$ and $\overline{h}(\lambda)$, i.e.:

- for some $h > \underline{h}(\lambda)$ we see an exponential growth of $\mathcal{Z}_N$
- for some $h < \overline{h}(\lambda)$ we see a vanishing behavior of $\mathcal{Z}_N$

How to give more confidence to these observations?

Qualitative results

It is numerically rather evident that the true critical line $h_c(\lambda)$ lies strictly in between $\underline{h}(\lambda)$ and $\overline{h}(\lambda)$, i.e.:

- for some $h > \underline{h}(\lambda)$ we see an exponential growth of $\mathcal{Z}_N$
- for some $h < \overline{h}(\lambda)$ we see a vanishing behavior of $\mathcal{Z}_N$

How to give more confidence to these observations?

Beating the LB: $h_c(\lambda) > \underline{h}(\lambda)$

Rigorous statistical test for $\mathcal{L}$ocalization with explicit error bound (superadditivity + concentration of measure)

Qualitative results

It is numerically rather evident that the true critical line $h_c(\lambda)$ lies strictly in between $\underline{h}(\lambda)$ and $\overline{h}(\lambda)$, i.e.:

- for some $h > \underline{h}(\lambda)$ we see an exponential growth of $\mathcal{Z}_N$
- for some $h < \overline{h}(\lambda)$ we see a vanishing behavior of $\mathcal{Z}_N$

How to give more confidence to these observations?

Beating the LB: $h_c(\lambda) > \underline{h}(\lambda)$

Rigorous statistical test for $\mathcal{L}$ocalization with explicit error bound (superadditivity + concentration of measure)

Beating the UB: $h_c(\lambda) < \overline{h}(\lambda)$

Quantitative criterion to measure the convergence under diffusive rescaling to the Brownian meander

$\mathcal{L}$ocalization in a finite volume

Markov property of S $\Rightarrow$ for $N, M \in 2\mathbb{N}$

$$\mathcal{Z}_{N+M,\omega}(0) \;\geq\; \mathcal{Z}_{N,\omega}(0) \,\cdot\, \mathcal{Z}_{M,\theta^N\omega}(0) \qquad \big[\, (\theta^N\omega)_n := \omega_{N+n} \,\big]$$

$\Rightarrow$ the sequence $N \mapsto \mathbb{E}\big[\log \mathcal{Z}_{N,\omega}(0)\big]$ is superadditive

$\mathcal{L}$ocalization in a finite volume

Markov property of S $\Rightarrow$ for $N, M \in 2\mathbb{N}$

$$\mathcal{Z}_{N+M,\omega}(0) \;\geq\; \mathcal{Z}_{N,\omega}(0) \;\cdot\; \mathcal{Z}_{M,\theta^N\omega}(0) \qquad \big[\,(\theta^N\omega)_n := \omega_{N+n}\,\big]$$

$\Rightarrow$ the sequence $N \mapsto \mathbb{E}\big[\log \mathcal{Z}_{N,\omega}(0)\big]$ is superadditive

Since $\mathcal{Z}_N(0) \asymp \mathcal{Z}_N$ we have the basic formula:

$$\mathtt{F}(\lambda, h) \;=\; \sup_{N\in\mathbb{N}} \frac{1}{N}\,\mathbb{E}\big[\log \mathcal{Z}_{N,\omega}(0)\big]$$

In a suggestive way:

$$(\lambda, h) \in \mathcal{L} \qquad \Longleftrightarrow \qquad \exists\; N \in \mathbb{N} \,:\; \mathbb{E}\big[\log \mathcal{Z}_{N,\omega}(0)\big] > 0$$

$\mathcal{L}$ocalization in a finite volume

Markov property of S $\Rightarrow$ for $N, M \in 2\mathbb{N}$

$$\mathcal{Z}_{N+M,\omega}(0) \geq \mathcal{Z}_{N,\omega}(0) \cdot \mathcal{Z}_{M,\theta^N\omega}(0) \qquad \left[(\theta^N\omega)_n := \omega_{N+n} \right]$$

$\Rightarrow$ the sequence $N \mapsto \mathbb{E}\left[\log \mathcal{Z}_{N,\omega}(0) \right]$ is superadditive

Since $\mathcal{Z}_N(0) \asymp \mathcal{Z}_N$ we have the basic formula:

$$\mathtt{F}(\lambda, h) = \sup_{N \in \mathbb{N}} \frac{1}{N} \mathbb{E}\left[\log \mathcal{Z}_{N,\omega}(0) \right]$$

