# Polynomial Chaos and Scaling Limits of Disordered Systems

## 4. Free energy estimates. Introduction to marginal relevance.

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# Overview

In the previous lecture we constructed continuum partition functions $\mathcal{Z}^W$ and we used them to define a continuum disordered model $\mathcal{P}^W$

In this lecture we show how the continuum objects $\mathcal{Z}^W$ and $\mathcal{P}^W$ yield quantitative information on the discrete model (free energy estimates)

We will focus on Pinning models (rather than DPRE)

In the last part we will introduce marginally relevant models
(Pinning for $\alpha = \frac{1}{2}$, DPRE for $d = 2$, 2d Stochastic Heat Equation)

# Outline

1. Pinning models
2. Weak disorder regime
3. The marginally relevant regime

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1. Pinning models

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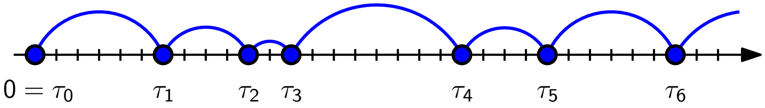
# Ingredients: renewal process & disorder




Discrete renewal process $\tau = \{0 = \tau_0 < \tau_1 < \tau_2 < \ldots\} \subseteq \mathbb{N}_0$

Gaps $(\tau_{i+1} - \tau_i)_{i \geq 0}$ are i.i.d. with polynomial-tail distribution:

$$\mathbf{P}^{\mathrm{ref}}(\tau_1 = n) \sim \frac{c_K}{n^{1+\alpha}}, \qquad c_K > 0, \quad \alpha \in (0,1)$$

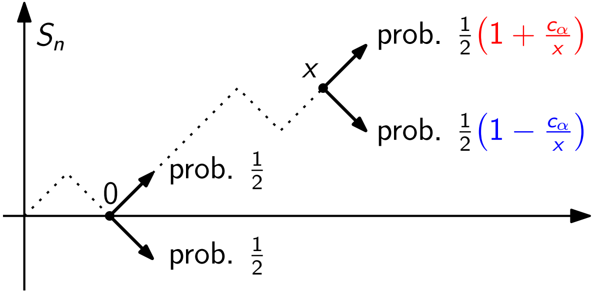
$\tau = \{n \in \mathbb{N}_0 : \ S_n = 0\}$ zero level set of a Markov chain $S = (S_n)_{n \geq 0}$

Disorder $\omega = (\omega_i)_{i \in \mathbb{N}}$: i.i.d. real random variables with law $\mathbb{P}$

$$\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_1}] < \infty \qquad \mathbb{E}[\omega_1] = 0 \qquad \mathbb{V}\mathrm{ar}[\omega_1] = 1$$

# Bessel random walks

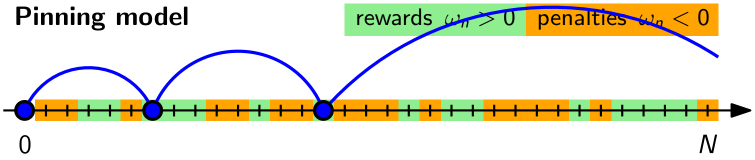
For $\alpha \in (0,1)$ the $\alpha$-Bessel random walk is defined as follows:




$$c_\alpha := \frac{1}{2} - \alpha$$

- $(\alpha = \frac{1}{2})$ no drift $(c_\alpha = 0)$ $\rightsquigarrow$ simple random walk
- $(\alpha < \frac{1}{2})$ drift away from the origin $(c_\alpha > 0)$
- $(\alpha > \frac{1}{2})$ drift toward the origin $(c_\alpha < 0)$

# Disordered pinning model




$N \in \mathbb{N}$ (system size) $\beta \geq 0$, $h \in \mathbb{R}$ (disorder strength, bias)

