Polynomial Chaos and Scaling Limits of Disordered Systems

3. Continuum disordered models

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Levico Terme $\sim$ September 30 - October 2, 2015

Overview

In the previous lecture we developed techniques that allow to construct continuum partition functions $\boldsymbol{Z}_\delta^\omega \xrightarrow[\delta \to 0]{d} \mathcal{Z}^W$
(Lindeberg Principle $\rightsquigarrow$ Multi-linear CLT)

In this lecture we use continuum partition functions $\mathcal{Z}^W$ to build a continuum disordered model $\mathcal{P}^W$

We will focus on the DPRE [Alberts, Khanin, Quastel 2014b] building "Continuum directed polymer (BM) in random environment"

The approach can also be applied to Pinning [C., Sun, Zygouras 2015+b] from which we draw inspiration

(Remark: the DPRE laws of different size are not consistent!)

Outline

1. Continuum partition functions
2. The continuum DPRE
3. Proof
4. Universality

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2. The continuum DPRE

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Continuum partition function for DPRE

Recall the partition function of 1d DPRE $(N = 1/\delta)$

$$\begin{aligned}
Z_\delta^\omega &= \mathbf{E}^{\mathrm{ref}}\left[\exp\left(\mathcal{H}^\omega\right)\right] = \mathbf{E}^{\mathrm{ref}}\left[\exp\left(\sum_{n=1}^{N}\left(\beta\omega_{(n,S_n)} - \lambda(\beta)\right)\right)\right] \\
&= \mathbf{E}^{\mathrm{ref}}\left[\exp\left(\sum_{n=1}^{N}\sum_{z\in\mathbb{Z}}\left(\beta\omega_{(n,z)} - \lambda(\beta)\right)\mathbb{1}_{\{S_n=z\}}\right)\right] \\
&= \mathbf{E}^{\mathrm{ref}}\left[\exp\left(\sum_{(t,x)\in\mathbb{T}_\delta}\left(\beta\omega_{(t,x)} - \lambda(\beta)\right)\mathbb{1}_{\{S_t^\delta=x\}}\right)\right] \\
&= \mathbf{E}^{\mathrm{ref}}\left[\prod_{(t,x)\in\mathbb{T}_\delta} e^{(\beta\omega_{(t,x)}-\lambda(\beta))\mathbb{1}_{\{S_t^\delta=x\}}}\right] = \mathbf{E}^{\mathrm{ref}}\left[\prod_{(t,x)\in\mathbb{T}_\delta}\left(1 + X_{t,x}\mathbb{1}_{\{S_t^\delta=x\}}\right)\right]
\end{aligned}$$

- $S_t^\delta := \sqrt{\delta}\, S_{t/\delta}$ lives on $\mathbb{T}_\delta = \left([0,1]\cap\delta\mathbb{N}_0\right)\times\sqrt{\delta}\,\mathbb{Z}$
- $X_{t,x} = e^{(\beta\omega_{(t,x)}-\lambda(\beta))} - 1 \qquad \mathbb{E}[X_{t,x}] = 0 \qquad \mathbb{V}\mathrm{ar}[X_{t,x}] \sim \beta^2$

Continuum partition function for DPRE

Developing the product yields a polynomial chaos expansion

$$Z_N^\omega = 1 + \sum_{(t,x)\in\mathbb{T}_\delta} \mathbf{P}^{\mathrm{ref}}(S_t^\delta = x)\, X_{t,x} + \frac{1}{2} \sum_{(t,x)\neq(t',x')\in\mathbb{T}_\delta} \mathbf{P}^{\mathrm{ref}}(S_t^\delta = x, S_{t'}^\delta = x')\, X_{t,x}\, X_{t',x'} + \ldots$$

Recall the LLT: $\mathbf{P}^{\mathrm{ref}}(S_n = x) \sim \frac{1}{\sqrt{n}}\, g\big(\frac{x}{\sqrt{n}}\big)$ with $g(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$

$$\mathbf{P}^{\mathrm{ref}}(S_t^\delta = x) = \mathbf{P}^{\mathrm{ref}}(S_{\frac{t}{\delta}} = \tfrac{x}{\sqrt{\delta}}) \sim \sqrt{\delta}\, g_t(x) \qquad g_t(x) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}$$

Replacing $X_{t,x} \approx \beta Y_{t,x}$ with $Y_{t,x}$ i.i.d. $\mathcal{N}(0,1)$ yields