In a suggestive way:

$$(\lambda, h) \in \mathcal{L} \qquad \Longleftrightarrow \qquad \exists\, N \in \mathbb{N} : \ \mathbb{E}\left[\log \mathcal{Z}_{N,\omega}(0) \right] > 0$$

$\mathcal{L}$ocalization can be proven by looking at *finite systems*

Statistical test for $\mathcal{L}$ocalization

To prove $(\lambda, h) \in \mathcal{L}$ it suffices to find N s.t. $\mathbb{E}\big[\log \mathcal{Z}_{N,\omega}^{\lambda,h}(0)\big] > 0$

Statistical test for $\mathcal{L}$ocalization

To prove $(\lambda, h) \in \mathcal{L}$ it suffices to find N s.t. $\mathbb{E}\big[\log \mathcal{Z}_{N,\omega}^{\lambda,h}(0)\big] > 0$

Drawback: $\mathbb{E}$ means sum over 2^N possible charges ...

Statistical test for $\mathcal{L}$ocalization

To prove $(\lambda, h) \in \mathcal{L}$ it suffices to find N s.t. $\mathbb{E}\big[\log \mathcal{Z}_{N,\omega}^{\lambda,h}(0)\big] > 0$

Drawback: $\mathbb{E}$ means sum over 2^N possible charges ...
Could small values of N suffice? No: back to this later

Statistical test for $\mathcal{L}$ocalization

To prove $(\lambda, h) \in \mathcal{L}$ it suffices to find N s.t. $\mathbb{E}\big[\log \mathcal{Z}_{N,\omega}^{\lambda,h}(0)\big] > 0$

Drawback: $\mathbb{E}$ means sum over 2^N possible charges ...
Could small values of N suffice? No: back to this later

Problem: N large $\longrightarrow$ we can sample only a small number of ω

Statistical test for $\mathcal{L}$ocalization

To prove $(\lambda, h) \in \mathcal{L}$ it suffices to find N s.t. $\mathbb{E}\big[\log \mathcal{Z}_{N,\omega}^{\lambda,h}(0)\big] > 0$

Drawback: $\mathbb{E}$ means sum over 2^N possible charges ...
Could small values of N suffice? No: back to this later

Problem: N large $\longrightarrow$ we can sample only a small number of ω

Solution: Concentration of Measure – few ω may suffice!

Statistical test for $\mathcal{L}$ocalization

To prove $(\lambda, h) \in \mathcal{L}$ it suffices to find N s.t. $\mathbb{E}\big[\log \mathcal{Z}_{N,\omega}^{\lambda,h}(0)\big] > 0$

Drawback: $\mathbb{E}$ means sum over 2^N possible charges ...
Could small values of N suffice? No: back to this later

Problem: N large $\longrightarrow$ we can sample only a small number of ω

Solution: Concentration of Measure – few ω may suffice!

Lipschitz function $G : \{-1, +1\}^N \to \mathbb{R}$ with Lipschitz constant C_{Lip}:

$$\mathbb{P}\big[G > \mathbb{E}(G) + u\big] \;\leq\; \exp\left(-\frac{u^2}{4\, C_{Lip}^2}\right)$$

More generally for an i.i.d. family $\{G_i\}_i$:

$$\mathbb{P}\left[\frac{\sum_{i=1}^n G_i}{n} > \mathbb{E}(G_1) + u\right] \;\leq\; \exp\left(-\frac{n\, u^2}{4\, C_{Lip}^2}\right)$$

Statistical test for $\mathcal{L}$ocalization

By direct computation for $G := \log \mathcal{Z}_{N,\omega}$ we get $C_{Lip} = 2\lambda\sqrt{N}$

Statistical test for $\mathcal{L}$ocalization

By direct computation for $G := \log \mathcal{Z}_{N,\omega}$ we get $C_{Lip} = 2\lambda\sqrt{N}$

Statistical test for $\mathcal{L}$

- Null hypothesis: $H_0 : \mathbb{E}\big[\log \mathcal{Z}_{N,\omega}^{\lambda,h}\big] \leq 0$

Statistical test for $\mathcal{L}$ocalization

By direct computation for $G := \log \mathcal{Z}_{N,\omega}$ we get $C_{Lip} = 2\lambda\sqrt{N}$

Statistical test for $\mathcal{L}$

- Null hypothesis: $H_0 : \mathbb{E}\big[\log \mathcal{Z}_{N,\omega}^{\lambda,h}\big] \leq 0$
- $\widehat{u}_n :=$ average of a sample of n independent realizations of $\log \mathcal{Z}_{N,\omega}^{\lambda,h}$.

