Polynomial Chaos and Scaling Limits of Disordered Systems

2. Lindeberg Principle and multi-linear CLT

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Overview

In the first lecture we saw the key role of the partition function

$$Z_N^\omega = \mathbf{E}^{\mathrm{ref}}\big[e^{\mathcal{H}_N^\omega(S)}\big] = \mathbf{E}^{\mathrm{ref}}\Big[e^{\sum_{n=1}^N \sum_{x\in\mathbb{Z}^d}(\beta\omega(n,S_n)-\lambda(\beta))}\Big] \qquad \text{(directed polymer)}$$

Z_N^ω is a complicated function of the i.i.d. random field $\omega = (\omega(n,x))$

Z_N^ω is a simple function of another i.i.d. random field $X = (X(n,x))$

Multi-linear polynomial: $$Z_N^\omega = \sum_{A\subseteq\{1,\ldots,N\}} \phi(A) \prod_{i\in A} X_i$$

Goal of this lecture

Study convergence in distribution of multi-linear polynomials

Outline

1. Polynomial chaos
2. Lindeberg Principle
3. White noise and Wiener chaos
4. CLT for polynomial chaos
5. Proofs

Outline

Polynomial chaos

$\mathbb{T}$ = finite or countable index set $(\mathbb{T} = \{1, \ldots, N\},\ \mathbb{T} = \mathbb{N},\ \mathbb{T} = \mathbb{Z}^d)$

Multi-linear polynomial $\Psi(x)$ in the variables $(x_i)_{i \in \mathbb{T}}$

$$\Psi(x) = \sum_{I \subseteq \mathbb{T}} \psi(I) \prod_{i \in I} x_i \qquad \text{(sum restricted to } |I| < \infty)$$

[$\Psi(x)$ is a formal polynomial $\longleftrightarrow$ kernel $(\psi(I))_{I \subseteq \mathbb{T}}$]

Polynomial chaos

$$Z = \Psi(X) = \sum_{I \subseteq \mathbb{T}} \psi(I) \prod_{i \in I} X_i$$

$$= \psi(\emptyset) + \sum_{i \in \mathbb{T}} \psi(i)\, X_i + \frac{1}{2} \sum_{i \neq j \in \mathbb{T}} \psi(i,j)\, X_i\, X_j + \ldots$$

with $X = (X_i)_{i \in \mathbb{T}}$ independent (possibly non i.d.) random variables in L^2

Polynomial chaos

$$Z = \Psi(X) = \sum_{I \subseteq \mathbb{T}} \psi(I)\, X^I \qquad \text{with} \quad X^I := \prod_{i \in I} X_i$$

- In case $|\mathbb{T}| < \infty$ no problem
- In case $|\mathbb{T}| = \infty$ we mean $Z = \lim_{N\to\infty} \Psi_{\mathbb{T}_N}(X)$ in prob. $(\mathbb{T}_N \uparrow \mathbb{T})$

If $\mathbb{E}[X_i] = 0$ then $\mathbb{E}[X^I X^J] = \mathbb{1}_{\{I=J\}}$ $\rightsquigarrow$ $\Psi(X)$ well-defined in L^2 if

$$\sum_{I \subseteq \mathbb{T}} \psi(I)^2 < \infty$$

If $\mathbb{E}[X_i] = \mu_i \in \mathbb{R}$ $\rightsquigarrow$ $\Psi(X)$ well-defined in L^2 if

$$\sum_{i \in \mathbb{T}} \mu_i^2 < \infty \qquad \text{and} \qquad \sum_{I \subseteq \mathbb{T}} (1+\varepsilon)^{|I|}\, \psi(I)^2 < \infty \quad \text{for some} \quad \varepsilon > 0$$

Variance and influences

Fix a multi-linear polynomial

$$\Psi(x) = \sum_{I \subseteq \mathbb{T}} \psi(I)\, x^I \qquad \text{with} \quad x^I := \prod_{i \in I} x_i$$

$$\mathsf{C}_\Psi := \sum_{I \subseteq \mathbb{T},\ I \neq \emptyset} \psi(I)^2 = \mathbb{V}\mathrm{ar}[\Psi(X)]$$

$$\mathrm{Inf}_i[\Psi] := \sum_{I \subseteq \mathbb{T},\, I \ni i} \psi(I)^2 = \mathbb{E}\Big[\mathbb{V}\mathrm{ar}\big[\,\Psi(X)\,\big|\,X_{\mathbb{T}\setminus\{i\}}\,\big]\Big]$$

