Polynomial Chaos and Scaling Limits of Disordered Systems

1. Introduction

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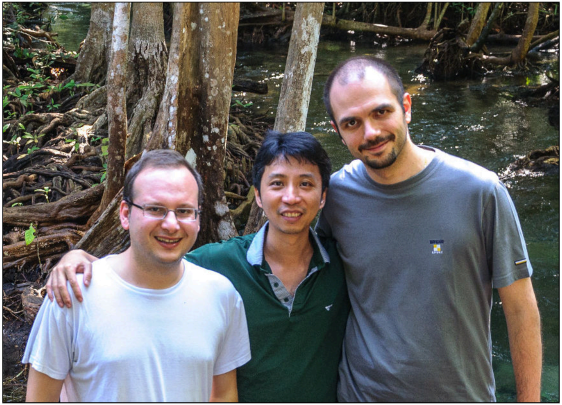
Coworkers



Joint work with Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

Overview

Consider a homogeneous system, described by a probability measure $\mathbf{P}^{\mathrm{ref}}$ (with "interesting" large scale properties)

Perturb it in a inhomogeneous way, defining a disordered system $\mathbf{P}^{\omega}$

$$\mathbf{P}^{\omega}(\mathrm{d}\sigma) \propto e^{\boldsymbol{H}^{\omega}(\sigma)} \, \mathbf{P}^{\mathrm{ref}}(\mathrm{d}\sigma) \qquad \qquad \omega \text{ random process}$$

Are large scale properties affected by (a small amount of) disorder?
Is the law $\mathbf{P}^{\omega}$ radically different from $\mathbf{P}^{\mathrm{ref}}$?

Disorder relevance vs. irrelevance

We are going to look at this problem in the weak disorder regime

General framework emerges ("model independent") $\rightsquigarrow$ Universality

Overview

General framework $\leftrightsquigarrow$ concrete examples

1. Directed polymer in random environment (perturb. of random walk)
2. Disordered pinning models (perturb. of renewal process)
3. Random-field Ising model

1'. Stochastic Heat Equation

(Inspired by [Alberts, Khanin, Quastel 2014] on directed polymers)

- This lecture is general introduction (motivation, key ideas, heuristic arguments, no proof) $\rightsquigarrow$ a lot of entropy, don't be scared!
- Next lectures devoted to specific issues (precise statements, proofs)

Ready to start! Introduce our key examples 1. 2. 3.

Outline

1. Homogeneous systems

2. Disordered systems

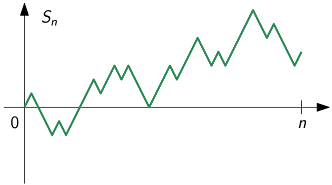
3. Main results

4. Stochastic Heat Equation

Outline

1. Homogeneous systems
2. Disordered systems
3. Main results
4. Stochastic Heat Equation

1. Random walk



$\mathbf{P}^{\mathrm{ref}}$ = law of symm. random walk on $\mathbb{Z}^d$

$$S = (S_n)_{n \ge 0}$$

with i.i.d. increments $S_n - S_{n-1}$

S attracted to α-stable Lévy process

Brownian motion

$$\begin{cases} \mathbf{E}^{\mathrm{ref}}[|S_1|^2] < \infty & \text{if } \alpha = 2 \\ \mathbf{P}^{\mathrm{ref}}(|S_1| > x) \sim \frac{C}{x^\alpha} & \text{if } 0 < \alpha < 2 \end{cases}$$

Alternative "language"

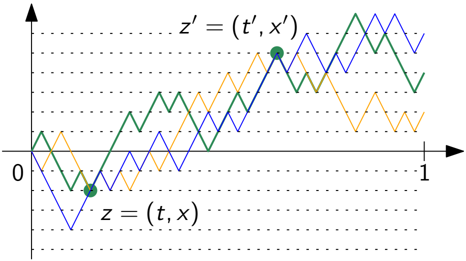
Define "spin" variable $\sigma_{n,x}$ in each space-time point

$$\sigma_{n,x} := \mathbb{1}_{\{S_n = x\}} \in \{0, 1\}$$

The random field $(\sigma_{n,x})_{(n,x) \in \mathbb{N}_0 \times \mathbb{Z}^d}$ is far from independent!