Statistical test for $\mathcal{L}$ocalization

By direct computation for $G := \log \mathcal{Z}_{N,\omega}$ we get $C_{Lip} = 2\lambda\sqrt{N}$

Statistical test for $\mathcal{L}$

- Null hypothesis: $H_0 : \mathbb{E}\left[\log \mathcal{Z}_{N,\omega}^{\lambda,h}\right] \leq 0$
- $\widehat{u}_n :=$ average of a sample of n independent realizations of $\log \mathcal{Z}_{N,\omega}^{\lambda,h}$.
- If $\widehat{u}_n > 0$ then we refuse H_0 (that is $(\lambda, h) \in \mathcal{L}$!) with a level of error not larger than

$$p := \exp\left(-\frac{\widehat{u}_n^2 n}{16\lambda^2 N}\right)$$

Statistical test for $\mathcal{L}$ocalization

Numerical results: the LB is strict

λ	0.3	0.6	1
h	0.22	0.41	0.58
$\underline{h}(\lambda)$	0.195	0.363	0.530
$\overline{h}(\lambda)$	0.286	0.495	0.662
p–value	1.5×10^{-6}	9.5×10^{-3}	1.6×10^{-5}
N	300000	500000	160000
n	225000	330000	970000

$$p := \exp\left(-\frac{\widehat{u}_n^2 n}{16\lambda^2 N}\right)$$

Computer–assisted proof?

Back to naïve idea: can we hope that

$$\mathbb{E}\left[\log Z_{N,\omega}^{\lambda,h}\right] > 0 \quad \text{for small values of } N \quad (\text{up to } N \approx 20)$$

for $h > \underline{h}(\lambda)$, to perform an explicit computation?

Computer–assisted proof?

Back to naïve idea: can we hope that

$$\mathbb{E}\big[\log Z_{N,\omega}^{\lambda,h}\big] > 0 \quad \text{for small values of } N \quad (\text{up to } N \approx 20)$$

for $h > \underline{h}(\lambda)$, to perform an explicit computation?

NO!

λ	0.05($\star$)	0.1	0.2	0.4	0.6	1	2($\star$)	4($\star\star$)
N_+	750000	190000	40000	9500	4250	1800	900	800
N_-	600000	130000	33000	7500	3650	1550	750	700

$p = 10^{-5} \sim 10^{-6}$ ($\star$) $p = 10^{-2} \sim 10^{-3}$ ($\star\star$) limit model ($\lambda \to \infty$)

With the stated p–value and for $h = h^{(2/3)}(\lambda)$, both

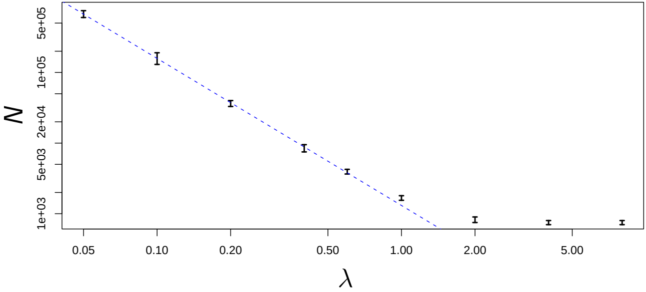
$$\mathbb{E}\big[\log Z_{N_+,\omega}^{\lambda,h}\big] > 0 \qquad \mathbb{E}\big[\log Z_{N_-,\omega}^{\lambda,h}\big] < 0$$

Computer–assisted proof?

Back to naïve idea: can we hope that

$$\mathbb{E}\left[\log Z_{N,\omega}^{\lambda,h}\right] > 0 \quad \text{for small values of } N \quad (\text{up to } N \approx 20)$$

for $h > \underline{h}(\lambda)$, to perform an explicit computation?