## The pinning model

Gibbs change of measure $\mathbf{P}_N^\omega = \mathbf{P}_{N,\beta,h}^\omega$ of the renewal distribution $\mathbf{P}^{\mathrm{ref}}$

$$\frac{\mathrm{d}\mathbf{P}_N^\omega}{\mathrm{d}\mathbf{P}^{\mathrm{ref}}}(\tau) := \frac{1}{Z_N^\omega} \exp\left( \sum_{n=1}^{N} (\beta\omega_n + h - \lambda(\beta)) \mathbb{1}_{\{n\in\tau\}} \, \mathbb{1}_{\{S_n=0\}} \right)$$

# The phase transition

How are the typical paths $\tau$ of the pinning model $\mathbf{P}_N^\omega$?

Contact number $\quad \mathcal{C}_N := \big|\tau \cap (0,N]\big| = \sum_{n=1}^N \mathbb{1}_{\{n \in \tau\}} = \sum_{n=1}^N \mathbb{1}_{\{S_n = 0\}}$

## Theorem (phase transition)

$\exists$ *continuous, non decreasing, deterministic critical curve* $\mathbf{h}_c(\beta)$:

- *Localized regime: for* $h > \mathbf{h}_c(\beta)$ *one has* $\mathcal{C}_N \approx N$

$$\exists \mu = \mu_{\beta,h} > 0 : \quad \mathbf{P}_N^\omega\left( \left| \frac{\mathcal{C}_N}{N} - \mu \right| > \varepsilon \right) \xrightarrow[N\to\infty]{} 0 \qquad \omega\text{–a.s.}$$

- *Deocalized regime: for* $h < \mathbf{h}_c(\beta)$ *one has* $\mathcal{C}_N = O(\log N)$

$$\exists A = A_{\beta,h} > 0 : \quad \mathbf{P}_N^\omega\left( \frac{\mathcal{C}_N}{\log N} > A \right) \xrightarrow[N\to\infty]{} 0 \qquad \omega\text{–a.s.}$$

# Estimates on the critical curve

For $\beta = 0$ (homogeneous pinning, no disorder) one has $\mathbf{h}_c(0) = 0$

What is the behavior of $\mathbf{h}_c(\beta)$ for $\beta > 0$ small ?

**Theorem ( $\mathrm{P}(\tau_1 = n) \sim \frac{c_K}{n^{1+\alpha}}$ )**

- ($\alpha < \frac{1}{2}$) *disorder is irrelevant:* $\mathbf{h}_c(\beta) = 0$ *for* $\beta > 0$ *small*
  [Alexander] [Toninelli] [Lacoin] [Cheliotis, den Hollander]
- ($\alpha \geq \frac{1}{2}$) *disorder is relevant:* $\mathbf{h}_c(\beta) > 0$ *for all* $\beta > 0$
  - ($\alpha > 1$) $\mathbf{h}_c(\beta) \sim C\,\beta^2$ *with explicit* $C = \frac{\alpha}{1+\alpha}\frac{1}{2\mathrm{E}(\tau_1)}$
    [Berger, C., Poisat, Sun, Zygouras]
  - ($\frac{1}{2} < \alpha < 1$) $C_1\,\beta^{\frac{2\alpha}{2\alpha-1}} \leq \mathbf{h}_c(\beta) \leq C_2\,\beta^{\frac{2\alpha}{2\alpha-1}}$ $\mathbf{h}_c(\beta) \sim \hat{C}\,\beta^{\frac{2\alpha}{2\alpha-1}}$
    *using continuum model!*
    [Derrida, Giacomin, Lacoin, Toninelli] [Alexander, Zygouras] [C., Torri, Toninelli]
  - ($\alpha = \frac{1}{2}$) $\mathbf{h}_c(\beta) = e^{-\frac{c+o(1)}{\beta^2}}$ [Giacomin, Lacoin, Toninelli] [Berger, Lacoin]