1. Random walk - large scale properties

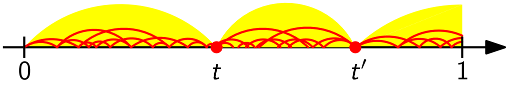
Diffusive rescaling $S^\delta = (\sqrt{\delta}\, S_{\frac{t}{\delta}})_{t \ge 0}$ $\mathbb{T}_\delta := \delta\mathbb{N}_0 \times (\sqrt{\delta}\mathbb{Z})^d$



S^δ converges in law to BM as $\delta \to 0$ (Donsker)

$$\frac{\mathbf{P}^{\mathrm{ref}}_\delta(\sigma_z = 1,\, \sigma_{z'} = 1)}{(\delta^{\frac{d}{2}})^2} \xrightarrow[\delta \to 0]{} \psi(z, z') = \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{\frac{d}{2}}} \, \frac{e^{-\frac{|x'-x|^2}{2(t'-t)}}}{(2\pi(t'-t))^{\frac{d}{2}}}$$

2. Renewal process



$\mathbf{P}^{\mathrm{ref}}$ = law of a renewal process (= RW with positive increments)

$$\mathbf{P}^{\mathrm{ref}}\big((\tau_{i+1} - \tau_i) = n\big) \sim \frac{c}{n^{1+\alpha}} \qquad \text{tail exponent } \alpha \in (0,1)$$

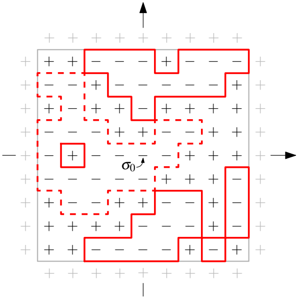
$\tau = \{0 = \tau_0 < \tau_1 < \tau_2 < \ldots\} \subseteq \mathbb{N}_0$ viewed as a random subset

"spins" $\sigma_n := \mathbb{1}_{\{n \in \tau\}} \in \{0,1\}$

$\mathbb{T}_\delta = \delta\mathbb{N}_0$ $\delta\tau \xrightarrow{d}$ α-stable regenerative set (as $\delta \to 0$)

$$\frac{\mathbf{E}_\delta^{\mathrm{ref}}[\sigma_t \, \sigma_{t'}]\mathbf{P}_\delta^{\mathrm{ref}}(t \in \tau,\ t' \in \tau)}{(\delta^{1-\alpha})^2} \xrightarrow[\delta \to 0]{} \psi(t,t') = \frac{c'}{t^{1-\alpha}} \, \frac{c'}{(t'-t)^{1-\alpha}}$$

3. Critical 2-dim. Ising model



Rescaled lattice $\mathbb{T}_\delta := (\delta\mathbb{Z})^2 \cap [-1,+1]^2$

Spin configurations $\sigma = (\sigma_x)_{x\in\mathbb{T}_\delta} \in \{-1, 1\}^{\mathbb{T}_\delta}$

$\mathbf{P}_\delta^{\mathrm{ref}} =$ critical 2d Ising model with "+" b.c.

$$\mathbf{P}_\delta^{\mathrm{ref}}(\sigma) \propto \exp\left(\beta_c \sum_{x\sim y\in\Omega} \sigma_x\sigma_y\right)$$

$$\beta_c = \tfrac{1}{2}\log(1+\sqrt{2})$$

Convergence to (distrib. valued) continuum field $(\boldsymbol{\sigma}_x)_{x\in[0,1]^2}$ as $\delta \to 0$

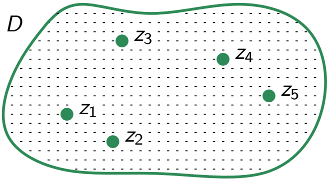
[Camia, Garban, Newman 2015]

$$\frac{\mathbf{E}_\delta^{\mathrm{ref}}[\sigma_z\,\sigma_{z'}]}{(\delta^{\frac{1}{8}})^2} \xrightarrow[\delta\to 0]{} \psi(z,z')$$