Continuum partition function for DPRE

$$Z_N^\omega = 1 + \beta\sqrt{\delta} \sum_{(t,x)\in\mathbb{T}_\delta} g_t(x)\, Y_{t,x}$$
$$+ \frac{1}{2}(\beta\sqrt{\delta})^2 \sum_{(t,x)\neq(t',x')\in\mathbb{T}_\delta} g_t(x)\, g_{t'-t}(x'-x)\, Y_{t,x}\, Y_{t',x'} + \dots$$

Cells in $\mathbb{T}_\delta$ have volume $v_\delta = \delta\sqrt{\delta} = \delta^{\frac{3}{2}}$ $\rightsquigarrow$ "Stochastic Riemann sums" converge to stochastic integrals if $\beta\sqrt{\delta} \approx \sqrt{v_\delta}$ (check the variance!)

For $\boxed{\beta \sim \hat\beta\,\delta^{\frac{1}{4}}} = \frac{\hat\beta}{N^{\frac{1}{4}}}$ we get

$$Z_N^\omega \xrightarrow[\delta\to 0]{d} \mathcal{Z}^W = 1 + \hat\beta \int_{[0,1]\times\mathbb{R}} g_t(x)\, W(\mathrm{d}t\mathrm{d}x)$$
$$+ \frac{\hat\beta^2}{2}\int_{([0,1]\times\mathbb{R})^2} g_t(x)\, g_{t'-t}(x'-x)\, W(\mathrm{d}t\mathrm{d}x)\, W(\mathrm{d}t'\mathrm{d}x')$$
$$+ \dots$$

Free and constrained partition functions

We have constructed $\mathcal{Z}^W$ = "free" partition function on $[0,1] \times \mathbb{R}$
RW paths starting at $(0,0)$ with no constraint on right endpoint

$$\mathcal{Z}^W = \mathcal{Z}^W((0,0),(1,\star)) \qquad \mathbb{E}[\mathcal{Z}^W] = 1$$

Consider also constrained partition functions: for $(s,y),(t,x) \in [0,1] \times \mathbb{R}$

$$\boldsymbol{Z}_\delta^\omega((s,y),(t,x)) = \mathbf{E}^{\mathrm{ref}}\Big[\exp\Big(\mathcal{H}^\omega\Big)\, \mathbb{1}_{\{S_t^\delta = x\}} \Big| S_s^\delta = y\Big]$$

which (divided by $\sqrt{\delta}$) converges to a limit that we call

$$\mathcal{Z}^W((s,y),(t,x)) \qquad \mathbb{E}[\mathcal{Z}^W((s,y),(t,x))] = g_{t-s}(x-y)$$

This is a function of white noise in the stripe $W([s,t] \times \mathbb{R})$

Key properties

Key properties

For a.e. realization of W the following properties hold:

- Continuity: $\mathcal{Z}^W((s,y),(t,x))$ is jointly continuous in (s,y,t,x) (on the domain $s < t$)
- Positivity: $\mathcal{Z}^W((s,y),(t,x)) > 0$ for all (s,y,t,x) satisfying $s < t$
- Semigroup (Chapman-Kolmogorov): for all $s < r < t$ and $x, y \in \mathbb{R}$

$$\mathcal{Z}^W((s,y),(t,x)) = \int_{\mathbb{R}} \mathcal{Z}^W((s,y),(r,z))\, \mathcal{Z}^W((r,z),(t,x))\, \mathrm{d}z$$

(Inherited from discrete partition functions: drawing!)

How to prove these properties?

The four-parameter field $\mathcal{Z}^W((s,y),(t,x))$ solves the 1d SHE

$$\begin{cases} \partial_t \mathcal{Z}^W = \frac{1}{2}\Delta_x \mathcal{Z}^W + \hat{\beta}\, W\, \mathcal{Z}^W \\ \lim_{t\downarrow s} \mathcal{Z}^W((s,y),(t,x)) = \delta(y-x) \end{cases}$$

Checked directly from Wiener chaos expansion (mild solution)

It is known that solutions to the SHE satisfy the properties above

Alternative approach (to check, OK for pinning [C., Sun, Zygouras 2015+b])

- Prove continuity by Kolmogorov criterion, showing that

$$\frac{\mathcal{Z}^W((s,y),(t,x))}{g_{t-s}(x-y)} \quad \text{is continuous also for } t = s$$

- Use continuity to prove semigroup for all times
- Use continuity to deduce positivity for close times, then bootstrap to arbitrary times using semigroup

Outline

Partition functions and f.d.d.

