Scaling and Universality in Probability

Francesco Caravenna

Università degli Studi di Milano-Bicocca

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Overview

A more expressive (but less fancy) title would be

Convergence of Discrete Probability Models to a Universal Continuum Limit

This is a key topic of classical and modern probability theory

I will present a (limited) selection of representative results, in order to convey the main ideas and give the flavor of the subject

Outline

Reminders (I). Probability spaces

Fix a set Ω. A probability P is a map from subsets of Ω to $[0,1]$ s.t.

$$\mathrm{P}(\Omega) = 1\,, \qquad \mathrm{P}\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mathrm{P}(A_i) \quad \text{for disjoint } A_i$$

[P is only defined on a subclass (σ-algebra) $\mathcal{A}$ of "measurable" subsets of Ω]

$(\Omega, \mathcal{A}, \mathrm{P})$ is an abstract probability space. We will be "concrete":

(Metric space E, "Borel σ-algebra", Probability μ)

- Integral $\int_E \varphi \, \mathrm{d}\mu$ for bounded and continuous $\varphi : E \to \mathbb{R}$
- Discrete probability $\mu = \sum_i p_i \, \delta_{x_i}$ with $x_i \in E$, $p_i \in [0,1]$

$$\int_E \varphi \, \mathrm{d}\mu := \sum_i p_i \, \varphi(x_i)$$

Riemann sums and integral on $[0,1]$

- Partition $\underline{t} = (t_0, t_1, \ldots, t_k)$ of $[0,1]$

$$0 = t_0 < t_1 < \ldots < t_k = 1 \qquad (k \in \mathbb{N})$$

- Riemann sum of a function $\varphi : [0,1] \to \mathbb{R}$ relative to $\underline{t}$

$$R(\varphi, \underline{t}) := \sum_{i=1}^{k} \varphi(t_i)\,(t_i - t_{i-1})$$

Theorem

Let $\underline{t}^{(n)}$ be partitions with

$$\mathrm{mesh}(\underline{t}^{(n)}) := \max_{1 \le i \le k_n} \left(t_i^{(n)} - t_{i-1}^{(n)}\right) \xrightarrow[n\to\infty]{} 0$$

If $\varphi : [0,1] \to \mathbb{R}$ is continuous, then

$$R(\varphi, \underline{t}^{(n)}) \xrightarrow[n\to\infty]{} \int_0^1 \varphi(x)\,\mathrm{d}x$$

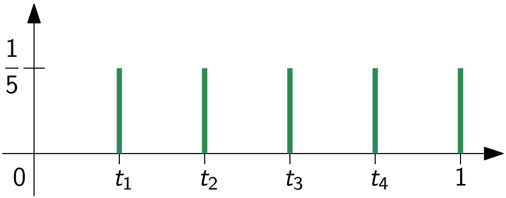
A probabilistic reformulation

Partition $\underline{t} = \big(t_0, t_1, \ldots, t_k\big) \quad \rightsquigarrow \quad$ discrete probability $\mu_{\underline{t}}$ on $[0,1]$

$$\mu_{\underline{t}}(\cdot) := \sum_{i=1}^{k} p_i \, \delta_{t_i}(\cdot) \qquad \text{where} \qquad p_i := t_i - t_{i-1}$$

Uniform partition

$\underline{t} = \big(0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\big) \qquad \rightsquigarrow \qquad \mu_{\underline{t}} =$ uniform probability on $\big\{\frac{1}{n}, \frac{2}{n}, \ldots, 1\big\}$



A probabilistic reformulation

Key observation: Riemann sum is . . . integral w.r.t. $\mu_{\underline{t}}$

$$R(\varphi, \underline{t}) = \sum_{i=1}^{k} \varphi(t_i)\, p_i = \int_{[0,1]} \varphi \, \mathrm{d}\mu_{\underline{t}}$$

