Polynomial Chaos and Scaling Limits of Disordered Systems

5. Marginally relevant systems

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Levico Terme $\sim$ September 30 - October 2, 2015

Overview

In the previous lectures we focused on systems that are disorder relevant (in particular DPRE with $d = 1$ and Pinning model with $\alpha > \frac{1}{2}$)

- We constructed continuum partition functions $\mathcal{Z}^W$
- We used $\mathcal{Z}^W$ to build continuum disordered models $\mathcal{P}^W$
- We used $\mathcal{Z}^W$ to get estimates on the free energy $\mathbf{F}(\beta, h)$

In this last lecture we consider the subtle marginally relevant regime (in particular DPRE with $d = 2$, Pinning model with $\alpha = \frac{1}{2}$, 2d SHE)

We present some results on the the continuum partition function

Outline

1. The marginally relevant regime

Outline

The 2d Stochastic Heat Equation

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x) + \beta\, W(t,x)\, u(t,x) \\ u(0,x) \equiv 1 \end{cases} \qquad (t,x) \in [0,\infty) \times \mathbb{R}^2$$

where $W(t,x)$ is (space-time) white noise on $[0,\infty) \times \mathbb{R}^2$

Mollification in space: fix $j \in C_0^\infty(\mathbb{R}^d)$ and set $j_\delta(z) := \delta^{-1} j(\sqrt{\delta} z)$

$$W_\delta(t,x) := \int_{\mathbb{R}^d} j_\delta(x-y)\, W(t,y)\, \mathrm{d}y$$

Then $\qquad u_\delta(t,x) = \mathrm{E}_x\left[\exp\left\{\beta \int_0^t W_\delta(t-s, B_s)\, \mathrm{d}s - \frac{1}{2}\beta^2 t \|j_\delta\|_2^2\right\}\right]$

By soft arguments $u_{\delta,\beta} \overset{d}{\approx} Z_{N,\beta}^\omega$ (partition function of 2d DPRE)

Scaling limit of marginal partition function

Theorem 1. [C., Sun, Zygouras '15b]

Consider DPRE $d = 2$ or Pinning $\alpha = \frac{1}{2}$ or 2d SHE

(or long-range DPRE with $d = 1$ and Cauchy tails)

Rescaling $\beta := \dfrac{\hat{\beta}}{\sqrt{\log N}}$ (and $h \equiv 0$) the partition function converges in law to an explicit limit:

$$Z_N^\omega \xrightarrow[N\to\infty]{d} \mathcal{Z}^W = \begin{cases} \text{log-normal} & \text{if } \hat{\beta} < 1 \\ 0 & \text{if } \hat{\beta} \geq 1 \end{cases}$$

$$\mathcal{Z}^W \overset{d}{=} \exp\left\{ \sigma_{\hat{\beta}} W_1 - \frac{1}{2}\sigma_{\hat{\beta}}^2 \right\} \qquad \text{with} \quad \sigma_{\hat{\beta}} = \log \frac{1}{1 - \hat{\beta}^2}$$

The regime $\hat{\beta} = 1$ (in progress)

What happens for $\hat{\beta} \geq 1$? $\quad$ $\boldsymbol{Z}_N^W(t,x) \to 0$ in law for *fixed* (t,x)

However, $\boldsymbol{Z}_N^W(t,x)$ should have a non-zero limit if we look at it as a space-time distribution (actually a measure) $\quad$ cf. [Bertini, Cancrini 95]

Although $\mathbb{V}\mathrm{ar}[\boldsymbol{Z}_N^W(t,x)] \to \infty$ covariances are bounded on diffusive scale

$$\mathbb{C}\mathrm{ov}[\boldsymbol{Z}_N^W(t,x)\,,\,\boldsymbol{Z}_N^W(t',x')] \underset{N\to\infty}{\sim} K\Big(\big(\tfrac{t}{N},\tfrac{x}{\sqrt{N}}\big)\,,\,\big(\tfrac{t'}{N},\tfrac{x'}{\sqrt{N}}\big)\Big)$$

Then $\quad \mathbb{V}\mathrm{ar}\big[\langle \boldsymbol{Z}_N^W\,,\,\phi\rangle\big] \to (\phi, K\phi) < \infty \quad$ for every $\phi \in C_0([0,1]\times\mathbb{R}^2)$

$$\langle \boldsymbol{Z}_N^W\,,\,\phi\rangle := \frac{1}{N^{3/2}} \sum_{(t,x)\in\mathbb{N}\times\mathbb{Z}^2} \phi\left(\frac{t}{N},\frac{x}{\sqrt{N}}\right)\boldsymbol{Z}_N^W(t,x)$$