[Chelkak, Hongler, Izyurov 2015]

The general setup

Lattice $\mathbb{T}_\delta \subseteq D \subseteq \mathbb{R}^d$ (mesh $\approx \delta$)



$z \longmapsto$ two-valued field $\sigma_z \in \begin{cases} \pm 1 \\ \{0,1\} \end{cases}$

- $\mathcal{S} = \{0,1\}^{\mathbb{T}_\delta}$ space of spin configurations $\sigma = (\sigma_z)_{z \in \mathbb{T}_\delta}$
- $\mathbf{P}_\delta^{\mathrm{ref}}$ "interesting" probability on $\mathcal{S}$: non-trivial correlations

$$\underline{\mathbf{E}_\delta^{\mathrm{ref}}\left[\sigma_{z_1}\sigma_{z_2}\cdots\sigma_{z_k}\right]\mathbf{P}_\delta^{\mathrm{ref}}\left(\sigma_{\{z_1,z_2,\ldots,z_k\}}=1\right)}$$

$\exists\gamma >$

$$0: \quad \frac{\mathbf{P}_\delta^{\mathrm{ref}}\left(\sigma_{\{z_1,z_2,\ldots,z_k\}}=1\right)}{(\delta^\gamma)^k} \xrightarrow[\delta\to 0]{} \psi_k(z_1,\ldots,z_k)$$

Typically $\mathbf{P}_\delta^{\mathrm{ref}}$ has a non-trivial continuum limit as $\delta \to 0$

Keep in mind your favorite example (e.g. random walk)

Outline

Enters disorder

$(\omega_z)_{z \in \mathbb{T}_\delta}$ i.i.d. random variables (e.g. $\mathcal{N}(0,1)$)

$\mathbb{E}[\omega_z] = 0 \qquad \mathbb{E}[\omega_z^2] = 1 \qquad \lambda(\beta) = \log \mathbb{E}[e^{\beta\omega_z}] < \infty$

Each site $z \in \mathbb{T}_\delta$ carries a charge ω_z that can be $\begin{cases} > 0 & \text{reward} \\ < 0 & \text{penalty} \end{cases}$

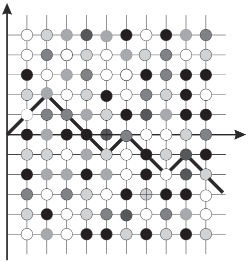
Spatial inhomogeneities in $\mathbf{P}_\delta^{\mathrm{ref}}$ $\rightsquigarrow$ new probability law $\mathbf{P}_\delta^{\omega}$

Gibbs measure: $\mathbf{P}_\delta^{\omega}(\mathrm{d}\sigma) := \frac{1}{\boldsymbol{Z}_\delta^{\omega}} e^{\boldsymbol{H}^{\omega}(\sigma)} \, \mathbf{P}_\delta^{\mathrm{ref}}(\mathrm{d}\sigma)$

$(-)$ Energy: $\sigma \mapsto \boldsymbol{H}^{\omega}(\sigma) := \sum_{z \in \mathbb{T}_\delta} \big(\beta\,\omega_z \quad + \quad h \quad - \lambda(\beta) + h\big)\, \sigma_z$

$\beta \geq 0$ disorder strength $\qquad h \in \mathbb{R}$ disorder bias

1. Directed Polymer in Random Environment (random walk)



- Symmetric random walk $S = (S_n)_{n \geq 0}$ on $\mathbb{Z}^d$ attracted to BM (finite variance)
- (Forget rescaled lattice $\mathbb{T}_\delta = \delta\mathbb{N}_0 \times (\sqrt{\delta}\mathbb{Z})^d$)
- $\omega_{n,x} > 0$ reward $\omega_{n,x} < 0$ penalty
- "spin" $\sigma_{n,x} := \mathbb{1}_{\{S_n = x\}} \in \{0, 1\}$

Directed polymer in random environment ($N = 1/\delta$ steps)

$$\mathbf{P}^\omega(S) = \frac{1}{Z_\delta^\omega} e^{\sum_{n=1}^N (\beta\omega_{n,S_n} - \lambda(\beta) + h)} \mathbf{P}^{\mathrm{ref}}(S)$$

RW paths in corridors of large $\omega > 0$ have high probability (energy gain)

...but such paths are few! (entropy loss) $\rightsquigarrow$ Who wins?