$\mathcal{D}$elocalized observations

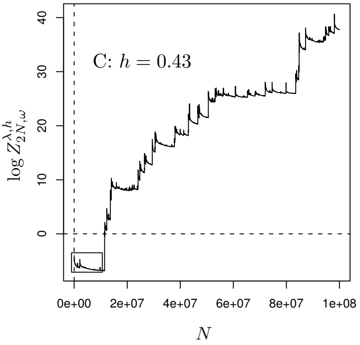
Difficult to make rigorous numerical claims on $\mathcal{D}$:

$\mathcal{D}$elocalization is not a finite-volume issue

$\mathcal{D}$elocalized observations

Difficult to make rigorous numerical claims on $\mathcal{D}$:

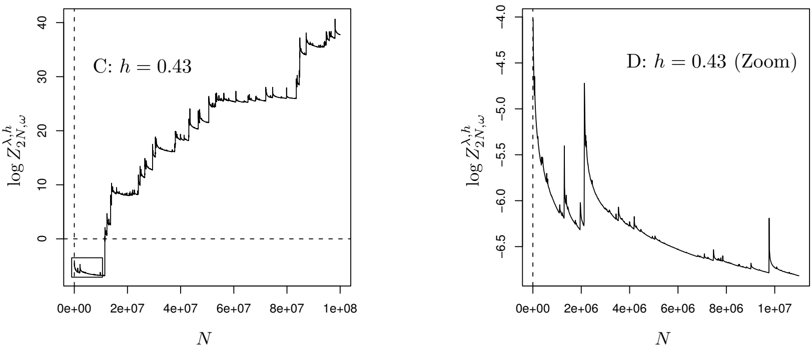
$\mathcal{D}$elocalization is not a finite-volume issue



$\mathcal{D}$elocalized observations

Difficult to make rigorous numerical claims on $\mathcal{D}$:

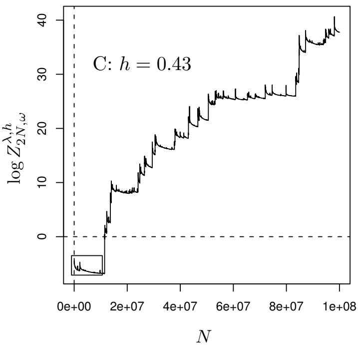
$\mathcal{D}$elocalization is not a finite-volume issue

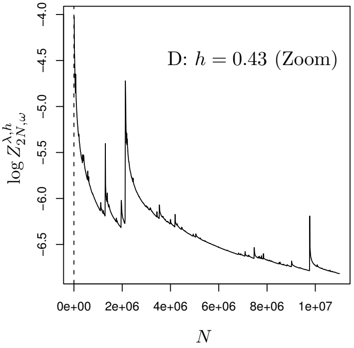


$\mathcal{D}$elocalized observations

Difficult to make rigorous numerical claims on $\mathcal{D}$:

$\mathcal{D}$elocalization is not a finite-volume issue





Jumps correspond to atypical stretches in the environment ω

$\mathcal{D}$elocalized path analysis

Assume the convergence to Brownian meander in $\mathring{\mathcal{D}}$:

$$\text{under} \quad \mathbf{P}_{N,\omega} \qquad \frac{S_N}{\sqrt{N}} \Longrightarrow x\, e^{-x^2/2}\, \mathrm{d}x \quad =: \ \varphi^+(x)\, \mathrm{d}x$$

$\mathcal{D}$elocalized path analysis

Assume the convergence to Brownian meander in $\mathring{\mathcal{D}}$:

$$\text{under} \quad \mathbf{P}_{N,\omega} \qquad \frac{S_N}{\sqrt{N}} \implies x\, e^{-x^2/2}\, \mathrm{d}x \quad =: \; \varphi^+(x)\, \mathrm{d}x$$

The law of S_N under $\mathbf{P}_{N,\omega}$ is computed in terms of $\{\mathcal{Z}_{N,\omega}(x)\}_x$:

$$\mathbf{P}_{N,\omega}\big[S_N = x\big] \;=\; \frac{\mathcal{Z}_{N,\omega}(x)}{\mathcal{Z}_{N,\omega}}\,.$$

Delocalized path analysis

Assume the convergence to Brownian meander in $\mathring{\mathcal{D}}$:

$$\text{under} \quad \mathbf{P}_{N,\omega} \qquad \frac{S_N}{\sqrt{N}} \Longrightarrow x\, e^{-x^2/2}\,\mathrm{d}x \quad =: \; \varphi^+(x)\,\mathrm{d}x$$

The law of S_N under $\mathbf{P}_{N,\omega}$ is computed in terms of $\{\mathcal{Z}_{N,\omega}(x)\}_x$:

$$\mathbf{P}_{N,\omega}\left[S_N = x\right] \;=\; \frac{\mathcal{Z}_{N,\omega}(x)}{\mathcal{Z}_{N,\omega}}\,.$$