Multi-scale correlations for $\hat\beta < 1$

Theorem 2. [C., Sun, Zygouras '15b]

Consider DPRE with $d = 2$ or 2d SHE (fix $\hat\beta < 1$)

Fix space-time points $\mathrm{X} = (t_N, x_N)$ and $\mathrm{X}' = (t'_N, x'_N)$ with

$$\|X - X'\| := |t_N - t'_N| + \sqrt{|x_N - x'_N|} \sim N^{\zeta} \qquad \zeta \in (0,1)$$

Then $$\left(Z_N^{\omega}(\mathrm{X}) \, , \, Z_N^{\omega}(\mathrm{X}')\right) \xrightarrow[N\to\infty]{d} \left(e^{Y - \frac{1}{2}\mathbb{V}\mathrm{ar}[Y]} \, , \, e^{Y' - \frac{1}{2}\mathbb{V}\mathrm{ar}[Y']}\right)$$

Y, Y' joint $\mathcal{N}(0, \sigma^2_{\hat\beta})$ with $$\mathbb{C}\mathrm{ov}\left[Y, Y'\right] = \log \frac{1 - \zeta \hat\beta^2}{1 - \hat\beta^2}$$

Fluctuations for $\hat{\beta} < 1$

Theorem 3. [C., Sun, Zygouras '15b]

Consider DPRE with $d = 2$ or 2d SHE (fix $\hat{\beta} < 1$)

$$Z_N^W(t,x) \approx 1 + \frac{1}{\sqrt{\log N}} G\left(\tfrac{t}{N}, \tfrac{x}{N}\right) \qquad (\text{in } \mathcal{S}')$$

where G(t,x) is a Gaussian field on $[0,1] \times \mathbb{R}^2$ with

$$\mathbb{C}\mathrm{ov}\left[G(\mathrm{X}), G(\mathrm{X}')\right] \sim C \log \frac{1}{\|\mathrm{X} - \mathrm{X}'\|}$$

Proof of Theorem 1. for pinning

$$\mathbf{Z}_N^{\omega} = \sum_{k=0}^{N} \beta^k \sum_{0<n_1<\ldots<n_k\le N} \frac{X_{n_1} X_{n_2} \cdots X_{n_k}}{\sqrt{n_1}\sqrt{n_2-n_1}\cdots\sqrt{n_k-n_{k-1}}}$$

$$= 1 + \frac{\hat{\beta}}{\sqrt{\log N}} \sum_{0<n\le N} \frac{X_n}{\sqrt{n}} + \left(\frac{\hat{\beta}}{\sqrt{\log N}}\right)^2 \sum_{0<n<n'\le N} \frac{X_n X_{n'}}{\sqrt{n}\sqrt{n'-n}} + \ldots$$

Goal: find the joint limit in distribution of all these sums

$\rightsquigarrow$ blackboard!

Fourth moment theorem

4th Moment Theorem [de Jong 90] [Nualart, Peccati, Reinert 10]

Consider homogeneous (deg. ℓ) polynomial chaos $Y_N = \sum_{|I|=\ell} \psi_N(I) \prod_{i \in I} X_i$

- $\max_i \psi_N(i) \xrightarrow[N\to\infty]{} 0$ (in case $\ell = 1$) [Small influences!]
- $\mathbb{E}[(Y_N)^2] \xrightarrow[N\to\infty]{} \sigma^2$
- $\mathbb{E}[(Y_N)^4] \xrightarrow[N\to\infty]{} 3\,\sigma^4$

Then $$Y_N \xrightarrow[N\to\infty]{d} \mathcal{N}(0, \sigma^2)$$

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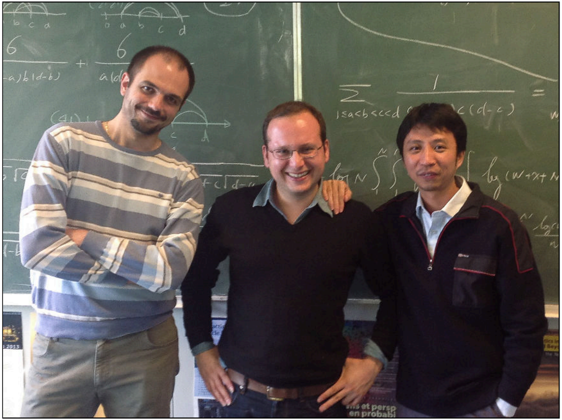
Collaborators



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