# Discrete free energy and critical curve

Partition function $\quad Z_N^{\omega} := \mathrm{E}\left[e^{H_N(\tau)}\right] = \mathrm{E}\left[e^{\sum_{n=1}^N (h+\beta\omega_n-\Lambda(\beta))\mathbb{1}_{\{n\in\tau\}}}\right]$

Consider first the regime of $N \to \infty$ with fixed $\beta, h$

- Free energy $\quad \mathbf{F}(\beta,h) := \lim_{N\to\infty} \frac{1}{N}\log Z_N^{\omega} \geq 0 \quad \mathbb{P}(\mathrm{d}\omega)$-a.s.

$$Z_N^{\omega} \geq \mathrm{E}\left[e^{H_N(\tau)}\,\mathbb{1}_{\{\tau\cap(0,N]=\emptyset\}}\right] = \mathrm{P}(\tau\cap(0,N]=\emptyset) \sim \frac{(const.)}{N^{\alpha}}$$

- Critical curve $\quad \mathbf{h}_c(\beta) = \sup\{h\in\mathbb{R}:\ \mathbf{F}(\beta,h)=0\} \quad$ non analiticity!

$$\text{(convexity)} \quad \frac{\partial \mathbf{F}(\beta,h)}{\partial h} = \lim_{N\to\infty} \mathbf{E}_N^{\omega}\left[\frac{\mathcal{C}_N}{N}\right] \begin{cases} >0 & \text{if } h>\mathbf{h}_c(\beta) \\ =0 & \text{if } h<\mathbf{h}_c(\beta) \end{cases}$$

$\mathbf{F}(\beta,h)$ and $\mathbf{h}_c(\beta)$ depend on the law of $\tau$ and $\omega$

Universality as $\beta, h \to 0$ ? $\quad$ YES, connected to continuum model

# A word on critical exponents

The free energy $\mathbf{F}(\beta, h)$ is non analytic at the critical point $h = \mathbf{h}_c(\beta)$

$$\mathbf{F}(\beta, h) = 0 \quad (h < \mathbf{h}_c(\beta)) \qquad \mathbf{F}(\beta, h) > 0 \quad (h > \mathbf{h}_c(\beta))$$

What is the behavior of $\mathbf{F}(\beta, h)$ as $h \downarrow \mathbf{h}_c(\beta)$ ?

For $\beta = 0$ the model is exactly solvable: $\mathbf{h}_c(0) = 0$ and

$$\mathbf{F}(0, h) - \mathbf{F}(0, \mathbf{h}_c(0)) \sim C\,(h - \mathbf{h}_c(0))^{\frac{1}{\alpha}} \qquad (\alpha \in (0,1))$$

## Smoothing inequality [Giacomin, Toninelli]

$$\mathbf{F}(\beta, h) - \mathbf{F}(\beta, \mathbf{h}_c(\beta)) \leq \frac{C}{\beta^2}\,(h - \mathbf{h}_c(0))^2$$

- For $\alpha > \frac{1}{2}$ disorder makes phase transition smoother!
  Also $\mathbf{h}_c(\beta) \neq \mathbf{h}_c(0)$ for every $\beta > 0$ $\rightsquigarrow$ disorder relevance
- For $\alpha < \frac{1}{2}$ and for $\beta > 0$ small $\mathbf{F}(\beta, h) \approx \mathbf{F}(0, h)$ $\rightsquigarrow$ irrelevance

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2. Weak disorder regime

3. The marginally relevant regime

# Continuum partition functions

Build continuum partition functions for Pinning Model with $\alpha \in (\frac{1}{2}, 1)$ (disorder relevant) following "usual" approach [C, Sun, Zygouras 2015+]

We need to rescale

$$\beta = \beta_N = \frac{\hat{\beta}}{N^{\alpha - 1/2}} \qquad h = h_N = \frac{\hat{h}}{N^{\alpha}}$$