Start from discrete: distribution of DPRE at two times $0 < t < t' < 1$

$$\mathbf{P}_\delta^\omega(S_t^\delta = x, S_{t'}^\delta = x') = \frac{\boldsymbol{Z}_\delta^\omega\big((0,0),(t,x)\big)\,\boldsymbol{Z}_\delta^\omega\big((t,x),(t',x')\big)\,\boldsymbol{Z}_\delta^\omega\big((t',x'),(1,\star)\big)}{\boldsymbol{Z}_\delta^\omega\big((0,0),(1,\star)\big)}$$

(drawing!) Analogous formula for any finite number of times

Idea: Replace $\boldsymbol{Z}_\delta^\omega \rightsquigarrow \mathcal{Z}^W$ to *define* the law of continuum DPRE

Recall: to define a process $(X_t)_{t\in[0,1]}$ it is enough (Kolmogorov) to assign finite-dimensional distributions (f.d.d.)

$$\mu_{t_1,\ldots,t_k}(A_1,\ldots,A_k) \text{ ``} = \mathrm{P}(X_{t_1} \in A_1, \ldots, X_{t_k} \in A_k) \text{ ''}$$

that are consistent

$$\mu_{t_1,\ldots,t_j,\ldots,t_k}(A_1,\ldots,\mathbb{R},\ldots,A_k) = \mu_{t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_k}(A_1,\ldots,A_{j-1},A_{j+1},\ldots,A_k)$$

The continuum 1d DPRE

- Fix $\hat{\beta} \in (0, \infty)$ (on which $\mathcal{Z}^W$ depend) [recall that $\beta \sim \hat{\beta}\delta^{\frac{1}{4}}$]
- Fix space-time white noise W on $[0,1] \times \mathbb{R}$ and a realization of continuum partition functions $\mathcal{Z}^W$ satisfying the key properties (continuity, strict positivity, semigroup)

The Continuum DPRE is the process $(X_t)_{t \in [0,1]}$ with f.d.d.

$$\frac{\mathcal{P}^W(X_t \in \mathrm{d}x, X_{t'} \in \mathrm{d}x')}{\mathrm{d}x\,\mathrm{d}x'} := \frac{\mathcal{Z}^W((0,0),(t,x))\,\mathcal{Z}^W((t,x),(t',x'))\,\mathcal{Z}^W((t',x'),(1,\star))}{\mathcal{Z}^W((0,0),(1,\star))}$$

- Well-defined by strict positivity of $\mathcal{Z}^W$
- Consistent by semigroup property

Relation with Wiener measure

The law of the continuum DPRE is a random probability

$$\mathcal{P}^{W}(X \in \cdot\,) \qquad \text{(quenched law)}$$

for the process $X = (X_t)_{t\in[0,1]}$ [Probab. kernel $\mathcal{S}'(\mathbb{R}) \to \mathbb{R}^{[0,1]}$]

Define a new law $\tilde{\mathbb{P}}$ (mutually absolutely continuous) for disorder W by

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(W) = \mathcal{Z}^{W}\big((0,0),(1,\star)\big)$$

Key Lemma

$$\mathcal{P}^{\mathrm{ann}}(X \in \cdot) := \int_{\mathcal{S}'(\mathbb{R})} \mathcal{P}^{W}(X \in \cdot\,)\,\tilde{\mathbb{P}}(\mathrm{d}W) = \mathrm{P}(BM \in \cdot\,)$$

Proof. The factor $\mathcal{Z}^{W}$ in $\tilde{\mathbb{P}}$ cancels the denominator in the f.d.d. for $\mathcal{P}^{W}$

Since $\mathbb{E}\big[\mathcal{Z}^{W}\big((s,y),(t,x)\big)\big] = g_{t-s}(x-y)$ one gets f.d.d. of BM □

Absolute continuity properties

Any given a.s. property of BM is an a.s. property of continuum DPRE, for a.e. realization of the disorder W

Theorem

$$\forall A : \quad \mathrm{P}(BM \in A) = 1 \quad \Rightarrow \quad \mathcal{P}^W(X \in A) = 1 \quad \text{for } \mathbb{P}\text{-a.e. } W$$

Corollary

$$\mathcal{P}^W(X \text{ has Hölder paths with exp. } \tfrac{1}{2}-) = 1 \qquad \text{for } \mathbb{P}\text{-a.e. } W$$

We can thus realize $\mathcal{P}^W$ as a law on $C([0,1], \mathbb{R})$, for $\mathbb{P}$-a.e. W

(More precisely: $\mathcal{P}^W$ *admits a modification* with Hölder paths)

Perhaps $\mathcal{P}^W$ absolutely continuous w.r.t. Wiener measure, for $\mathbb{P}$-a.e. W ?