Quantitative measure of $\mathcal{D}$elocalization (ℓ_1 distance)

$$\triangle_N^{\lambda,h}(\omega) \;:=\; \sum_{x\in\mathbb{Z}} \left| \mathbf{P}_{N,\omega}^{\lambda,h}\left[S_N = x\right] \;-\; \frac{1}{\sqrt{N}}\,\varphi^+\!\left(\frac{x}{\sqrt{N}}\right) \right|$$

$\mathcal{D}$elocalized path analysis

We work at $\lambda = 0.6$ $[\underline{h} = 0.36, \quad h_{test} = 0.41, \quad \overline{h} = 0.49]$

- $(\omega^r)_n := \omega_{N-n}$ environment attached backwards (fluctuations)

$\mathcal{D}$elocalized path analysis

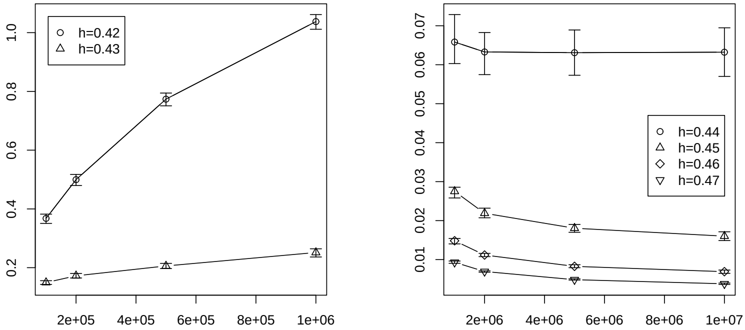
We work at $\lambda = 0.6$ $[\underline{h} = 0.36, \quad h_{test} = 0.41, \quad \overline{h} = 0.49]$

- $(\omega^r)_n := \omega_{N-n}$ environment attached backwards (fluctuations)
- Plot: 95% C.I. for the median of a sample $\{\triangle_N^{\lambda,h}(\omega^r)\}$

Delocalized path analysis

We work at $\lambda = 0.6 \quad [\underline{h} = 0.36, \quad h_{test} = 0.41, \quad \overline{h} = 0.49]$

- $(\omega^r)_n := \omega_{N-n}$ environment attached backwards (fluctuations)
- Plot: 95% C.I. for the median of a sample $\{\triangle_N^{\lambda,h}(\omega^r)\}$



A conjecture (?) on the true critical line

How to define a finite volume critical line?

A conjecture (?) on the true critical line

How to define a finite volume critical line?

Possible criteria: for fixed N and λ, take the value of h s.t.

- $\mathcal{Z}_{N,\omega}^{\lambda,h}(0) = 1$ (monotonicity in h)
- $\triangle_N^{\lambda,h}(\omega) = 0.1$ (empirical monotonicity in h)

Changing the (rather arbitrary) threshold does not affect the results.

A conjecture (?) on the true critical line

How to define a finite volume critical line?

Possible criteria: for fixed N and λ, take the value of h s.t.

- $\mathcal{Z}_{N,\omega}^{\lambda,h}(0) = 1$ (monotonicity in h)
- $\triangle_N^{\lambda,h}(\omega) = 0.1$ (empirical monotonicity in h)

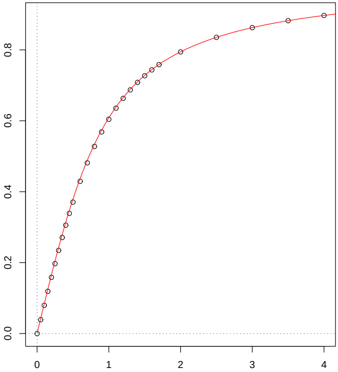
Changing the (rather arbitrary) threshold does not affect the results.

Numerical computations show that

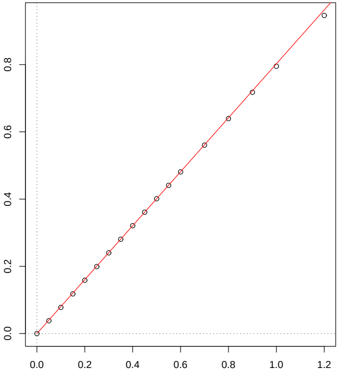
$$h_c(\cdot) \simeq h^{(m)}(\cdot) \qquad m = 0.82 - 0.83$$

with remarkable precision (value of m somewhat criterion dependent)