For any family of r.v.'s $X = (X_i)_{i \in \mathbb{T}}$ with $\mathbb{E}[X_i] = 0$ $\mathbb{V}\mathrm{ar}[X_i] = 1$

$\mathrm{Inf}_i[\Psi]$ quantifies how much $\Psi(x)$ depends on the variable x_i

Noise sensitivity [Benjaimini, Kalai, Schramm 2001] [Garban, Steif 2012]

Outline

1. Polynomial chaos

2. Lindeberg Principle

3. White noise and Wiener chaos

4. CLT for polynomial chaos

5. Proofs

Lindeberg Principle

If influences $\mathrm{Inf}_i(\Psi)$ are small, the law of $\Psi(X)$ is insensitive to the details of the laws of the individual X_i's

- Fix a multi-linear polynomial $\Psi(x) = \sum_{I \subseteq \mathbb{T},\, |I| \le \ell} \psi(I)\, x^I$ of degree ℓ
- $X = (X_i)_{i \in \mathbb{T}}$, $X' = (X'_i)_{i \in \mathbb{T}}$ indep. with zero mean, unit variance

$$m_3 := \max_{i \in \mathbb{T}} \left(\mathbb{E}[|X_i|^3] \vee \mathbb{E}[|X'_i|^3] \right) < \infty$$

Theorem [Mossel, O'Donnel, Oleszkiewicz 2010]

$$\mathrm{dist}\big(\Psi(X), \Psi(X')\big) := \sup_{f \in C^3:\ \|f'\|_\infty, \|f''\|_\infty, \|f'''\|_\infty \le 1} \Big| \mathbb{E}\big[f(\Psi(X))\big] - \mathbb{E}\big[f(\Psi(X'))\big] \Big|$$

$$\le 30^\ell\, C_\Psi\, {m_3}^\ell \sqrt{\max_{i \in \mathbb{T}} \left(\mathrm{Inf}_i[\Psi] \right)}$$

Lindeberg Principle

We can go beyond finite 3rd moment. Define the truncated moments

$$m_2^{>M} := \sup_{X \in \{X_i, X_i'\}} \mathbb{E}\left[X^2 \, \mathbb{1}_{\{|X|>M\}}\right] \qquad m_3^{\leq M} := \sup_{X \in \{X_i, X_i'\}} \mathbb{E}\left[|X|^3 \, \mathbb{1}_{\{|X|\leq M\}}\right]$$

Theorem [C., Sun, Zygouras 2015+]

$$\mathrm{dist}\big(\Psi(X), \Psi(X')\big)$$
$$\leq e^{\frac{2}{\varepsilon}\sum_{i\in\mathbb{T}} \mu_i^2} \, 70^{\ell+1} \, \mathrm{C}_{\Psi^\varepsilon} \left\{ m_2^{>M} + \left(m_3^{\leq M}\right)^{\ell} \sqrt{\max_{i\in\mathbb{T}} \left(\mathrm{Inf}_i[\Psi^\varepsilon]\right)} \right\}$$

- Explicit, non-asymptotic estimate!
- Extension to the case $\mathbb{E}[X_i] = \mathbb{E}[X_i'] = \mu_i \neq 0$

$$\Psi^\varepsilon(x) = \sum_{I \subseteq \mathbb{T}} (1+\varepsilon)^{|I|} \, \psi(I) \, x^I$$

Lindeberg Principle

$$\mathrm{dist}\Big(\Psi(X),\Psi(X')\Big) \;\le\; 70^{\ell+1}\,C_\Psi\left\{m_2^{>M} + \Big(m_3^{\le M}\Big)^{\ell}\sqrt{\max_{i\in\mathbb{T}}\big(\mathrm{Inf}_i[\Psi]\big)}\right\}$$

Corollary

Consider a family $(\Psi_\delta)_{\delta>0}$ of multi-linear polynomials

- Assume $\sup_{\delta>0} C_{\Psi_\delta} < \infty$ $\qquad \max_{i\in\mathbb{T}_\delta}\big(\mathrm{Inf}_i[\Psi_\delta]\big) \xrightarrow[\delta\to 0]{} 0$
- Take $(X_{\delta,i})$, $(X'_{\delta,i})$ with zero mean, unit variance and u.i. squares

$$\lim_{M\to\infty} m_2^{>M} := \sup_{X\in\{X_{\delta,i},X'_{\delta,i}\}} \mathbb{E}\big[X^2\,\mathbb{1}_{\{|X|>M\}}\big] = 0$$

Then

$$\boxed{\mathrm{dist}\Big(\Psi_\delta(X_\delta),\Psi_\delta(X'_\delta)\Big) \xrightarrow[\delta\to 0]{} 0}$$

Does $\Psi_N(X_\delta)$ have a limit in law as $\delta\to 0$? Check it for Gaussian X_δ's !