1. Directed Polymer in Random Environment (random walk)

- $[d \geq 3, \;\; \beta > 0$ small$]$ $\quad$ $\mathbf{P}^\omega$ "similar" to $\mathbf{P}^{\text{ref}}$ $\quad$ (entropy wins)

$$\frac{S_N}{\sqrt{N}} \text{ under } \mathbf{P}^\omega \xrightarrow[N\to\infty]{d} \mathcal{N}(0,1) \qquad (\mathbb{P}(\mathrm{d}\omega)\text{-a.s.})$$

i.e. the same under $\mathbf{P}^{\text{ref}}$ $\quad$ [Imbrie, Spencer 1988] [Bolthausen 1989]

- $[d \leq 2, \;\;$ any $\beta > 0]$ $\quad$ $\mathbf{P}^\omega$ "different" from $\mathbf{P}^{\text{ref}}$ $\quad$ (energy wins)

$$\max_{x\in\mathbb{Z}^d} \mathbf{P}^\omega\big(S_N = x\big) \geq c > 0 \qquad (\mathbb{P}(\mathrm{d}\omega)\text{-a.s.})$$

unlike $\mathbf{P}^{\text{ref}}(S_N = x) = O(N^{-\frac{1}{2}}) = o(1)$ $\quad$ [Carmona, Hu 2002]
[Comets, Shiga, Yoshida 2003]

For DPRE disorder is irrelevant for $d \geq 3$ and relevant for $d \leq 2$
($d = 2$ is actually marginally relevant, cf. below)

Disorder Relevance vs. Irrelevance

Does arbitrarily small disorder affect large scale properties?

Is $\mathbf{P}_\delta^\omega$ qualitatively different from $\mathbf{P}_\delta^{\mathrm{ref}}$?

[$\delta \to 0$ ($N \to \infty$) with fixed $\beta > 0$ (suitable h)]

YES: model is disorder relevant NO: model is disorder irrelevant

2. Disordered Pinning Model (renewal process + disorder)

$$\mathbf{P}^{\mathrm{ref}}(\tau_1 = n) \sim \frac{c}{n^{1+\alpha}}$$

$\left[\alpha > \frac{1}{2}\right]$ disorder relevant $\left[\alpha < \frac{1}{2}\right]$ disorder irrelevant

$\left[\alpha = \frac{1}{2}\right]$ marginal: (ir)relevance depends on finer details

(cf. free energy and critical exponents) [References: ...]

3. Random Field Ising Model (2d critical Ising + disorder) relevant

What are we going to do?

We focus on models $\mathbf{P}_\delta^\omega$ which are disorder relevant

Any fixed disorder strength $\beta > 0$, no matter how small, has dramatic effects in the large scale regime $\delta \to 0$ (i.e. $N \to \infty$)

Weak disorder regime

Can we tune $\beta \to 0$ as $\delta \to 0$ ("keep disorder under control")

and still see interesting effects on $\mathbf{P}_\delta^\omega$?

YES! This is the goal of our course

Very robust approach $\leftrightsquigarrow$ Universality

Before describing the results, let us present a concrete problem

Disordered continuum model?

Consider DPRE in $d = 1$ (random walk + disorder)

$$\mathbf{P}^{\omega}(S) \propto e^{\sum_{n=1}^{N} \beta\, \omega(n,S_n)}\, \mathbf{P}^{\mathrm{ref}}(S)$$

Can we define its continuum analogue (BM + disorder)? Naively

$$\mathcal{P}^{W}(\mathrm{d}B) \propto e^{\int_0^1 \hat{\beta}\, W(t,B_t)\, \mathrm{d}t}\, \mathcal{P}^{\mathrm{ref}}(\mathrm{d}B)$$

$\mathcal{P}^{\mathrm{ref}}$ = law of BM $\qquad W(t,x)$ = white noise on $\mathbb{R}^2$ (space-time)

- $\int_0^1 W(t,B_t)\,\mathrm{d}t$ ill-defined. Regularization? NO! [$\mathcal{P}^{W} \not\ll \mathcal{P}^{\mathrm{ref}}$!]