NO ! "$\forall A$" and "for $\mathbb{P}$-a.e. W" cannot be exchanged!

Singularity properties

Any given a.s. property of BM is an a.s. property of continuum DPRE, for a.e. realization of the disorder W. However:

Theorem

The law $\mathcal{P}^W$ is singular w.r.t. Wiener measure, for $\mathbb{P}$-a.e. W.

$$\text{for } \mathbb{P}\text{-a.e. } W \qquad \exists\, A = A_W \subseteq C([0,1],\mathbb{R}) :$$
$$\mathcal{P}^W(X \in A) = 1 \qquad \text{vs.} \qquad \mathrm{P}(BM \in A) = 0$$

Unlike discrete DPRE, there is no continuum Hamiltonian

$$\mathcal{P}^W(X \in \cdot\,) \not\propto e^{\mathcal{H}^W(\,\cdot\,)}\mathrm{P}(BM \in \cdot\,)$$

Absolute continuity is lost in the scaling limit

In a sense, the laws $\mathcal{P}^W$ are just *barely* not absolutely continuous w.r.t. Wiener measure ("stochastically absolutely continuous")

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Proof of singularity

Let $(X_t)_{t\in[0,1]}$ be the canonical process on $C([0,1],\mathbb{R})$ $\quad [\, X_t(f) = f(t) \,]$

Let $\mathcal{F}_n := \sigma(X_{t_i^n} : \ t_i^n = \frac{i}{2^n}, \ 0 \le i \le 2^n)$ be the dyadic filtration

Fix (a typical realization of) W. Setting $\mathcal{P}^{\mathrm{ref}}$ = Wiener measure

$$R_n^W(X) := \frac{\mathrm{d}\mathcal{P}^W|_{\mathcal{F}_n}}{\mathrm{d}\mathcal{P}^{\mathrm{ref}}|_{\mathcal{F}_n}}(X)$$

The process $(R_n^W)_{n\in\mathbb{N}}$ is a martingale w.r.t. $\mathcal{P}^{\mathrm{ref}}$ (exercise!)

Since $R_n^W \ge 0$, the martingale converges: $\quad R_n^W \xrightarrow[n\to\infty]{\text{a.s.}} R_\infty^W$

- $\mathcal{P}^W \ll \mathcal{P}^{\mathrm{ref}}$ if and only if $\mathcal{E}^{\mathrm{ref}}[R_\infty^W] = 1$ $\qquad$ (the martingale is UI)
- $\mathcal{P}^W$ is singular w.r.t. $\mathcal{P}^{\mathrm{ref}}$ if and only if $R_\infty^W = 0$

Proof of singularity

Known: for $\mathbb{P}$-a.e. W we have $R_n^W(X) \xrightarrow[n\to\infty]{} R_\infty^W(X)$ for $\mathcal{P}^{\mathrm{ref}}$-a.e. X

It suffices to show that $R_n^W(X) \xrightarrow[n\to\infty]{} 0$ in $\mathbb{P} \otimes \mathcal{P}^{\mathrm{ref}}$-probability

Fractional moment

For $\mathcal{P}^{\mathrm{ref}}$-a.e. X $\quad \mathbb{E}\tilde{\mathbb{E}}\big[\big(R_n^W(X)\big)^\gamma\big] \xrightarrow[n\to\infty]{} 0 \quad$ for some $\gamma \in (0,1)$

$$R_n^W(X) = \frac{1}{\mathcal{Z}^W\big((0,0),(1,\star)\big)} \prod_{i=0}^{2^n-1} \frac{\mathcal{Z}^W\big((t_i^n, X_{t_i^n}),(t_{i+1}^n, X_{t_{i+1}^n})\big)}{g_{\frac{1}{2^n}}(X_{t_{i+1}^n} - X_{t_i^n})}$$