Outline

White noise (1 dim.)

We are familiar with (1-dim.) Brownian motion $B = (B(t))_{t \geq 0}$

We are interested in its derivative "$W(t) := \frac{\mathrm{d}}{\mathrm{d}t} B(t)$" called white noise

[Well-defined as a (random) Schwarz distribution $\rightsquigarrow$ Max's course]

Think of W as a stochastic process $W = (W(\cdot))$ indexed by

$$\text{Intervals } I = [a,b] \quad \longmapsto \quad W(I) = B(b) - B(a) \sim \mathcal{N}(0, b-a)$$

$$\text{Borel sets } A \in \mathcal{B}(\mathbb{R}) \quad \longmapsto \quad W(A) = \int_{\mathbb{R}} \mathbb{1}_A(t)\, \mathrm{d}B(t) \sim \mathcal{N}(0, |A|)$$

W is a Gaussian process with

$$\mathbb{E}[W(A)] = 0 \qquad \mathbb{C}\mathrm{ov}[W(A), W(B)] = |A \cap B|$$

This can be taken as the definition of W $\rightsquigarrow$ multi-dimensional W

White noise

White noise on $\mathbb{R}^d$

It is a Gaussian process $W = (W(A))_{A \in \mathcal{B}(\mathbb{R}^d)}$ with

$$\mathbb{E}[W(A)] = 0 \qquad \mathbb{C}\mathrm{ov}[W(A), W(B)] = |A \cap B|$$

It is the continuum analogue of i.i.d. field $W(A) = \sum_{z \in A} X_z$ for $A \subseteq \mathbb{Z}^d$

- Existence OK (Kolmogorov)
- $W(A) \sim \mathcal{N}(0, |A|)$ $\quad W(A)$ indep. of $W(B)$ for $A \cap B = \emptyset$
- $\forall\ (A_n)_{n \in \mathbb{N}}$ disjoint $\implies W\left(\bigcup_{n \in \mathbb{N}} A_n\right) \overset{\text{a.s.}}{=} \sum_{n \in \mathbb{N}} W(A_n)$

  Almost a random signed measure on $\mathbb{R}^d$... but not quite:
  "$\forall$" and "a.s." cannot be exchanged! ($\rightsquigarrow$ infinite variation)

[$W(A)$ is equivalence class of random variables]

Single stochastic integrals w.r.t. white noise

We can define single stochastic integrals w.r.t. white noise $\int f(x)\, W(\mathrm{d}x)$
(next we consider multiple ones: $\int f(x,y)\, W(\mathrm{d}x) W(\mathrm{d}y)$, etc.)

For simple functions $f = \sum_{i=1}^k c_i \mathbb{1}_{A_i}$ with $c_i \in \mathbb{R}$

$$\int_{\mathbb{R}^d} f(x)\, W(\mathrm{d}x) \;:=\; \sum_{i=1}^{k} c_i W(A_i) \;\sim\; \mathcal{N}(0, \|f\|^2_{L^2(\mathbb{R}^d)})$$

Isometry $L^2(\mathbb{R}^d) \longrightarrow L^2(\Omega_W)$ $\rightsquigarrow$ Extends to all $f \in L^2(\mathbb{R}^d)$

Set for short $W(f) := \int f(x)\, W(\mathrm{d}x)$ and keep in mind the Ito isometry

$$\mathbb{E}[W(f)] = 0 \qquad \mathbb{E}[W(f)^2] = \|f\|^2_{L^2(\mathbb{R}^d)}$$

Multiple stochastic integrals w.r.t. white noise

In a “product measure” fashion, we define

$$W^{\otimes 2}(g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x,y)\, W(\mathrm{d}x) W(\mathrm{d}y) := \sum_{i=1}^{k} c_i W(A_i) W(B_i)$$

for $g(x,y) = \sum_{i=1}^k c_i \mathbb{1}_{A_i \times B_i}(x,y)$ with $A_i \cap B_i = \emptyset$ (“avoid diagonals”)