(Note that $h_N \approx \beta_N^{\frac{2\alpha}{2\alpha-1}} \approx \mathbf{h}_c(\beta_N)$)

One has $\boldsymbol{Z}_N^{\omega} \xrightarrow[N\to\infty]{d} \mathcal{Z}^W$ with

$$\mathcal{Z}^W := 1 + C \int_{0<t<1} \frac{\mathrm{d}W_t^{\hat{\beta},\hat{h}}}{t^{1-\alpha}} + C^2 \int_{0<t<t'<1} \frac{\mathrm{d}W_t^{\hat{\beta},\hat{h}}\,\mathrm{d}W_{t'}^{\hat{\beta},\hat{h}}}{t^{1-\alpha}(t'-t)^{1-\alpha}} + \ldots$$

where $W_t^{\hat{\beta},\hat{h}} := \hat{\beta} W_t + \hat{h}\, t$ and $C = \frac{\alpha \sin(\alpha\pi)}{\pi\, c_K}$

# Continuum partition functions

**Exercise**

Recalling that

$$\mathbf{P}^{\mathrm{ref}}(n \in \tau) = \mathbf{P}^{\mathrm{ref}}(S_n = 0) \sim \frac{c}{n^{1-\alpha}}$$

check that $\beta_N$ and $h_N$ are the correct scaling (polynomial chaos)

Like for DPRE we build constrained partition functions: $0 \le s < t < \infty$

$$\mathcal{Z}^W(s,t) = \text{scaling limit of } \mathbf{E}^{\mathrm{ref}}\left[e^{H^{\omega}_{[Ns,Nt]}} \mathbb{1}_{\{Nt \in \tau\}} \middle| Ns \in \tau\right]$$

We show that they satisfy continuity, strict positivity, semigroup

**Theorem** [C., Sun, Zygouras 2015+b]

We can build a continuum disordered Pinning model $\mathcal{P}^W$

as a random probability law on the space of closed subsets of $[0, 1]$

# Continuum free energy

In analogy with the discrete model, define

Continuum free energy $$\mathcal{F}(\hat\beta, \hat h) := \lim_{t\to\infty} \frac{1}{t} \log \mathcal{Z}^W_{\hat\beta,\hat h}(0,t) \quad \text{a.s.}$$

(existence and self-averaging need some work)

Again $\mathcal{F}(\hat\beta, \hat h) \geq 0$ and define

Continuum critical curve $$\mathcal{H}_c(\hat\beta) := \sup\left\{\hat h \in \mathbb{R} : \ \mathcal{F}(\hat\beta, \hat h) = 0\right\}$$

## Scaling relations

$$\forall c > 0 : \qquad \mathcal{Z}^W_{\hat\beta,\hat h}(c\,t) \overset{d}{=} \mathcal{Z}^W_{c^{\alpha-\frac{1}{2}}\hat\beta,\, c^\alpha \hat h}(t) \qquad \text{(Wiener chaos exp.)}$$

$$\mathcal{F}\left(c^{\alpha-\frac{1}{2}}\hat\beta, c^\alpha\right) = c\,\mathcal{F}(\hat\beta, \hat h) \qquad \boxed{\mathcal{H}_c(\hat\beta) = \mathcal{H}_c(1)\,\hat\beta^{\frac{2\alpha}{2\alpha-1}}}$$

# Interchanging the limits

Can we relate continuum free energy to the discrete one?

