Scaling Limits and Universality for Random Pinning Models

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IperMiB ∼ September 12, 2013

Outline

1. Hydrodynamics

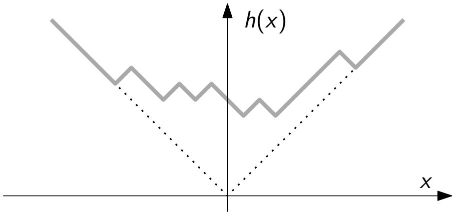
2. The link with "polymers"

3. The random pinning model

4. Weak disorder regime

Corner growth model (in dimension 1)

Random evolution of interface $h : \mathbb{Z} \to \mathbb{Z}$ $\qquad (|h(x+1) - h(x)| = 1)$



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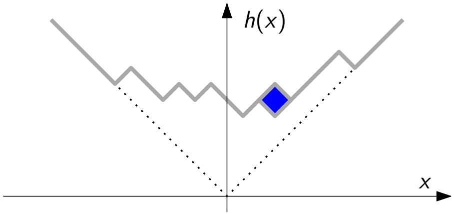

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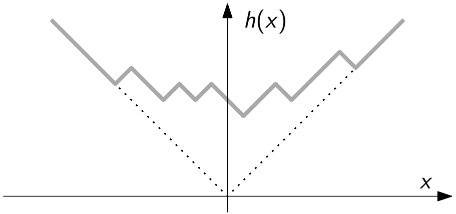

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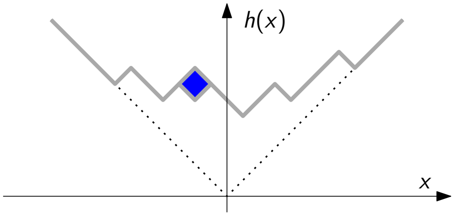

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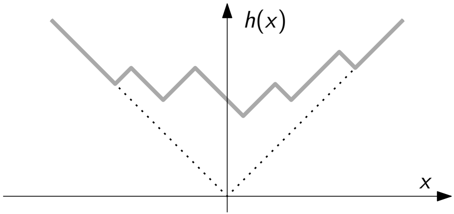

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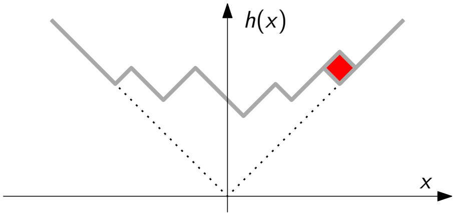

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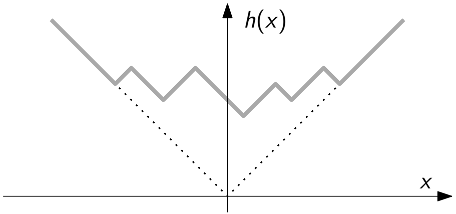

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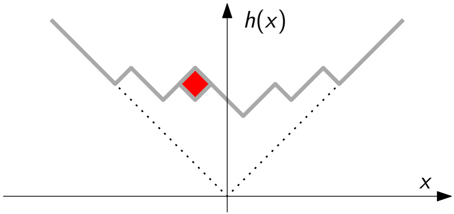

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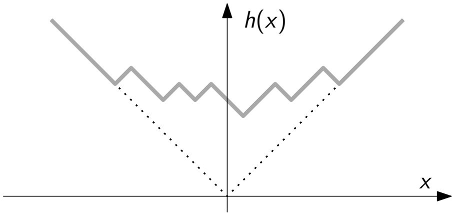

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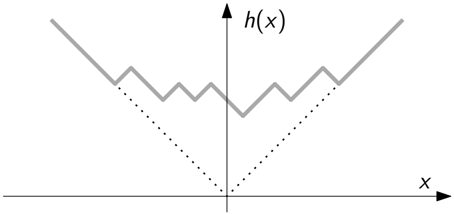

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$h(t, x) :=$ interface height at time $t \geq 0$, position $x \in \mathbb{Z}$.

Large scale evolution?