- Such simple functions are dense in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$

We can restrict to symmetric functions $g(x,y) = g(y,x)$ and note that

$$\mathbb{E}[W^{\otimes 2}(g)] = 0 \qquad \mathbb{E}[W^{\otimes 2}(g)^2] = 2\, \|g\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}$$

We can extend $W^{\otimes 2}(g)$ to every $g \in L^2_{\text{sym}}(\mathbb{R}^d \times \mathbb{R}^d)$

Note that $\mathbb{E}[W^{\otimes 1}(f), W^{\otimes 2}(g)] = 0$ for all f, g

Multiple stochastic integrals w.r.t. white noise

In a similar way we define

$$W^{\otimes k}(g) \;=\; \int_{(\mathbb{R}^d)^k} g(x_1, \ldots, x_k)\, W(\mathrm{d}x_1) \cdots W(\mathrm{d}x_k)$$

For symmetric functions we have

$$\mathbb{E}[W^{\otimes 2}(g)] = 0 \qquad \mathbb{E}[W^{\otimes 2}(g)^2] = k!\, \|g\|^2_{L^2((\mathbb{R}^d)^k)}$$

$$\mathbb{C}\mathrm{ov}[W^{\otimes k}(f), W^{\otimes k'}(g)] = k!\, \mathbb{1}_{\{k=k'\}}\, \langle f, g \rangle_{L^2((\mathbb{R}^d)^k)}$$

Wiener chaos expansion

Any r.v. $X \in L^2(\Omega_W)$ measurable w.r.t. $\sigma(W)$ can be written as

$$X = \sum_{k=0}^{\infty} \frac{1}{k!} W^{\otimes k}(f_k) \qquad \text{with} \qquad f_k \in L^2_{\mathrm{sym}}((\mathbb{R}^d)^k)$$

[Case $k = 0$: $\quad f_0 = \mathrm{E}[X] \quad W^{\otimes 0}(c) := c$]

Outline

Assumptions

[We use $\delta \to 0$ instead of $N \to \infty$]

Let $\mathbb{T}_\delta$ be a lattice in $\mathbb{R}^d$, all cells with the same volume v_δ

e.g. $\mathbb{T}_\delta = (\delta\mathbb{Z})^d, \; v_\delta = \delta^d$ $\qquad \mathbb{T}_\delta = (\delta\mathbb{Z}) \times (\sqrt{\delta}\mathbb{Z}), \; v_\delta = \delta^{3/2}$

A. Let $X_\delta = (X_{\delta,i})_{i \in \mathbb{T}_\delta}$ be independent random variables with

$$\mathbb{E}[X_{\delta,i}] = \mu_\delta(i) \qquad \mathbb{V}\mathrm{ar}[X_{\delta,i}] = 1$$

and such that $((X_{\delta,i} - \mathbb{E}[X_{\delta,i}])^2)_{\delta>0, i \in \mathbb{T}_\delta}$ are uniformly integrable

B. Let $\Psi_\delta(x)$ be a multi-linear polynomial such that for some $\varepsilon > 0$

$$\lim_{\ell \to \infty} \sup_{\delta > 0} \sum_{|I| > \ell} (1+\varepsilon)^{|I|} \, \psi_\delta(I)^2 \; = \; 0$$

i.e. $\Psi(x)$ approximated by finite degree polynomials (unif. in δ)

[If $\mu_\delta(i) \equiv 0$ one can take $\varepsilon = 0$]

CLT for polynomial chaos

Any function defined on $\mathbb{T}_\delta$ is extended (piecewise constant) to $\mathbb{R}^d$

$$\mu_\delta : \mathbb{R}^d \to \mathbb{R} \qquad \psi_\delta : (\mathbb{R}^d)^k \to \mathbb{R}$$

C. Assume that

$$\frac{\mu_\delta(x)}{v_\delta{}^{1/2}} \xrightarrow[\delta \to 0]{L^2(\mathbb{R}^d)} \boldsymbol{\mu}_0(x)$$

$$\frac{\psi_\delta(x_1, \ldots, x_k)}{v_\delta{}^{k/2}} \xrightarrow[\delta \to 0]{L^2((\mathbb{R}^d)^k)} \psi_0(x_1, \ldots, x_k)$$