By construction of continuum partition functions

$$\mathcal{Z}^{W}_{\hat\beta,\hat h}(t) \stackrel{d}{=} \lim_{N\to\infty} \boldsymbol{Z}^{\omega}_{\beta_N,h_N}(Nt)$$

Assuming uniform integrability of $\log \boldsymbol{Z}^{\omega}$ (OK)

$$\mathcal{F}(\hat\beta,\hat h) = \lim_{t\to\infty} \frac{1}{t}\, \mathbb{E}\big[ \log \mathcal{Z}^{W}_{\hat\beta,\hat h}(t) \big] = \lim_{t\to\infty} \frac{1}{t} \lim_{N\to\infty} \mathbb{E}\big[ \log \boldsymbol{Z}^{\omega}_{\beta_N,h_N}(Nt) \big]$$

Assuming we can interchange the limits $N\to\infty$ and $t\to\infty$

$$\mathcal{F}(\hat\beta,\hat h) = \lim_{N\to\infty} N \lim_{t\to\infty} \frac{1}{Nt\ t}\, \mathbb{E}\big[ \log \boldsymbol{Z}^{\omega}_{\beta_N,h_N}(Nt) \big] = \lim_{N\to\infty} N\, \mathbf{F}(\beta_N, h_N)$$

Setting $\delta = \frac{1}{N}$ for clarity, we arrive at...

# Interchanging the limits

## Conjecture

$$\mathcal{F}(\hat\beta, \hat h) \;=\; \lim_{\delta\to 0} \frac{\mathbf{F}\big(\hat\beta\delta^{\alpha-\frac{1}{2}}, \hat h\delta^{\alpha}\big)}{\delta}$$

## Theorem [C., Toninelli, Torri 2015]

For all $\hat\beta > 0$, $\hat h \in \mathbb{R}$ and $\eta > 0$ there is $\delta_0 > 0$ such that $\forall \delta < \delta_0$

$$\mathcal{F}(\hat\beta, \hat h - \eta) \;\le\; \frac{\mathbf{F}\big(\hat\beta\delta^{\alpha-\frac{1}{2}}, \hat h\delta^{\alpha}\big)}{\delta} \;\le\; \mathcal{F}(\hat\beta, \hat h + \eta)$$

This implies Conj. and $\boxed{\mathbf{h}_c(\beta) \sim \mathcal{H}_c(\beta) \sim \mathcal{H}_c(1)\,\beta^{\frac{2\alpha}{2\alpha-1}}}$

For any discrete Pinning model with $\alpha \in (\frac{1}{2}, 1)$, the free energy $\boldsymbol{F}(\beta, h)$ and the critical curve $\mathbf{h}_c(\beta)$ have a universal shape in the regime $\beta, h \to 0$

# Interchanging the limits

Very delicate result. How to prove it?

- Assume that there is a continuum Hamiltonian:

$$\boldsymbol{Z}^{\omega} = \mathbf{E}\left[e^{\boldsymbol{H}^{\omega}_{Nt}}\right] \qquad \mathcal{Z}^{W} = \mathcal{E}\left[e^{\mathcal{H}^{W}_{t}}\right]$$

- Couple $\boldsymbol{H}^{\omega}_{Nt}$ and $\mathcal{H}^{W}_{t}$ on the same probability space in such a way that the difference $\Delta_{N,t} := \boldsymbol{H}^{\omega}_{Nt} - \mathcal{H}^{W}_{t}$ is "small"
- Deduce that

$$\mathbb{E}\left[\log \boldsymbol{Z}^{\omega}\right] \leq \mathbb{E}\left[\log \mathcal{Z}^{W}\right] + \log \mathbb{E}\mathbf{E}\left[e^{\Delta_{N,t}}\right]$$

  and show that the last term is "negligible"

Problem: there is no continuum Hamiltonian!

Solution: perform coarse-graining and define an "effective" Hamiltonian (drawing!)

# The DPRE case

What about the DPRE?