Theorem

If mesh($\underline{t}^{(n)}$) $\to 0$ and $\varphi : [0,1] \to \mathbb{R}$ is continuous, then

$$\int_{[0,1]} \varphi \, \mathrm{d}\mu_{\underline{t}^{(n)}} \xrightarrow[n\to\infty]{} \int_{[0,1]} \varphi \, \mathrm{d}\boldsymbol{\lambda} \qquad (\star)$$

with $\boldsymbol{\lambda}$:= Lebesgue measure (probability) on $[0, 1]$

- Scaling Limit: convergence of $\mu_{\underline{t}^{(n)}}$ toward $\boldsymbol{\lambda}$
- Universality: the limit $\boldsymbol{\lambda}$ is the same, for any choice of $\underline{t}^{(n)}$

Weak convergence

- E is a Polish space (complete separable metric space), e.g.

$$[0,1]\,, \qquad C([0,1]) := \{\text{continuous } f : [0,1] \to \mathbb{R}\}\,, \qquad \ldots$$

- $(\mu_n)_{n\in\mathbb{N}}$, $\boldsymbol{\mu}$ are probabilities on E

Definition (weak convergence of probabilities)

We say that μ_n converges weakly to $\boldsymbol{\mu}$ (notation $\mu_n \Rightarrow \boldsymbol{\mu}$) if

$$\int_E \varphi \, \mathrm{d}\mu_n \xrightarrow[n\to\infty]{} \int_E \varphi \, \mathrm{d}\boldsymbol{\mu}$$

for every $\varphi \in C_b(E) := \{$continuous and bounded $\varphi : E \to \mathbb{R}\}$

[Analysts call this weak-$*$ convergence; note that $\mu_n, \boldsymbol{\mu} \in C_b(E)^*$]

A useful reformulation

- $\mu_n \Rightarrow \boldsymbol{\mu}$ does not imply $\mu_n(A) \to \boldsymbol{\mu}(A)$ for all meas. $A \subseteq E$?

Example

$$\mu_n = \text{uniform probability on } \left\{\tfrac{1}{n}, \tfrac{2}{n}, \ldots, 1\right\} \qquad A := \mathbb{Q} \cap [0,1]$$

$$\mu_n \Rightarrow \boldsymbol{\lambda} \ \text{(Lebesgue)} \qquad \text{but} \qquad 1 = \mu_n(A) \not\to \boldsymbol{\lambda}(A) = 0$$

- Weak convergence means $\mu_n(A) \to \boldsymbol{\mu}(A)$ for "nice" $A \subseteq E$

Theorem

$$\mu_n \Rightarrow \boldsymbol{\mu} \quad \text{iff} \quad \mu_n(A) \to \boldsymbol{\mu}(A) \ \forall \text{ meas. } A \subseteq E \ \text{ with } \ \boldsymbol{\mu}(\partial A) = 0$$

- Weak convergence links measurable and topological structures

Rest of the talk

Three interesting examples of weak convergence, leading to

- Brownian motion
- Schramm-Löwner Evolution (SLE)
- Continuum disordered pinning models

Common mathematical structure

- A Polish space E
- A sequence of discrete probabilities μ_n (easy) on E
- A "continuum" probability $\boldsymbol{\mu}$ (difficult!) such that $\mu_n \Rightarrow \boldsymbol{\mu}$

Outline

1. Weak Convergence of Probability Measures

2. Brownian Motion

3. A glimpse of SLE

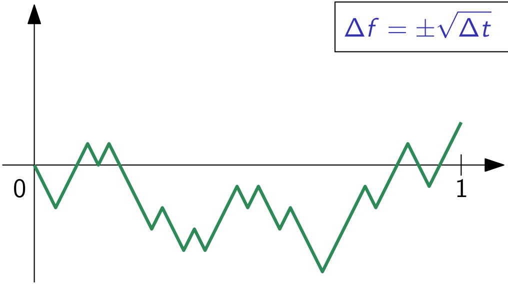
4. Scaling Limits in presence of Disorder

From random walk to Brownian motion

- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\}$ (with $\|\cdot\|_\infty$)
- $E_n := \Big\{$ piecewise linear $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and $f\big(\frac{i+1}{n}\big) = f\big(\frac{i}{n}\big) \pm \sqrt{\frac{1}{n}}\ \Big\} \subseteq C([0,1])$

  $|E_n| = 2^n$

$\Delta f = \pm\sqrt{\Delta t} \quad \rightsquigarrow \quad \text{slope}(f) = \pm\sqrt{n}$