Theorem [C., Sun, Zygouras 2015+]

Hp. A. B. C. yield $\Psi_\delta(X_\delta) \xrightarrow[\delta \to 0]{d} \boldsymbol{\Psi}_0$ with

$$\boldsymbol{\Psi}_0 := \sum_{k=0}^{\infty} \frac{1}{k!} \int \cdots \int_{(\mathbb{R}^d)^k} \psi_0(x_1, \ldots, x_k) \prod_{i=1}^{k} \Big(W(\mathrm{d}x_i) + \boldsymbol{\mu}_0(x_i)\, \mathrm{d}x_i \Big)$$

where W is white noise on $\mathbb{R}^d$

Outline

Proof of CLT for polynomial chaos

$$\Psi_\delta(X_\delta) = \sum_{I \subseteq \mathbb{T}_\delta} \psi_\delta(I) \prod_{x \in I} X_{\delta,x} = \approx \sum_{k=0}^{\infty \ell} \frac{1}{k!} \sum_{\substack{x_1,\ldots,x_k \in \mathbb{T}_\delta \\ \text{distinct points}}} \psi_\delta(x_1,\ldots,x_k) \prod_{i=1}^{k} X_{\delta,x_i}$$

1. Truncate the series at $k = \ell$. Choosing $\ell \in \mathbb{N}$ large, we make an error in L^2 which is small, *uniformly in* δ (recall Hp. B.)

2. Consider Gaussian disorder first. Since

$$\mathbb{E}[X_{\delta,x}] = \mu_\delta(x) \qquad \mathbb{V}\mathrm{ar}[X_{\delta,x}] = 1$$

we define Gaussians $X'_{\delta,x} \sim \mathcal{N}(\mu_\delta(x), 1)$. Using white noise W on $\mathbb{R}^d$

$$X'_{\delta,x} = \frac{W(\mathcal{C}_\delta(x))}{{v_\delta}^{1/2}} + \mu_\delta(x) = \frac{1}{{v_\delta}^{1/2}} \int_{\mathbb{R}^d} \mathbb{1}_{\mathcal{C}_\delta(x)}(z) \left(W(\mathrm{d}z) + \frac{\mu_\delta(z)}{{v_\delta}^{1/2}} \,\mathrm{d}z \right)$$

$\mathcal{C}_\delta(x) =$ cell containing $x \in \mathbb{T}_\delta$, with volume $|\mathcal{C}_\delta(x)| = v_\delta$

Proof of CLT for polynomial chaos

Replacing X_{δ,x_i} by X'_{δ,x_i} yields

$$\Psi_\delta(X'_\delta) \approx \sum_{k=0}^{\ell} \frac{1}{k!} W^{\otimes k}\left(\frac{\psi_\delta(z_1,\ldots,z_k)}{{v_\delta}^{k/2}}\right)$$

$$\boldsymbol{\Psi}_0 \approx \sum_{k=0}^{\ell} \frac{1}{k!} W^{\otimes k}\Big(\boldsymbol{\psi}_0(z_1,\ldots,z_k)\Big)$$

Assume that $\mu_\delta \equiv 0$ for simplicity

Terms with different k are orthogonal in L^2 $\rightsquigarrow$ by Ito isometry

$$\mathbb{E}\left[\left|\Psi_\delta(X'_\delta) - \boldsymbol{\Psi}_0\right|^2\right] = \sum_{k=0}^{\ell} \frac{1}{k!} \left\| \frac{\psi_\delta}{{v_\delta}^{k/2}} - \boldsymbol{\psi}_0 \right\|^2_{L^2((\mathbb{R}^d)^k)} \xrightarrow[\delta \to 0]{\text{by Hp. C.}} 0$$

Proof of CLT for polynomial chaos

3. Justify the replacement of $X_{\delta,x}$ by $X'_{\delta,x}$ using Lindeberg.

Assume $\mu_\delta \equiv 0$ for simplicity (hence $\varepsilon = 0$)