We can still define discrete $\mathbf{F}(\beta)$ and continuum $\mathcal{F}(\hat{\beta})$ free energy

Since $\mathcal{F}(\hat{\beta}) \sim \mathcal{F}(1)\,\beta^4$ we can hope that

$$\mathbf{F}(\beta) \sim \mathcal{F}(1)\,\beta^4 \qquad \text{as } \beta \to 0$$

provided the "interchanging of limits" is justified

N. Torri is currently working on this problem. A finer coarse-graining is needed, together with sharper estimates on continuum partition functions

# Outline

# The marginal case

We consider simultaneously different models that are marginally relevant:

- Pinning Models with $\alpha = \frac{1}{2}$
- DPRE with $d = 2$ (RW attracted to BM)
- DPRE with $d = 1$ (RW with Cauchy tails: $\mathrm{P}(|S_1| > n) \sim \frac{c}{n}$
- Stochastic Heat Equation in $d = 2$

All these different models share a crucial feature: logarithmic overlap

$$\mathbf{R}_N = \begin{cases} \displaystyle\sum_{1 \le n \le N} \mathbf{P}^{\mathrm{ref}}(n \in \tau)^2 \\ \displaystyle\sum_{1 \le n \le N} \sum_{x \in \mathbb{Z}^d} \mathbf{P}^{\mathrm{ref}}(S_n = x)^2 \end{cases} \sim C \log N$$

More generally: $\mathbf{R}_N$ diverges as a slowly varying function

# The marginal case

Analogy between Pinning model with $\alpha = \frac{1}{2}$ and DPRE with $d = 2$

$$\mathbf{Z}^{\omega}_{\mathrm{Pin}} = 1 + \sum_{n=1}^{N} \mathbf{P}^{\mathrm{ref}}(n \in \tau)\, X_n + \ldots$$

$$\mathbf{Z}^{\omega}_{\mathrm{DPRE}} = 1 + \sum_{n=1}^{N} \left( \sum_{x \in \mathbb{Z}^2} \mathbf{P}^{\mathrm{ref}}(S_n = x)\, X_{n,x} \right) + \ldots$$

Note that $\mathbf{P}^{\mathrm{ref}}(S_n = x) \sim \frac{1}{n}\, g_1(\frac{x}{\sqrt{n}})$ (recall that $d = 2$)

Then the random variable in parenthesis has variance

$$\sum_{x \in \mathbb{Z}^2} \mathbf{P}^{\mathrm{ref}}(S_n = x)^2 \;\sim\; \frac{1}{n}\,\frac{1}{n} \sum_{x \in \mathbb{Z}^2} g_1\big(\tfrac{x}{\sqrt{n}}\big)^2 \;\sim\; \frac{\|g_1\|_2^2}{n}$$

hence we can replace it by $\frac{\|g_1\|_2}{\sqrt{n}} X_n$ $\rightsquigarrow$ Pinning! ($\mathbf{P}^{\mathrm{ref}}(n \in \tau) \sim \frac{c}{\sqrt{n}}$)

# Relevant vs. marginal regime

For computation we focus on Pinning model (for simplicity $h = 0$)

Look at relevant case $\alpha > \frac{1}{2}$ $\left(\mathbf{P}^{\mathrm{ref}}(n \in \tau) \sim \frac{1}{n^{1-\alpha}}\right)$ $\beta \sim \frac{\hat{\beta}}{N^{\alpha - \frac{1}{2}}}$