Case $n = 40$

From random walk to Brownian motion

Let μ_n be the probability on $C([0,1])$ which is uniform on E_n:

$$\mu_n(\cdot) = \sum_{f \in E_n} \frac{1}{2^n} \, \delta_f(\cdot)$$

Theorem (Donsker)

The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly on $C([0,1])$: $\mu_n \Rightarrow \boldsymbol{\mu}$

The limiting probability $\boldsymbol{\mu}$ on $C([0,1])$ is called Wiener measure

- Deep result!
- Wiener measure is the law of Brownian motion
- Wiener measure is a "natural" probability on $C([0,1])$ (like Lebesgue for $[0,1]$)

Reminders (II). Random variables and their laws

A random variable (r.v.) is a measurable function $X : \Omega \to E$

[where $(\Omega, \mathcal{A}, \mathrm{P})$ is some abstract probability space]

The law (or distribution) μ_X of X is a probability on E

$$\mu_X(A) = \mathrm{P}(X^{-1}(A)) = \mathrm{P}(X \in A) \qquad \text{for } A \subseteq E$$

- X describes a random element of E
- μ_X describes the values taken by X and the resp. probabilities

Instead of a probability μ on E, it is often convenient to work with a random variable X with law μ

When $E = C([0,1])$, a r.v. $X = (X_t)_{t \in [0,1]}$ is a stochastic process

Simple random walk

Let us build a stochastic process $X^{(n)}$ with law μ_n

Fair coin tossing: independent random variables $Y_1, Y_2, \ldots$ with

$$\mathrm{P}(Y_i = +1) = \mathrm{P}(Y_i = -1) = \frac{1}{2}$$

Simple random walk: $S_0 := 0 \qquad S_n := Y_1 + Y_2 + \ldots + Y_n$

Diffusive rescaling: space $\propto \sqrt{\text{time}}$

$$X^{(n)}(t) := \text{linear interpol. of } \frac{S_{nt}}{\sqrt{n}} \qquad t \in [0,1]$$

The law of $X^{(n)}$ (r.v. in $C([0,1])$) is μ_n uniform probab. on E_n

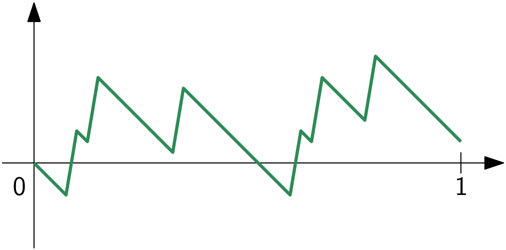
Donsker: The law of simple random walk, diffusively rescaled, converges weakly to the law of Brownian motion

General random walks

Instead of coin tossing, take independent random variables Y_i with a generic law, with zero mean and finite variance (say 1)

Define random walk S_n and its diffusive rescaling $X^{(n)}(t)$ as before

$$\text{E.g.} \qquad \mathrm{P}(Y_i = +2) = \frac{1}{3}, \quad \mathrm{P}(Y_i = -1) = \frac{2}{3}$$



The law μ_n of $X^{(n)}$ is a (non uniform!) probability on $C([0, 1])$

Universality of Brownian motion

Theorem (Donsker)

$$\mu_n \Rightarrow \boldsymbol{\mu} \ := \text{Wiener measure}$$

The law of any RW (zero mean, finite variance) diffusively rescaled converges weakly to the law of Brownian motion (Wiener measure)

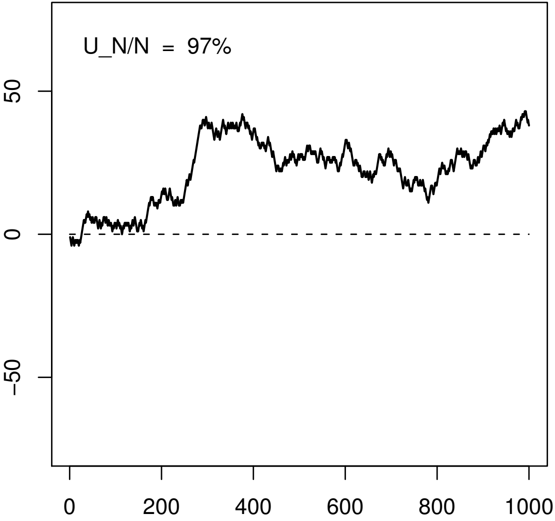
Universality: $\mu_n(A) \longrightarrow \boldsymbol{\mu}(A) \quad \forall A \subseteq C([0,1])$ with $\boldsymbol{\mu}(\partial A) = 0$