- $(X_{\delta,x})_{\delta>0,\, x\in\mathbb{T}_\delta}$ zero mean, unit variance, u.i. squares (by Hp. A.)
- $\sup_{\delta>0} C_{\Psi_\delta} = \sup_{\delta>0} \sum_{\emptyset \neq I \subseteq \mathbb{T}} \psi_\delta(I)^2 < \infty$ (by Hp. B. and C.)
- It remains to check that $\max_{x\in\mathbb{T}_\delta} \big(\mathrm{Inf}_x[\Psi_\delta]\big) \xrightarrow[\delta\to 0]{} 0$

$$\mathrm{Inf}_x[\Psi_\delta] = \sum_{I \ni x} \psi_\delta(I)^2 \approx \sum_{k=1}^{\ell} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} \frac{\psi_\delta(z_1,\ldots,z_k)^2}{{v_\delta}^k} \mathbb{1}_{\{\exists z_i \in \mathcal{C}_\delta(x)\}} \, \mathrm{d}z_1 \cdots \mathrm{d}z_k$$

$$\left\| \frac{\psi_\delta}{{v_\delta}^{k/2}} \mathbb{1}_{\{z_1 \in \mathcal{C}_\delta(x)\}} \right\|_{L^2}^2 \le \left\| \frac{\psi_\delta}{{v_\delta}^{k/2}} - \psi_0 \right\|_{L^2}^2 + \left\| \psi_0 \mathbb{1}_{\{z_1 \in \mathcal{C}_\delta(x)\}} \right\|_{L^2}^2 \xrightarrow[\delta\to 0]{\text{unif. in } x} 0 \qquad \square$$

Proof of Lindeberg Principle

Recall the assumptions:

- Fix a multi-linear polynomial $\Psi(x) = \sum_{I \subseteq \mathbb{T},\, |I| \le \ell} \psi(I)\, x^I$ of degree ℓ
- $X = (X_i)_{i \in \mathbb{T}}$, $X' = (X'_i)_{i \in \mathbb{T}}$ indep. with zero mean, unit variance

$$m_3 := \max_{i \in \mathbb{T}} \left(\mathbb{E}[|X_i|^3] \vee \mathbb{E}[|X'_i|^3] \right) < \infty$$

- For $f \in C^3(\mathbb{R} \to \mathbb{R})$ define $C_f := \max\{\|f'\|_\infty, \|f''\|_\infty, \|f'''\|_\infty\}$

Lindeberg Principle

$$\left| \mathbb{E}\big[f(\Psi(X))\big] - \mathbb{E}\big[f(\Psi(X'))\big] \right| \le 30^\ell\, C_f\, C_\Psi\, {m_3}^\ell \sqrt{\max_{i \in \mathbb{T}} \left(\mathrm{Inf}_i[\Psi] \right)}$$

Assume w.l.o.g. $\mathbb{T} = \{1, \ldots, n\}$ and set $g(\cdot) := f(\Psi(\cdot))$

Proof of Lindeberg Principle

1. Telescopic sum. Replace each X_i by X'_i, one by one:

$$g(X) - g(X')$$

$$= \sum_{i=1}^{n} \left\{ g(X_1, \ldots, X_i, X'_{i+1}, \ldots, X'_n) - g(X_1, \ldots, X_{i-1}, X'_i, \ldots, X'_n) \right\}$$

2. Taylor expansion. For $x_1 \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$

$$g(x_1, y) = g(0, y) + x_1 \, \partial_{x_1} g(0, y) + \frac{{x_1}^2}{2} \, \partial_{x_1}^2 g(0, y) + R_1(x_1, y)$$

Since $\mathbb{E}[X_1] = \mathbb{E}[X'_1]$ and $\mathbb{E}[(X_1)^2] = \mathbb{E}[(X'_1)^2]$

$$\left| \mathbb{E}[g(X_1, X')] - \mathbb{E}[g(X'_1, X')] \right| \le \mathbb{E}\left[|R_1(X_1, X')| \right] + \mathbb{E}\left[|R_1(X'_1, X')| \right]$$

Proof of Lindeberg Principle

3. Remainder estimate. We claim that for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

$$|R_1(x)| \le \frac{C_f}{6} \left|\hat{\Psi}_1(x)\right|^3 \qquad \hat{\Psi}_1(x) = \sum_{|I| \le \ell,\ I \ni 1} \psi(I)\, x^I$$

Proof. 3rd order Taylor remainder for $g(x_1, y)$ $(x_1 \in \mathbb{R} \quad y \in \mathbb{R}^{n-1})$

$$|R_1(x_1, y)| \le \frac{1}{6} \left| \sup_{t \in \mathbb{R}} \partial_t^3 g(t, y) \right| |x_1|^3 \le \frac{C_f}{6} \left| x_1\, \tilde{\Psi}_1(y) \right|^3$$