How do we proceed? Recall that $\mathbf{P}_{\delta}^{\mathrm{ref}} \xrightarrow{d} \mathcal{P}^{\mathrm{ref}}$ as $\delta \to 0$

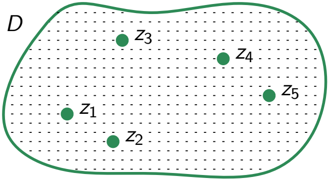
Theorem [Alberts, Khanin, Quastel 2014b] [C., Sun, Zygouras 2015+]

Tuning $\beta \sim \hat{\beta}\, \delta^{\frac{1}{4}}$ as $\delta \to 0$, the DPRE $\mathbf{P}_{\delta}^{\omega}$ has a non-trivial limit $\mathcal{P}^{W}$

Outline

Key assumption (disorder relevant vs. marginal)

- Lattice $\mathbb{T}_\delta \subseteq D \subseteq \mathbb{R}^d$ (mesh $\approx \delta$)
  Two-valued field $\sigma = (\sigma_z)_{z \in \mathbb{T}_\delta}$
- $\mathbf{P}_\delta^{\mathrm{ref}}$ interesting probability for σ



$$\exists \gamma > 0 : \quad \frac{\mathbf{E}_\delta^{\mathrm{ref}}\left[\sigma_{z_1}\,\sigma_{z_2}\cdots\sigma_{z_k}\right]}{(\delta^\gamma)^k} \xrightarrow[\delta\to 0]{} \xrightarrow[\delta\to 0]{L^2(D)} \psi_k(z_1,\ldots,z_k) \quad (\star)$$

L^2 characterizes disorder relevant regime! (Harris criterion)

1. DPRE. $\psi(t,x) = \dfrac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{\frac{d}{2}}}$ $\qquad L^2([0,1]\times\mathbb{R}^d) \rightsquigarrow d < 2$
2. Pinning. $\psi(t) = \dfrac{c'}{t^{1-\alpha}}$ $\qquad L^2([0,1]) \rightsquigarrow \alpha > \frac{1}{2}$ $\qquad$ marginal!

The partition function

Recall the definition of the disordered system

$$\mathbf{P}_{\delta}^{\omega}(\mathrm{d}\sigma) := \frac{1}{\boldsymbol{Z}_{\delta}^{\omega}} e^{\boldsymbol{H}^{\omega}(\sigma)} \, \mathbf{P}_{\delta}^{\mathrm{ref}}(\mathrm{d}\sigma)$$

The normalizing constant $\boldsymbol{Z}_{\delta}^{\omega}$ is called partition function

$$\boldsymbol{Z}_{\delta}^{\omega} = \mathbf{E}^{\mathrm{ref}}\left[e^{\boldsymbol{H}^{\omega}(\sigma)}\right] = \mathbf{E}^{\mathrm{ref}}\left[\exp\left(\sum_{z \in \mathbb{T}_{\delta}} (\beta\omega_z - \lambda(\beta) + h)\, \sigma_z\right)\right]$$

$\boldsymbol{Z}_{\delta}^{\omega}$ is a complicated function of i.i.d. random variables $(\omega_x)_{x \in \mathbb{T}_{\delta}}$

DPRE $\qquad \boldsymbol{Z}_{\delta}^{\omega} = \mathbf{E}^{\mathrm{ref}}\left[\exp\left(\sum_{n=1}^{N} (\beta\omega_{(n,S_n)} - \lambda(\beta) + h)\right)\right]$