Asymmetric case

Assume that $\gamma := p - q > 0$ is fixed ("upward drift")

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Law of Large Numbers

Random (micro) model $\rightsquigarrow$ Deterministic (macro) behavior

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Theorem (Bertini, Giacomin '97)

As $\epsilon \downarrow 0$, $\mathcal{H}_\epsilon(T, X) := \frac{1}{\sqrt{\epsilon}}\{H_\epsilon(T, X) - \frac{T}{2\sqrt{\epsilon}}\}$ converges in distrib. to the "solution" of a non-linear stochastic PDE: the KPZ equation

KPZ equation (Kardhar-Parisi-Zhang '86)

The KPZ equation is a non-linear stochastic PDE, believed to describe the statistics of several physical systems (Universality)

$$\partial_T \mathcal{H} = \frac{1}{2}\left(\Delta \mathcal{H} - (\nabla \mathcal{H})^2\right) + \dot{W} \qquad \text{(KPZ)}$$

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$\mathcal{Z}$ solves the Stochastic Heat Equation, a linear stochastic PDE

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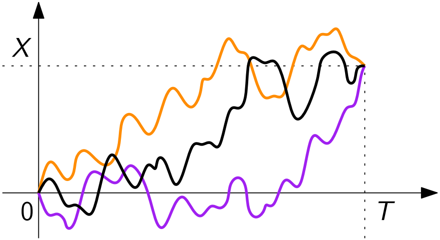

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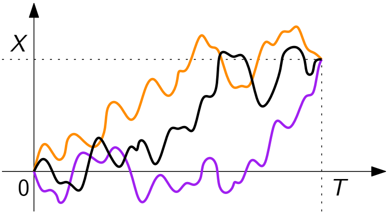

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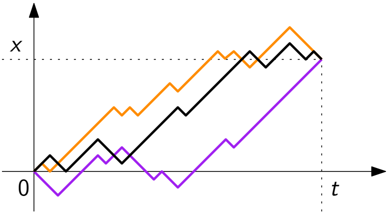

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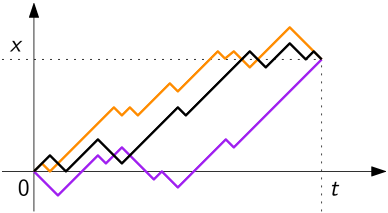

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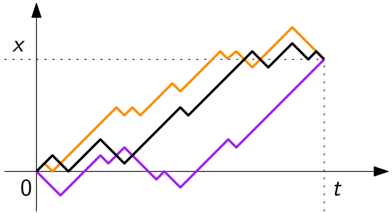

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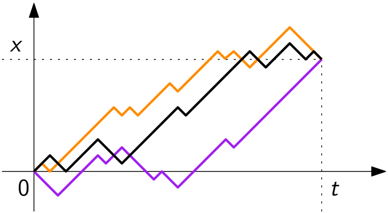
- Converges in the continuum limit to the SHE solution
- Discretization retains the essence of the problem



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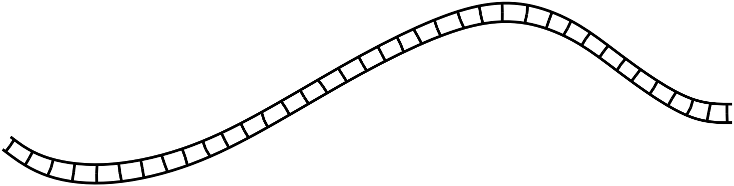

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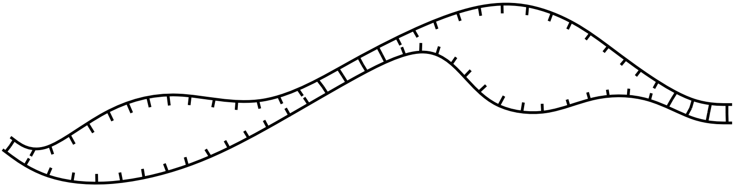

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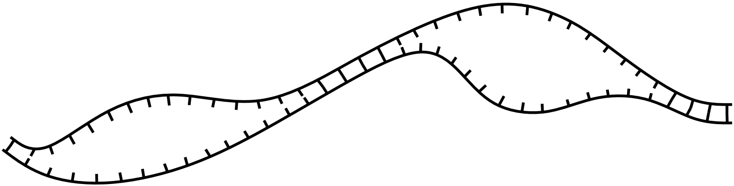

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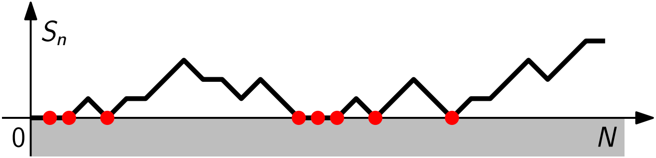
How to model such a situation?