Since $g(\cdot) = f(\Psi(\cdot))$

$$\partial_t^3 g(t, y) = f'''(\Psi(t, y)) \left(\tilde{\Psi}_1(y)\right)^3$$

$$(\partial_t \Psi)(t, y) = \sum_{|I| \le \ell,\ I \ni 1} \psi(I)\, y^{I \setminus \{1\}} =: \tilde{\Psi}_1(y) \quad \text{no dependence on } t\,!$$

Proof of Lindeberg Principle

4. Hypercontractivity. For any multi-linear polynomial Ψ of degree ℓ

$$\forall\, 2 < q < \infty : \qquad \|\Psi(Y)\|_{L^q} \;\le\; (B_q)^{\ell}\, \|\Psi(Y)\|_{L^2}$$

$$B_q = 2\sqrt{q-1}\,\max_{i \in \mathbb{T}} \frac{\|Y_i\|_{L^q}}{\|Y_i\|_{L^2}}$$

In our case

$$\mathbb{E}\Big[\big|\hat{\Psi}_1\big|^3\Big] \le (B_3)^{3\ell}\, \mathbb{E}\Big[\big|\hat{\Psi}_1\big|^2\Big]^{3/2} = 2^{\frac{9}{2}\ell}\, {m_3}^{\ell} \left(\sum_{|I| \le \ell,\, I \ni 1} \psi(I)^2 \right)^{3/2}$$

$$= 2^{\frac{9}{2}\ell}\, {m_3}^{\ell} \Big(\mathrm{Inf}_1[\Psi]\Big)^{3/2}$$

The influence $\mathrm{Inf}_1[\Psi]$ of x_1 on $\Psi(x)$ has appeared!

Proof of Lindeberg Principle

5. Conclusion. Recalling that $g(\cdot) = f(\Psi(\cdot))$

$$
\begin{aligned}
\big|\mathbb{E}[g(X)] - \mathbb{E}[g(X')]\big| &\le \big|\mathbb{E}[g(X_1, X')] - \mathbb{E}[g(X_1', X')]\big| + \ldots \\
&\le \mathbb{E}\big[|R_1(X_1, X')|\big] + \mathbb{E}\big[|R_1(X_1', X')|\big] + \ldots \\
&\le \frac{C_f}{6} \Big\{ \mathbb{E}\big[|\hat{\Psi}_1(X_1, X')|^3\big] + \mathbb{E}\big[|\hat{\Psi}_1(X_1, X')|^3\big] + \ldots \Big\} \\
&\le \frac{C_f}{6}\, 2\, 2^{\frac{9}{2}\ell}\, {m_3}^{\ell} \Big\{ \mathrm{Inf}_1[\Psi]^{3/2} + \ldots \Big\} \\
&\le C_f \, \frac{2^{\frac{9}{2}\ell}}{3}\, {m_3}^{\ell} \sqrt{\max_{i \in \mathbb{T}} \big(\mathrm{Inf}_i[\Psi]\big)} \Big\{ \mathrm{Inf}_1[\Psi] + \ldots \Big\}
\end{aligned}
$$

$$
\mathrm{Inf}_1[\Psi] + \mathrm{Inf}_2[\Psi] + \ldots = \sum_{i=1}^{n} \sum_{|I| \le \ell,\, I \ni i} \psi(I)^2 = \sum_{|I| \le \ell} |I|\, \psi(I)^2 \le \ell\, C_\Psi \qquad \square
$$

References

- I. Benjaimini, G. Kalai, O. Schramm
  Noise sensitivity of Boolean functions and applications to percolation
  Inst. Hautes Études Sci. Publ. Math. 90 (2001), 5–43
- F. Caravenna, R. Sun, N. Zygouras
  Polynomial chaos and scaling limits of disordered systems
  J. Eur. Math. Soc. (JEMS), to appear
- C. Garban, J. Steif
  Noise Sensitivity and Percolation
  In "Probability and Statistical Physics in Two and more Dimensions"
  Clay Mathematics Proceedings 15 (2012), 319–393
- E. Mossel, R. O'Donnell, and K. Oleszkiewicz
  Noise stability of functions with low influences: Variance and optimality
  Ann. Math. 171 (2010) 295–341