Sample the ω's along a path of the RW $(S_n)_{n \geq 0}$, then average their exp

Plan of the course

The partition function $\boldsymbol{Z}_{\delta}^{\omega}$ encodes the key properties of $\mathbf{P}_{\delta}^{\omega}$

Key Result (scaling limit of $\boldsymbol{Z}_{\delta}^{\omega}$)

The partition function $\boldsymbol{Z}_{\delta}^{\omega}$ has a non-trivial limit in distribution $\mathcal{Z}^{W}$ (continuum partition function) when $\beta, h \to 0$ at suitable rates as $\delta \downarrow 0$

- Lecture II. Key Result for disorder relevant systems
  Multi-linear CLT based on a Lindeberg principle
- Lecture III. $\mathcal{Z}^{W} \rightsquigarrow$ disordered continuum model
- Lecture IV. $\mathcal{Z}^{W} \rightsquigarrow$ free energy estimates (sketch)
  Intro to marginal relevance
- Lecture V. Key Result for marginal systems
  (DPRE $d = 2$, Pinning $\alpha = \frac{1}{2}$, 2d Stochastic Heat Equation)

Sketch of the approach: Polynomial Chaos

1. Linearization. Since $\sigma_x \in \{0,1\}$, every function of σ_x is linear

$$Z_\delta^\omega = \mathbf{E}_\delta^{\mathrm{ref}}\left[e^{\sum_{x \in \mathbb{T}_\delta} (\beta\omega_x - \lambda(\beta) + h)\sigma_x} \right] = \mathbf{E}_\delta^{\mathrm{ref}}\left[\prod_{x \in \mathbb{T}_\delta} e^{(\beta\omega_x - \lambda(\beta) + h)\sigma_x} \right] = \mathbf{E}_\delta^{\mathrm{ref}}\left[\prod_{x \in \mathbb{T}_\delta} (1$$

where $X_x := e^{\beta\omega_x - \lambda(\beta) + h} - 1$. New random variables (X_x) with

$$\mathbb{E}[X_x] \simeq h \qquad \mathbb{V}\mathrm{ar}[X_x] \simeq \beta^2$$

2. High-temperature expansion. By a binomial expansion of the product

$$Z_\delta^\omega = \sum_{k=0}^{|\mathbb{T}_\delta|} \frac{1}{k!} \sum_{(x_1,\ldots,x_k) \in (\mathbb{T}_\delta)^k} \mathbf{E}_\delta^{\mathrm{ref}}\left[\sigma_{x_1} \cdots \sigma_{x_k}\right] X_{x_1} \cdots X_{x_k}$$

Multilinear polynomial! Formally replace $\sum \rightsquigarrow \int$ and $X_{x_i} \rightsquigarrow W(\mathrm{d}x_i)$

Justified by Lindeberg principle (Lecture II)

Key Result A (disorder relevant systems) – Lecture II

Theorem A [C., Sun, Zygouras '15+]

Let $\mathbf{P}_\delta^{\mathrm{ref}}$ satisfy $(\star)$ with exponent γ (dimension d). Assume $\sigma_x \in \{0,1\}$.

The partition function has a non-trivial limit in law: $\boldsymbol{Z}_\delta^\omega \xrightarrow[\delta\downarrow 0]{d} \mathcal{Z}^W$

provided we scale $\beta, h \to 0$ appropriately:

$$\beta := \hat\beta\, \delta^{d/2-\gamma} \qquad h := \hat h\, \delta^{d-\gamma} \qquad (\hat\beta,\ \hat h \text{ fixed})$$

The limit $\mathcal{Z}^W$ is explicit function of $W(\mathrm{d}x) :=$ white noise on $\mathbb{R}^d$

$$\mathcal{Z}^W := \sum_{k=0}^{\infty} \frac{1}{k!} \int \cdots \int_{\Omega^k} \psi_k(z_1,\ldots,z_k) \prod_{i=1}^{k} \big(\hat\lambda\, W(\mathrm{d}z_i) + \hat h\, \mathrm{d}z_i\big)$$