Example (Feller I, Chapter III)

- $U_+(f) := \mathrm{Leb}\{t \in [0,1] : f(t) > 0\}$
  $= \{\text{amount of time in which } f > 0\}$
- $A := \{f : \ U_+(f) \geq 0.95 0.99 \ \text{ or } U_+(f) \leq 0.05 0.01\} \subseteq C([0,1])$

Then $\mu_n(A) \to \boldsymbol{\mu}(A) \simeq \mathbf{0.29 0.13}$. Random walk has a chance of 29%13% of spending 95%99% or more of its time on the same side!

Some sample paths of the SRW



Outline

A glimpse of SLE

Even the simplest randomness (coin tossing) can lead to interesting models, such as random walks and Brownian motion

Brownian motion is at the heart of Schramm-Löwner Evolution (SLE), one of the greatest achievements of modern probability

[Fields Medal awarded to W. Werner (2006) and S. Smirnov (2010)]

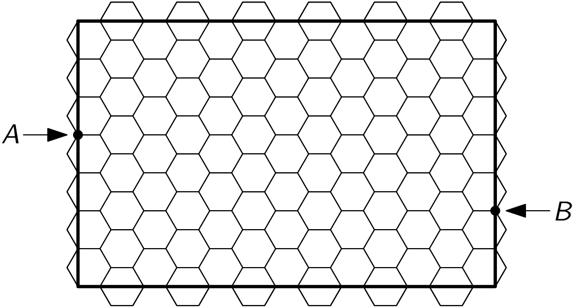
We present an instance of SLE, which emerges as the scaling limit of percolation (spatial version of coin tossing)

Fix a simply connected Jordan domain $D \subseteq \mathbb{R}^2$ and $A, B \in \partial D$

$$E := \big\{ \text{continuous } f : [0,1] \to \overline{D} \text{ with } f(0) = A,\ f(1) = B \big\}$$
$$= \big\{ \text{curves in } \overline{D} \text{ joining } A \text{ to } B \big\} \qquad [\|\cdot\|_\infty \text{ norm, up to reparam.}]$$

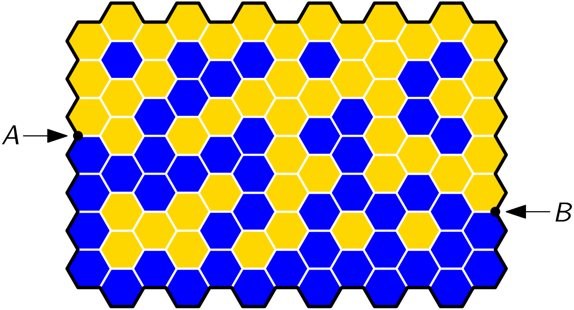
We now introduce discrete probabilities μ_n on E

1. The rescaled hexagonal lattice



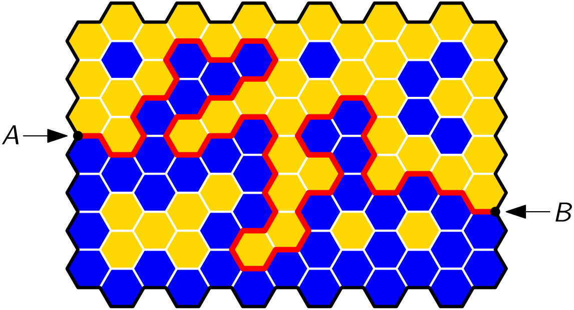
- Fix $n \in \mathbb{N}$ and consider the hexagonal lattice of side $\frac{1}{n}$
- Approximate ∂D with a closed loop in the lattice