The pinning model

1. Configuration space $\Omega = (\mathbb{N}_0)^N$ (N = system size)

   S_n = distance of the n-th monomers of the two strands



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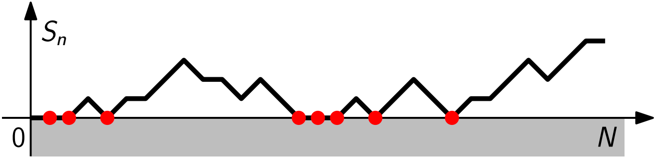

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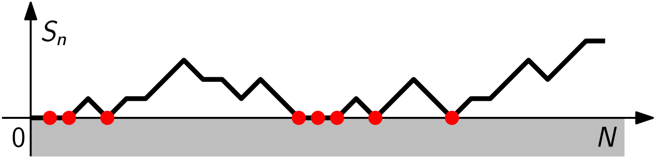

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Gibbs measure: $$\mathrm{P}_{N,\beta}^{\xi}(\{S\}) = \frac{1}{Z_{N,\beta}} \exp\big(-\beta H(S)\big)\, \mathrm{P}(\{S\})$$

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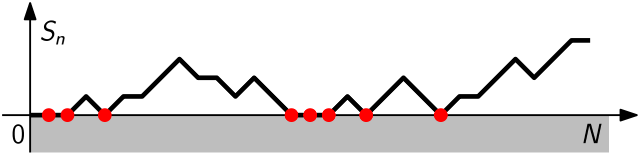

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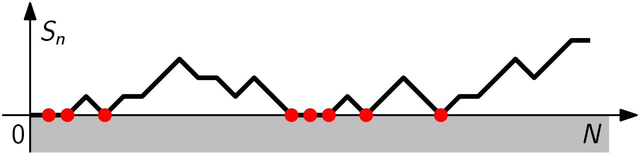

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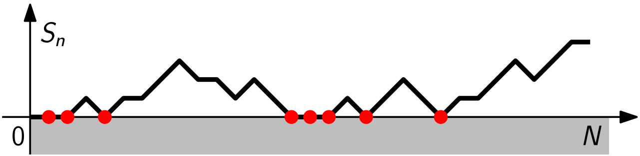

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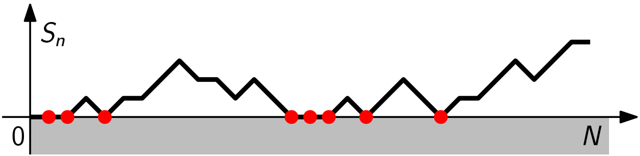

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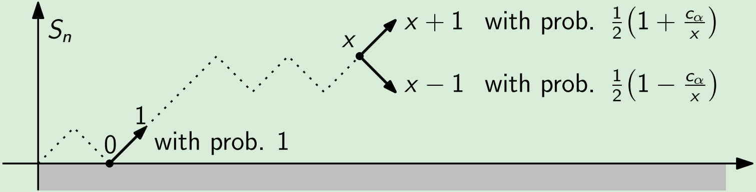

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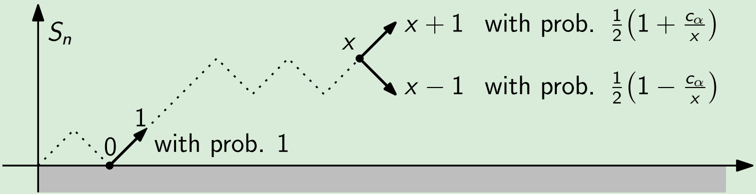

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Discretization of Bessel process: $\mathrm{d}S_t = \mathrm{d}B_t + \frac{c_\alpha}{S_t}\,\mathrm{d}t$

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Large scale properties of the Gibbs measure $\mathrm{P}^{\xi}_{N,\beta,h}$ (as $N \to \infty$)

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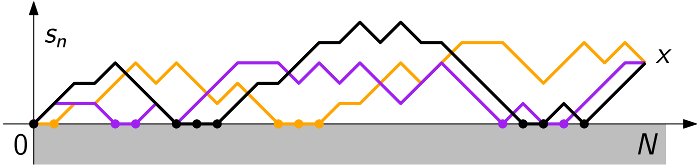

Outline

1. Hydrodynamics

2. The link with "polymers"

3. The random pinning model

4. Weak disorder regime

From discrete to continuum

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The diffusively rescaled Gibbs measure converges to a random distribution on $C([0,1], \mathbb{R})$ ("perturbation" of a Bessel processes)

[Joint work with N. Zygouras (Warwick) and R. Sun (Singapore)]

Thanks.