Wiener chaos expansion

A concrete example: Disordered Pinning Model

Pinning Models with $\alpha > \frac{1}{2}$ (disorder relevant) $[\,\delta = \frac{1}{N}\,]$

$$Z_\delta^\omega \approx 1 + \sum_{0<n\le N} \frac{X_n}{n^{1-\alpha}} + \sum_{0<m<n\le N} \frac{X_m \quad X_n}{m^{1-\alpha}(n-m)^{1-\alpha}} + \dots$$

Rescaling $\beta \sim \delta^{\alpha - \frac{1}{2}}$ ($h \equiv 0$ for simplicity)

$$\xrightarrow[\delta \to 0]{d} 1 + \int_{0<t<1} \frac{\mathrm{d}W_t}{t^{1-\alpha}} + \int_{0<s<t<1} \frac{\mathrm{d}W_s \quad \mathrm{d}W_t}{s^{1-\alpha}(t-s)^{1-\alpha}} + \dots$$

Intriguing question: what happens for $\alpha = \frac{1}{2}$?

This case is marginal, like DPRE for $d = 2$

(and also to 2d Stochastc Heat Equation, see below)

Key Result B (marginally relevant systems) – Lecture V

Theorem B [C., Sun, Zygouras '15b]

Consider DPRE $d = 2$ or Pinning $\alpha = \frac{1}{2}$ or 2d SHE
(or long-range DPRE with $d = 1$ and Cauchy tails)

Rescaling $\beta := \dfrac{\hat{\beta}}{\sqrt{\log \frac{1}{\delta}}}$ (and $h \equiv 0$) the partition function converges in law to an explicit limit:

$$\mathbf{Z}_\delta^\omega \xrightarrow[\delta \downarrow 0]{d} \mathcal{Z}^W = \begin{cases} \text{log-normal} & \text{if } \hat{\beta} < 1 \\ 0 & \text{if } \hat{\beta} \geq 1 \end{cases}$$

$$\mathcal{Z}^W = \exp\left\{ \int_0^1 \frac{\hat{\beta}}{\sqrt{1 - \hat{\beta}^2 t}} \mathrm{d}W_t - \frac{1}{2} \int_0^1 \frac{\hat{\beta}^2}{1 - \hat{\beta}^2 t} \mathrm{d}t \right\} \qquad (\hat{\beta} < 1)$$

Harris Criterion

Pointwise convergence of correlation typically implies

$$\psi_k(z_1,\ldots,z_k) \underset{z_i \to z_j}{\approx} \frac{1}{|z_i - z_j|^{\gamma}} \qquad \psi_k \in L^2_{\mathrm{loc}} \iff \gamma < \frac{d}{2}$$

This restriction is not technical, but substantial!

Harris Criterion (1974)

Decide relevance/irrelevance looking at homogeneous model $\mathbf{P}_\delta^{\mathrm{ref}}$ through its correlation length exponent ν

$\nu < \frac{2}{d}$ relevant $\qquad \nu = \frac{2}{d}$ marginal $\qquad \nu > \frac{2}{d}$ irrelevant

In our context $\nu = \frac{1}{d-\gamma}$ $\qquad \nu < \frac{2}{d}$ coincides with $\boxed{\gamma < \frac{d}{2}}$

Outline

The Stochastic Heat Equation

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x) + \tilde\beta\, W(t,x)\, u(t,x) \\ u(0,x) \equiv 1 \end{cases} \qquad (t,x) \in [0,\infty) \times \mathbb{R}^d$$

What if $W(t,x)$ is (space-time) white noise on $[0,\infty) \times \mathbb{R}^d$?

Mollification in space: fix $j \in C_0^\infty(\mathbb{R}^d)$ and set $j_\delta(z) := \delta^{-1} j(\delta^{\frac{1}{d}} z)$

$$W_\delta(t,x) := \int_{\mathbb{R}^d} j_\delta(x-y)\, W(t,y)\, \mathrm{d}y$$

($t \mapsto \int_0^t W_\delta(s,x)\,\mathrm{d}s$ is a one-dimensional BM with variance $\|j_\delta\|_2^2$)

Solution $u_\delta(t,x)$ admits generalized Feynman-Kac formula

$$u_\delta(t,x) = \mathrm{E}_x\left[\exp\left\{\tilde\beta \int_0^t W_\delta(t-s, B_s)\,\mathrm{d}s - \frac{1}{2}\tilde\beta^2 t \|j_\delta\|_2^2\right\}\right]$$

The Stochastic Heat Equation

Does $u_\delta(t,x)$ admit a limit as $\delta \to 0$?