2. Percolation



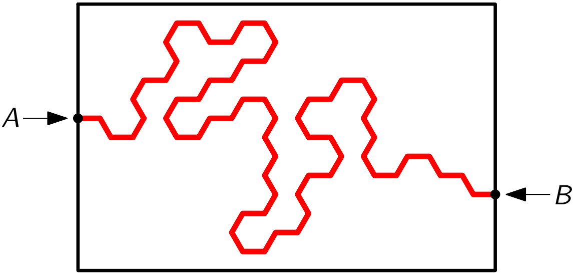
- Boundary hexagons colored yellow (A to B) and blue (B to A)
- Inner hexagons colored by coin tossing (critical percolation)

3. The exploration path



- Exploration path: start from A and follow the boundary between yellow and blue hexagons, eventually leading to B

4. The law μ_n



- Forgetting the colors, the exploration path is an element of E (continuous curve $A \to B$)
- It is a random element of E (determined by coin tossing)
- Its law μ_n is a discrete probability on E ($\frac{1}{n}$ = lattice mesh)

Scaling limit of the exploration path

Fix a (simply connected) Jordan domain D and points $A, B \in \partial D$

$$E := \{\text{curves in } \overline{D} \text{ joining } A \text{ to } B\}$$

Theorem (Schramm; Smirnov; Camia & Newman)

The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly on E: $\mu_n \Rightarrow \boldsymbol{\mu}$

The limiting probability $\boldsymbol{\mu}$ is the law of (the trace of) SLE(6)

- Extremely challenging!
- Universality? Independence of lattice (loop soup - conj.)
- Conformal Invariance. For another Jordan domain D'

$$\boldsymbol{\mu}_{D';A',B'} = \phi_{\#}(\boldsymbol{\mu}_{D;A,B})$$

where $\phi : D \to D'$ is conformal with $\phi(A) = A'$, $\phi(B) = B'$

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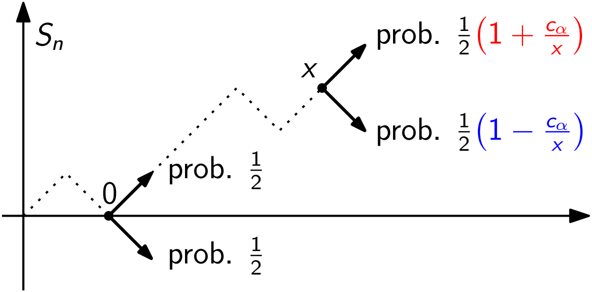
From simple to Bessel random walk

The simple random walk is $S_n := Y_1 + \ldots + Y_n$ [Y_i coin tossing]

Fix $\alpha \in (0,1)$ and define the α-Bessel random walk as follows:



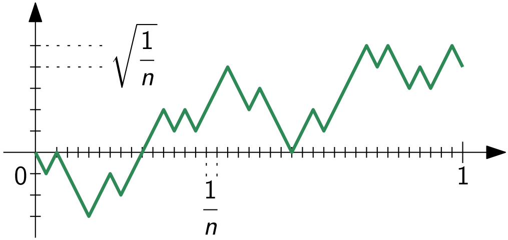
$$c_\alpha := \frac{1}{2} - \alpha$$

- $(\alpha = \frac{1}{2})$ no drift $(c_\alpha = 0)$ $\rightsquigarrow$ simple random walk
- $(\alpha < \frac{1}{2})$ drift away from the origin $(c_\alpha > 0)$
- $(\alpha > \frac{1}{2})$ drift toward the origin $(c_\alpha < 0)$

Diffusively rescaled α-Bessel RW

Definition

$$\mu_{n,\alpha} := \text{law of diffusively rescaled } \alpha\text{-Bessel RW}$$



Discrete probability on $E_n \subseteq C([0,1])$

Not uniform for $\alpha \neq \frac{1}{2}$

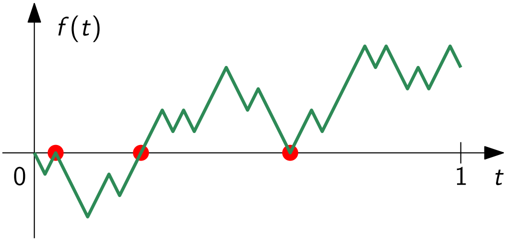
Theorem (Extension of Donsker)

$$\forall \alpha \in (0,1), \ \ \mu_{n,\alpha} \text{ converges weakly on } C([0,1]): \ \ \mu_{n,\alpha} \Rightarrow \boldsymbol{\mu}_\alpha$$

$$\left[\boldsymbol{\mu}_\alpha := \text{law of “}\alpha\text{-Bessel process”} \ \ (\text{Brownian motion for } \alpha = \tfrac{1}{2}) \right]$$