A conjecture (?) on the true critical line

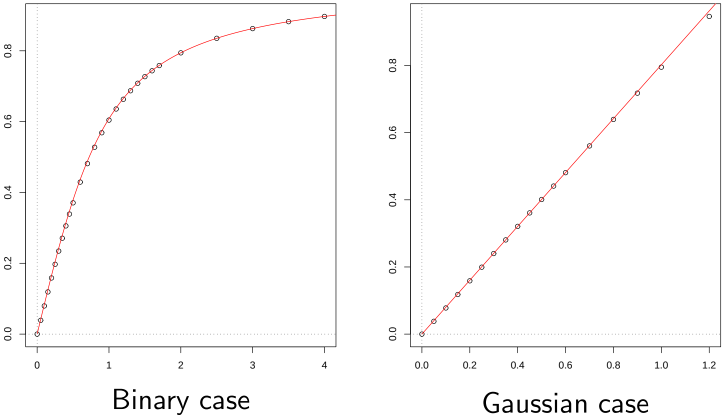


Binary case



Gaussian case

A conjecture (?) on the true critical line



Plotted points are obtained for one fixed realization of ω

Outline of the talk

The constrained annealing technique

Annealed bound on $Z_{N,\omega}$ (old partition function):

$$f(\lambda, h) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega}^{\lambda,h} \leq \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_{N,\omega}^{\lambda,h} =: f_a(\lambda, h)$$

The constrained annealing technique

Annealed bound on $Z_{N,\omega}$ (old partition function):

$$f(\lambda, h) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega}^{\lambda,h} \leq \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_{N,\omega}^{\lambda,h} =: f_a(\lambda, h)$$

Useless: $f^a(\lambda, h) > \lambda h$ for every λ, h.

The constrained annealing technique

Annealed bound on $Z_{N,\omega}$ (old partition function):

$$f(\lambda, h) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega}^{\lambda,h} \leq \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_{N,\omega}^{\lambda,h} =: f_a(\lambda, h)$$

Useless: $f^a(\lambda, h) > \lambda h$ for every λ, h.

Constrained annealing (Morita approximation): replace $Z_{N,\omega}$ by

$$Z'_{N,\omega} := Z_{N,\omega} \exp(A_N(\omega)) \qquad \mathbb{E}(A_N(\omega)) = 0$$

The constrained annealing technique

Annealed bound on $Z_{N,\omega}$ (old partition function):

$$f(\lambda, h) \;=\; \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega}^{\lambda,h} \;\leq\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_{N,\omega}^{\lambda,h} \;=:\; f_a(\lambda, h)$$

Useless: $f^a(\lambda, h) > \lambda h$ for every λ, h.

Constrained annealing (Morita approximation): replace $Z_{N,\omega}$ by

$$Z'_{N,\omega} \;:=\; Z_{N,\omega} \exp(A_N(\omega)) \qquad \mathbb{E}(A_N(\omega)) = 0$$

- New annealed free energy $f'_a(\lambda, h) \;:=\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z'_{N,\omega}$

The constrained annealing technique

Annealed bound on $Z_{N,\omega}$ (old partition function):

$$f(\lambda, h) \;=\; \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega}^{\lambda,h} \;\leq\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_{N,\omega}^{\lambda,h} \;=:\; f_a(\lambda, h)$$

Useless: $f^a(\lambda, h) > \lambda h$ for every λ, h.

Constrained annealing (Morita approximation): replace $Z_{N,\omega}$ by

$$Z'_{N,\omega} \;:=\; Z_{N,\omega} \exp(A_N(\omega)) \qquad \mathbb{E}(A_N(\omega)) = 0$$

- New annealed free energy $f'_a(\lambda, h) \;:=\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z'_{N,\omega}$
- Same quenched free energy $\frac{1}{N} \mathbb{E} \log Z'_{N,\omega} \to f(\lambda, h)$

The constrained annealing technique

Annealed bound on $Z_{N,\omega}$ (old partition function):

$$f(\lambda, h) \;=\; \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega}^{\lambda,h} \;\leq\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_{N,\omega}^{\lambda,h} \;=:\; f_a(\lambda, h)$$

Useless: $f^a(\lambda, h) > \lambda h$ for every λ, h.