Analogy with the partition function of DPRE

$$u_{\delta,\tilde\beta}(1,0) \overset{d}{\approx} Z^{\omega}_{\delta,\beta=\delta^{\frac{2-d}{4}}\tilde\beta} = \begin{cases} Z^{\omega}_{\delta,\beta=\delta^{\frac{1}{4}}\tilde\beta} & (d=1) \\ Z^{\omega}_{\delta,\beta=\tilde\beta} & (d=2) \end{cases}$$

Our results for DPRE can be transfered to SHE

By a time-reversal in W and a space-time rescaling, for $(t,x)=(1,0)$

$$u_\delta(1,0) = \mathrm{E}_0\left[:\exp: \left\{ \tilde\beta\delta^{\frac{2-d}{4}} \int_0^{\frac{1}{\delta}} \left(\int_{\mathbb{R}^d} j(B_s - z)\, W(s,z)\, \mathrm{d}z \right) \mathrm{d}s \right\} \right]$$

using the shorthand $:\exp:\{Y\} = \exp(Y - \frac{1}{2}\mathbb{V}\mathrm{ar}[Y])$

The Stochastic Heat Equation

1d SHE

Fix $\tilde{\beta} > 0$. The regularized solution $u_\delta(t,x)$ has a limit in law as $\delta \to 0$ (explicit Wiener chaos expansion)

This is well-know in the literature. This is possibly new.

2d SHE

Fix $\hat{\beta} > 0$ and rescale $\tilde{\beta} := \frac{\hat{\beta}}{\sqrt{\log \frac{1}{\delta}}}$. The regularized solution $u_\delta(t,x)$

has a limit in law as $\delta \to 0$ that is $\begin{cases} \text{log-normal} & \text{if } \hat{\beta} < 1 \\ 0 & \text{if } \hat{\beta} \geq 1 \end{cases}$

$u_\delta(t,x)$ and $u_\delta(t',x')$ are asympt. independent for $(t,x) \neq (t',x')$
($\hat{\beta} < 1$ is weak disorder) Interesting multi-scale correlations (Lecture V)

References

- T. Alberts, K. Khanin, J. Quastel
  The intermediate disorder regime for directed polymers in dimension 1 + 1
  Ann. Probab. 42 (2014), 1212–1256
- T. Alberts, K. Khanin, J. Quastel
  The Continuum Directed Random Polymer
  J. Stat. Phys. 154 (2014), 305–326
- E. Bolthausen
  A note on diffusion of directed polymers in a random environment
  Commun. Math. Phys. 123 (1989), 529534.
- F. Camia, C. Garban, C.M. Newman
  Planar Ising magnetization field I. Uniqueness of the critical scaling limit
  Ann. Probab. 43 (2015), 528–571

References

- F. Caravenna, R. Sun, N. Zygouras
  Polynomial chaos and scaling limits of disordered systems
  J. Eur. Math. Soc. (JEMS), to appear
- F. Caravenna, R. Sun, N. Zygouras
  Universality in marginally relevant disordered systems
  preprint (2015)
- P. Carmona, Y. Hu
  On the partition function of a directed polymer in a random environment
  Probab.Theory Related Fields 124 (2002), 431–457
- D. Chelkak, C. Hongler, C. Izyurov
  Conformal invariance of spin correlations in the planar Ising model
  Ann. Math. 181 (2015), 1087–1138

References

- F. Comets, T. Shiga, N. Yoshida
  Directed Polymers in Random Environment: Path Localization and Strong Disorder
  Bernoulli 9 (2003), 705–723
- J.Z. Imbrie, T. Spencer
  Diffusion of directed polymer in a random environment
  J. Stat. Phys. 52 (1988), 609–626