$$\begin{aligned}
\boldsymbol{Z}_N^{\omega} &= \sum_{k=0}^{N} \beta^k \sum_{0<n_1<\ldots<n_k \le N} \prod_{i=1}^{k} \mathbf{P}^{\mathrm{ref}}(n_i - n_{i-1} \in \tau)\, X_{n_i} \\
&= \sum_{k=0}^{N} \beta^k \sum_{0<n_1<\ldots<n_k \le N} \frac{X_{n_1} X_{n_2} \cdots X_{n_k}}{n_1^{1-\alpha}(n_2-n_1)^{1-\alpha}\cdots(n_k-n_{k-1})^{1-\alpha}} \\
&= \sum_{k=0}^{N} \left(\frac{\beta\sqrt{N}}{N^{1-\alpha}}\right)^k \sum_{0<n_1<\ldots<n_k \le N} \frac{\frac{1}{\sqrt{N}} X_{n_1} \frac{1}{\sqrt{N}} X_{n_2} \cdots \frac{1}{\sqrt{N}} X_{n_k}}{(\frac{n_1}{N})^{1-\alpha}(\frac{n_2}{N}-\frac{n_1}{N})^{1-\alpha}\cdots(\frac{n_k}{N}-\frac{n_{k-1}}{N})^{1-\alpha}} \\
&\xrightarrow[N\to\infty]{d} \sum_{k=0}^{\infty} \hat{\beta}^k \int_{0<t_1<\ldots<t_k \le 1} \frac{\mathrm{d}W_{t_1}\, \mathrm{d}W_{t_2} \cdots \mathrm{d}W_{t_k}}{t_1^{1-\alpha}(t_2-t_1)^{1-\alpha}\cdots(t_k-t_{k-1})^{1-\alpha}}
\end{aligned}$$

For $\alpha = \frac{1}{2}$ last step breaks down $\frac{1}{\sqrt{t}} \notin L^2([0,1])$ How to make sense?

# Relevant vs. marginal regime

Always in the relevant case $\alpha > \frac{1}{2}$ (setting $\chi = 2(1-\alpha) < 1$ )

$$\mathbb{V}\mathrm{ar}[\boldsymbol{Z}_N^{\omega}] \xrightarrow[N\to\infty]{} \sum_{k=0}^{\infty} \hat{\beta}^k \int_{0<t_1<\ldots<t_k\le 1} \frac{\mathrm{d}t_1\, \mathrm{d}t_2 \cdots \mathrm{d}t_k}{t_1^{\chi}(t_2-t_1)^{\chi}\cdots(t_k-t_{k-1})^{\chi}}$$

$$\le \sum_{k=0}^{\infty} \hat{\beta}^k \frac{\Gamma(1-\chi)^{k+1}}{\Gamma((1-\chi)(k+1))} \le \sum_{k=0}^{\infty} \hat{\beta}^k \frac{c_1^k}{(c_2 k)!} < \infty$$

The $k!$ makes the series converge for all $\hat{\beta} > 0$

It arises from the constraint $0 < t_1 < \ldots < t_k \le 1$

## Exercise

Prove "by bare hands" that (probabilistic argument!)

$$\int_{0<t_1<\ldots<t_k\le 1} \frac{\mathrm{d}t_1\, \mathrm{d}t_2 \cdots \mathrm{d}t_k}{t_1^{\chi}(t_2-t_1)^{\chi}\cdots(t_k-t_{k-1})^{\chi}} \le e^{-Ck\log k}$$

# Relevant vs. marginal regime

In the marginal regime $\alpha = \frac{1}{2}$

$$\boldsymbol{Z}_N^\omega = \sum_{k=0}^{N} \beta^k \sum_{0<n_1<\ldots<n_k\le N} \frac{X_{n_1} X_{n_2} \cdots X_{n_k}}{\sqrt{n_1}\sqrt{n_2-n_1}\cdots\sqrt{n_k-n_{k-1}}}$$

$$= 1 + \beta \sum_{0<n\le N} \frac{X_n}{\sqrt{n}} + \beta^2 \sum_{0<n<n'\le N} \frac{X_n X_{n'}}{\sqrt{n}\sqrt{n'-n}} + \ldots$$

Goal: find the joint limit in distribution of all these sums

Linear term is easy ($X_n \sim \mathcal{N}(0,1)$ by Lindeberg): asympt. $\mathcal{N}(0,\sigma^2)$

$$\sigma^2 = \beta^2 \sum_{0<n\le N} \frac{1}{n} \sim \beta^2 \log N$$

We then rescale $\boxed{\beta = \beta_N \sim \frac{\hat{\beta}}{\sqrt{\log N}}}$ Other terms converge?

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