The disordered pinning model

Idea: reward/penalize α-Bessel RW $\mu_{n,\alpha}$ each time it visits zero



- Fix a real sequence $\omega = (\omega_i)_{i\in\mathbb{N}}$ (charges attached to $t = \frac{i}{n}$)
- Total charge (energy) of a path $H_n^\omega(f) := \sum_{i=1}^n \omega_i \, \mathbb{1}_{\{f(\frac{i}{n})=0\}}$

Disordered pinning model $\mu_{n,\alpha}^\omega$ (Gibbs measure)

$$\mu_{n,\alpha}^\omega(f) := \frac{1}{\text{(normaliz.)}} \, e^{H_n^\omega(f)} \, \mu_{n,\alpha}(f) \,, \qquad \forall f \in E_n$$

The disordered pinning model

$\mu_{n,\alpha}^{\omega}$ is a probability on $C([0,1])$ that depends on the sequence ω

How to choose the charges ω? In a random way!

$(\omega_i)_{i \in \mathbb{N}}$ independent $\mathcal{N}(h, \beta^2)$ [mean $h \in \mathbb{R}$, variance $\beta^2 > 0$]

Disordered systems: two sources of randomness!

- First we sample a typical ω, called (quenched) disorder
- Then we have a probability $\mu_{n,\alpha}^{\omega}$ on the space E_n of RW paths

The disordered pinning model $\mu_{n,\alpha}^{\omega}$ is a random probability on E_n

[i.e. a random variable $\omega \mapsto \mu_{n,\alpha}^{\omega}$ taking values in $\mathcal{M}_1(E_n)$]

Weak convergence of $\mu_{n,\alpha}^{\omega}$ [of its law] to some random probab. $\boldsymbol{\mu}_{\alpha}^{\omega}$?

Scaling limits of disordered pinning model

Inspired by [Alberts, Khanin, Quastel 2014]

Theorem (F. Caravenna, R. Sun, N. Zygouras)

Rescale suitably β, h (disorder mean and variance) and let $n \to \infty$

- ($\alpha < \frac{1}{2}$) Disorder disappears in the scaling limit!

  $\mu^{\omega}_{n,\alpha} \Rightarrow \boldsymbol{\mu}_{\alpha}$ law of α-Bessel process (as if $\omega \equiv 0$)

- ($\alpha > \frac{1}{2}$) Disorder survives in the scaling limit!

  $\mu^{\omega}_{n,\alpha} \Rightarrow \boldsymbol{\mu}^{\omega}_{\alpha}$ truly random probability on $C([0,1])$

  Recall that $\mu^{\omega}_{n,\alpha} \ll \mu_{n,\alpha}$ for every $n \in \mathbb{N}$ (Gibbs measure)

  However $\boldsymbol{\mu}^{\omega}_{\alpha} \not\ll \boldsymbol{\mu}_{\alpha}$ for a.e. ω ! (no continuum Gibbs meassure)

- ($\alpha = \frac{1}{2}$) Work in progress. . .

Thanks

Weak convergence in presence of disorder

- E is a Polish space (complete separable metric space)
- $\mathcal{M}_1(E) :=$ probability measures on E
- Notion of convergence $\mu_n \Rightarrow \mu$ (weak convergence) in $\mathcal{M}_1(E)$

What if μ_n^ω, μ^ω are random probabilities on E?
[$\omega \in \Omega$ probability space]

- The space $\tilde{E} := \mathcal{M}_1(E)$ is also Polish
- Random probabilities μ_n^ω, μ^ω are $\tilde{E}$-valued random variables
- Their laws are probabilities on $\tilde{E}$: weak convergence applies!

We still write $\mu_n^\omega \Rightarrow \mu^\omega$ for this convergence
(heuristics/intuition analogous to the non-disordered case)