- Switch from $\mathbb{E}$ to equivalent law $\tilde{\mathbb{E}}$ to cancel the denominator
- For fixed X, the $\mathcal{Z}^W\big((t_i^n, X_{t_i^n}),(t_{i+1}^n, X_{t_{i+1}^n})\big)$'s are independent

We need to exploit translation and scale invariance of their laws

Proof of singularity

Lemma 1 (Translation and scale invariance)

If we set $\Delta_i^n := \dfrac{X_{t_{i+1}^n} - X_{t_i^n}}{\sqrt{t_{i+1}^n - t_i^n}}$ we have

$$\frac{\mathcal{Z}_{\hat\beta}^{W}\big((t_i^n, X_{t_i^n}), (t_{i+1}^n, X_{t_{i+1}^n})\big)}{g_{\frac{1}{2^n}}(X_{t_{i+1}^n} - X_{t_i^n})} \overset{d}{=} \frac{\mathcal{Z}_{\frac{\hat\beta}{2^{n/4}}}^{W}\big((0,0),(1,\Delta_i^n)\big)}{g_1(\Delta_i^n)}$$

Lemma 2 (Expansion)

For $z \in \mathbb{R}$ and $\varepsilon \in [0,1]$ (say)

$$\frac{\mathcal{Z}_{\varepsilon}^{W}\big((0,0),(1,z)\big)}{g_1(z)} = 1 + \varepsilon\, X_z + \varepsilon^2\, Y_{\varepsilon,z}$$

$\mathbb{E}[X_z] = 0 \quad \mathbb{E}[X_{\varepsilon,z}] = 0 \quad \mathbb{E}\big[X_z^2\big] \le C \quad \mathbb{E}\big[Y_{\varepsilon,z}^2\big] \le C$ unif. in ε, z

Proof of singularity

By Taylor expansion, for fixed $\gamma \in (0,1)$

$$\mathbb{E}\left[\left(\frac{\mathcal{Z}_\varepsilon^W((0,0),(1,z))}{g_1(z)}\right)^\gamma\right] = \mathbb{E}\left[\left(1+\varepsilon X_z + \varepsilon^2 Y_{\varepsilon,z}\right)^\gamma\right]$$

$$= 1 + \gamma\left\{\varepsilon\mathbb{E}[X_z] + \varepsilon^2\mathbb{E}[Y_{\varepsilon,z}]\right\} + \frac{\gamma(\gamma-1)}{2}\left\{\varepsilon^2\mathbb{E}[(X_x)^2] + \ldots\right\} + \ldots$$

$$= 1 - c\,\varepsilon^2 \le e^{-c\,\varepsilon^2}$$

($\star$) First order terms vanish ($\star$) $\gamma(\gamma-1)<0$ ($\star$) For some $c>0$

Estimate is uniform over $z \in \mathbb{R}$ $\rightsquigarrow$ We can set $z = \Delta_i^n$ and $\varepsilon = \frac{1}{2^{n/4}}$

$$\tilde{\mathbb{E}}\left[\left(R_n^W(X)\right)^\gamma\right] = \prod_{i=0}^{2^n-1} \mathbb{E}\left[\left(\frac{\mathcal{Z}_\varepsilon^W((0,0),(1,\Delta_i^n))}{g_1(\Delta_i^n)}\right)^\gamma\right] \le e^{-c\,\varepsilon^2\,2^n} = e^{-c\,2^{n/2}}$$

which vanishes as $n \to \infty$ □

Proof of Lemma 1

Introducing the dependence on $\hat{\beta}$

$$\mathcal{Z}^{W}_{\hat{\beta}}((s,y),(t,x)) \overset{d}{=} \mathcal{Z}^{W}_{\hat{\beta}}((0,0),(t-s,x-y))$$

$$\mathcal{Z}^{W}_{\hat{\beta}}((0,0),(t,x)) \overset{d}{=} \frac{1}{\sqrt{t}}\,\mathcal{Z}^{W}_{\hat{\beta}t^{\frac{1}{4}}}\left((0,0),\left(1,\frac{x}{\sqrt{t}}\right)\right)$$

transl. invariance + diffusive rescaling (prefactor, new $\hat{\beta}$) (drawing!)