Constrained annealing (Morita approximation): replace $Z_{N,\omega}$ by

$$Z'_{N,\omega} \;:=\; Z_{N,\omega} \exp(A_N(\omega)) \qquad \mathbb{E}(A_N(\omega)) = 0$$

- New annealed free energy $f'_a(\lambda, h) \;:=\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z'_{N,\omega}$
- Same quenched free energy $\frac{1}{N} \mathbb{E} \log Z'_{N,\omega} \to f(\lambda, h)$

Therefore $\quad f(\lambda, h) \;\leq\; f'_a(\lambda, h)$

The constrained annealing technique

New partition function $\mathcal{Z}_{N,\omega}$ corresponds to $A_N(\omega) = -\lambda \sum_{n=1}^{N} \omega_n$ (a part from $-\lambda h$).

The constrained annealing technique

New partition function $\mathcal{Z}_{N,\omega}$ corresponds to $A_N(\omega) = -\lambda \sum_{n=1}^{N} \omega_n$ (a part from $-\lambda h$). Direct computation

$$\mathtt{F}_a'(\lambda, h) \;=\; \min\{\log \mathrm{M}(-2\lambda) - 2\lambda h,\, 0\}$$

The constrained annealing technique

New partition function $\mathcal{Z}_{N,\omega}$ corresponds to $A_N(\omega) = -\lambda \sum_{n=1}^{N} \omega_n$ (a part from $-\lambda h$). Direct computation

$$\mathtt{F}(\lambda, h) \;\leq\; \mathtt{F}'_a(\lambda, h) \;=\; \min\{\log \mathrm{M}(-2\lambda) - 2\lambda h,\, 0\}$$

The constrained annealing technique

New partition function $\mathcal{Z}_{N,\omega}$ corresponds to $A_N(\omega) = -\lambda \sum_{n=1}^N \omega_n$ (a part from $-\lambda h$). Direct computation

$$\mathtt{F}(\lambda, h) \;\le\; \mathtt{F}'_a(\lambda, h) \;=\; \min\{\log \mathrm{M}(-2\lambda) - 2\lambda h,\, 0\}$$

Hence we prove the UB:

$$\mathtt{F}(\lambda, h) \;=\; 0 \quad \text{if} \quad h \;\ge\; \frac{1}{2\lambda} \log \mathrm{M}(-2\lambda) \;=:\; \overline{h}(\lambda)$$

The constrained annealing technique

New partition function $\mathcal{Z}_{N,\omega}$ corresponds to $A_N(\omega) = -\lambda \sum_{n=1}^{N} \omega_n$ (a part from $-\lambda h$). Direct computation

$$\mathtt{F}(\lambda, h) \;\leq\; \mathtt{F}'_a(\lambda, h) \;=\; \min\{\log \mathrm{M}(-2\lambda) - 2\lambda h,\, 0\}$$

Hence we prove the UB:

$$\mathtt{F}(\lambda, h) \;=\; 0 \quad \text{if} \quad h \;\geq\; \frac{1}{2\lambda} \log \mathrm{M}(-2\lambda) \;=:\; \overline{h}(\lambda)$$

Natural idea: could we improve $\overline{h}(\lambda)$ with some smarter choice of $A_N(\omega) = \sum_{n=1}^{N} B(\theta^n \omega)$, with $B(\cdot)$ local?

The constrained annealing technique

New partition function $\mathcal{Z}_{N,\omega}$ corresponds to $A_N(\omega) = -\lambda \sum_{n=1}^{N} \omega_n$ (a part from $-\lambda h$). Direct computation

$$\mathrm{F}(\lambda, h) \ \le \ \mathrm{F}'_a(\lambda, h) \ = \ \min\{\log \mathrm{M}(-2\lambda) - 2\lambda h,\, 0\}$$

Hence we prove the UB:

$$\mathrm{F}(\lambda, h) \ = \ 0 \quad \text{if} \quad h \ge \frac{1}{2\lambda} \log \mathrm{M}(-2\lambda) \ =: \ \overline{h}(\lambda)$$

Natural idea: could we improve $\overline{h}(\lambda)$ with some smarter choice of $A_N(\omega) = \sum_{n=1}^{N} B(\theta^n \omega)$, with $B(\cdot)$ local?