$$\mathcal{Z}^{W}((0,0),(t,x)) = g_t(x) + \hat{\beta}\int_{[0,t]\times\mathbb{R}} g_s(z)\,g_{t-s}(x-z)\,W(\mathrm{d}s\mathrm{d}z) + \ldots$$

$$= \frac{1}{\sqrt{t}}\,g_1\!\left(\tfrac{x}{\sqrt{t}}\right) + \frac{1}{\sqrt{t}}\left(\frac{\hat{\beta}\,t^{\frac{3}{4}}}{\sqrt{t}}\right)\int_{[0,t]\times\mathbb{R}} g_{\frac{s}{t}}\!\left(\tfrac{z}{\sqrt{t}}\right) g_{1-\frac{s}{t}}\!\left(\tfrac{x-z}{\sqrt{t}}\right)\frac{W(\mathrm{d}s\mathrm{d}z)}{t^{\frac{3}{4}}} + \ldots$$

$= \text{OK}!$ □

Outline

Convergence of discrete DPRE

- $\mathbf{P}_\delta^\omega$ = law of discrete DPRE (recall that $S_t^\delta := \sqrt{\delta} S_{t/\delta}$)
  "Rescaled RW S^δ moving in an i.i.d. environment ω"
- $\mathcal{P}^W$ = law of continuum DPRE
  "BM moving in a white noise environment W"

Both $\mathbf{P}_\delta^\omega$ and $\mathcal{P}^W$ are random probability laws on $E := C([0,1], \mathbb{R})$
i.e. RVs (defined on different probab. spaces) taking values in $\mathcal{M}_1(E)$

Does $\mathbf{P}_\delta^\omega$ converge in distribution toward $\mathcal{P}^W$ as $\delta \to 0$?

$$\forall\, \psi \in C_b(\mathcal{M}_1(E) \to \mathbb{R}) : \qquad \mathbb{E}\big[\psi(\mathbf{P}_\delta^\omega)\big] \xrightarrow[\delta\to 0]{} \mathbb{E}\big[\psi(\mathcal{P}^W)\big]$$

The answer is positive... almost surely ;-)

Statement for Pinning model proved in [C., Sun, Zygouras 2015+b]

Details need to be checked for DPRE (stronger assumptions on RW ?)

Universality

The convergence of $\mathbf{P}_\delta^\omega$ toward $\mathcal{P}^W$ is an instance of universality

There are many discrete DPRE:

- any RW S (zero mean, finite variance + technical assumptions)
- any (i.i.d.) disorder ω (finite exponential moments)

In the continuum ($\delta \to 0$) and weak disorder ($\beta \to 0$) regime, all these microscopic models $\mathbf{P}_\delta^\omega$ give rise to a unique macroscopic model $\mathcal{P}^W$

Tomorrow we will see how the continuum model $\mathcal{P}^W$ can tell quantitative information on discrete models $\mathbf{P}_\delta^\omega$ (free energy estimates)

Convergence

How to prove convergence in distribution $\mathbf{P}_\delta^\omega \xrightarrow[\delta\to 0]{d} \mathcal{P}^W$?

Prove a.s. convergence through a suitable coupling of (ω, W)

Assume we have convergence in distribution of discrete partition functions to continuum ones, in the space of continuum functions of $(s, y), (t, x)$

$$\boldsymbol{Z}_\delta^\omega\big((s,y),(t,x)\big) \xrightarrow[\delta\to 0]{d} \mathcal{Z}^W\big((s,y),(t,x)\big)$$

By Skorokhod representation theorem, there is a coupling of (ω, W) under which this convergence holds a.s.

Fix such a coupling: for a.e. (ω, W) the f.d.d. of $\mathbf{P}_\delta^\omega$ converge weakly to those of $\mathcal{P}^W$. It only remains to prove tightness of $\mathbf{P}_\delta^\omega(\cdot)$.

Convergence

To prove tightness (also for the convergence of discrete partition functions) a key tool is the inequality of Garsia, Rodemich and Rumsey

$$\left(\sup_{s,t\in[0,1]^d,\, s\neq t} \frac{|f(t)-f(s)|}{|t-s|^{\mu-\frac{2d}{p}}} \right)^p \le C_{\mu,p,d} \int_{[0,1]^d\times[0,1]^d} \frac{|f(t)-f(s)|^p}{|t-s|^{p\mu}} \, \mathrm{d}s \, \mathrm{d}t$$

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