Theorem ([C. and Giacomin])

For every local function $B(\cdot)$ and for every $h < \overline{h}(\lambda)$ we have

$$F'_a(\lambda, h) \ > \ 0$$

The proof

$$\mathrm{F}'_a \;=\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left\{ \mathcal{Z}_{N,\omega} \exp\left(\sum_{n=1}^{N} B(\theta^n \omega) \right) \right\} \qquad (1)$$

The proof

$$\mathrm{F}'_a \;=\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left\{ \mathcal{Z}_{N,\omega} \exp\left(\sum_{n=1}^{N} B(\theta^n \omega) \right) \right\} \tag{1}$$

By the basic $\mathcal{D}$elocalization bound

$$\mathcal{Z}_{N,\omega} \;\geq\; P(S_n > 0,\, 1 \leq n \leq N) \;\approx\; N^{-1/2}$$

The proof

$$\mathrm{F}'_a \;=\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left\{ \mathcal{Z}_{N,\omega} \exp\left(\sum_{n=1}^{N} B(\theta^n \omega) \right) \right\} \qquad (1)$$

By the basic $\mathcal{D}$elocalization bound

$$\mathcal{Z}_{N,\omega} \;\geq\; P(S_n > 0,\, 1 \leq n \leq N) \;\approx\; N^{-1/2}$$

hence

$$\mathrm{F}'_a \;\geq\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} \exp\left(\sum_{n=1}^{N} B(\theta^n \omega) \right) \;=:\; \gamma$$

The proof

$$\mathrm{F}'_a = \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left\{ \mathcal{Z}_{N,\omega} \exp\left(\sum_{n=1}^{N} B(\theta^n \omega) \right) \right\} \qquad (1)$$

By the basic $\mathcal{D}$elocalization bound

$$\mathcal{Z}_{N,\omega} \geq P(S_n > 0, 1 \leq n \leq N) \approx N^{-1/2}$$

hence

$$\mathrm{F}'_a \geq \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} \exp\left(\sum_{n=1}^{N} B(\theta^n \omega) \right) =: \gamma$$

- if $\gamma > 0$ then $\mathrm{F}'_a > 0$ and we are done

The proof

$$\mathrm{F}'_a \;=\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left\{ \mathcal{Z}_{N,\omega} \exp\left(\sum_{n=1}^{N} B(\theta^n \omega) \right) \right\} \qquad (1)$$

By the basic $\mathcal{D}$elocalization bound

$$\mathcal{Z}_{N,\omega} \;\geq\; P(S_n > 0,\, 1 \leq n \leq N) \;\approx\; N^{-1/2}$$

hence

$$\mathrm{F}'_a \;\geq\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} \exp\left(\sum_{n=1}^{N} B(\theta^n \omega) \right) \;=:\; \gamma$$

- if $\gamma > 0$ then $\mathrm{F}'_a > 0$ and we are done
- if $\gamma = 0$ then $B(\cdot)$ is trivial, i.e. $\sup_\omega \sum_{n=1}^{N} B(\theta^n \omega) = o(N)$.

The proof

$$\mathrm{F}'_a \;=\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left\{ \mathcal{Z}_{N,\omega} \exp\left(\sum_{n=1}^{N} B(\theta^n \omega) \right) \right\} \qquad (1)$$

By the basic $\mathcal{D}$elocalization bound

$$\mathcal{Z}_{N,\omega} \;\geq\; P(S_n > 0,\, 1 \leq n \leq N) \;\approx\; N^{-1/2}$$

hence

$$\mathrm{F}'_a \;\geq\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} \exp\left(\sum_{n=1}^{N} B(\theta^n \omega) \right) \;=:\; \gamma$$

- if $\gamma > 0$ then $\mathrm{F}'_a > 0$ and we are done
- if $\gamma = 0$ then $B(\cdot)$ is trivial, i.e. $\sup_\omega \sum_{n=1}^{N} B(\theta^n \omega) = o(N)$. Hence by (1)

$$\mathrm{F}'_a \;=\; \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} \mathcal{Z}_{N,\omega} \;=\; \mathrm{F}_a > 0$$

A general statement

Theorem

Let $\mathcal{Z}_{N,\omega}$ be the partition function of a system of size N and with i.i.d. disorder ω. Assume that

- *for every* $\mathcal{Z}_{N,\omega} \geq c_N > 0$ *with* $\frac{1}{N} \log c_N \to 0$
- *the annealed free energy is positive:* $\textsf{F}_a > 0$

Then, for any choice of the local function $B(\omega)$ with

$$\mathbb{E}(B(\omega)) = 0 \qquad \mathbb{E}(\exp(\alpha B(\omega))) < \infty \quad \forall \alpha \in \mathbb{R}$$

the constrained annealed free energy with $A_N(\omega) = \sum_{n=1}^{N} B(\theta^n \omega)$ *is positive:* $\textsf{F}_a' > 0$.