Marginally relevant polymer models in the critical window

(joint work with R. Sun and N. Zygouras)

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Lyon $\sim$ 6 June 2017

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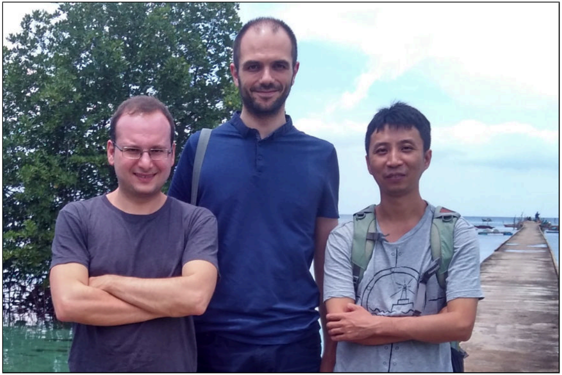


Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

Overview

I am going to talk about

- Directed Polymer in Random Environment in dim. $d = 2$

Our results also apply to other marginally relevant disordered systems

- Pinning Model with tail exponent $\alpha = 1/2$
- Directed Polymer with Cauchy tails in dim. $d = 1$
- Stochastic Heat Equation (SHE) with $d = 2$

Outline

1. Directed Polymer

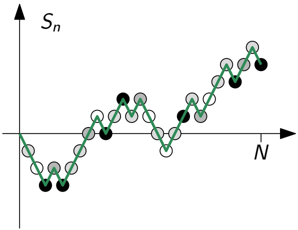
2. Known Results

3. Critical Window

4. Techniques and proofs

5. Additional results

Directed Polymer in Random Environment



- Reference Model: simple random walk on $\mathbb{Z}^d$

  $(S_n)_{n\geq 0} \qquad \mathbf{P}^{\mathrm{rw}}(S_n - S_{n-1} = \pm e_i) = \frac{1}{2d}$

- Disorder: i.i.d. random variables $\omega(n,x)$

  zero mean, unit variance, expon. moments

  $$\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega(n,x)}] < \infty$$

- (-) Hamiltonian $\quad H_N(S,\omega) := \sum_{n=1}^{N} \omega(n, S_n)$

Directed Polymer in Random Environment $\mathbf{P}_N^\omega = \mathbf{P}_{N,\beta}^\omega$

$$\frac{\mathrm{d}\mathbf{P}_N^\omega(S_1,\ldots,S_N)}{\mathrm{d}\mathbf{P}^{\mathrm{rw}}(S_1,\ldots,S_N)} \propto e^{\beta H_N(S,\omega)} = \frac{e^{\beta H_N(S,\omega)}}{\mathbf{Z}_N^\omega}$$

Weak and strong disorder

- ($d \geq 3$) There is a weak disorder phase: $\exists\, \beta_c > 0$ such that

for $0 \leq \beta < \beta_c$ $\quad$ $\mathbf{P}_N^\omega$ is "similar" to $\mathbf{P}^{\mathrm{rw}}$

CLT $$\mathbf{P}_N^\omega\left(\frac{S_N}{\sqrt{N}} \in \cdot\right) \xrightarrow[N\to\infty]{d} \mathcal{N}(0,1)$$

[Imbrie, Spencer 88] [Bolthausen 89] [Comets, Yoshida 06] [Chatterjee 16]

- ($d = 1$, $d = 2$) There is always strong disorder:

for any $\beta > 0$: $\quad$ $\mathbf{P}_N^\omega$ "very different" from $\mathbf{P}^{\mathrm{rw}}$

Conj. super-diffusivity $\quad |S_N| \gg \sqrt{N}$ under $\mathbf{P}_N^\omega$

Macroscopic atoms $\quad \max_{x\in\mathbb{Z}^d} \mathbf{P}_N^\omega(S_N = x) \geq c > 0$

[Carmona, Hu 02] [Comets, Shiga, Yoshida 03] [Vargas 07] [Lacoin 11] [Chatterjee 16]

Intermediate disorder

Henceforth we focus on the cases $d = 1$, $d = 2$

Any fixed disorder strength $\beta > 0$ has dramatic effects as $N \to \infty$

> Can we tune $\beta = \beta_N \to 0$ to see an interesting transition ?

This is called intermediate disorder regime, because it interpolates between weak and strong disorder

(cf. near-critical percolation)

We do not focus on the probability $\mathbf{P}_N^\omega$ but rather on partition functions

Partition function

Partition function (normalized)

$$Z_N^\omega = \mathbf{E}^{\mathrm{rw}}\left[e^{\beta H_N(S,\omega)}\right] = \mathbf{E}^{\mathrm{rw}}\left[e^{\beta \sum_{n=1}^N \omega(n,S_n)}\right] e^{-\lambda(\beta)N}$$

It amounts to redefine $Z_N^\omega \rightsquigarrow Z_N^\omega \,/\, \mathbb{E}[Z_N^\omega]$

- Z_N^ω is a positive random variable with $\mathbb{E}[Z_N^\omega] = 1$ (martingale!)
- ($d = 1$, $d = 2$) Strong disorder means

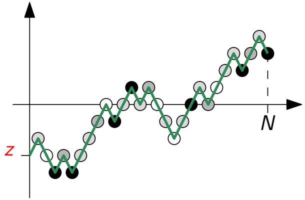
$$\forall \beta > 0: \qquad \lim_{N\to\infty} Z_N^\omega = 0 \qquad \mathbb{P}\text{-a.s.}$$

The random field of partition functions

$Z_N^\omega(z) :=$ partition function

for RW starting at $z \in \mathbb{Z}^d$

$$= \mathbf{E}^{\mathrm{rw}}\left[e^{\beta H_N} \,\middle|\, S_0 = z \right] e^{-\lambda(\beta) N}$$



Note that $Z_N^\omega(z) \overset{d}{=} Z_N^\omega(0) = Z_N^\omega \xrightarrow[N\to\infty]{} 0$ for every fixed $\beta > 0$

Can we tune $\beta = \beta_N \to 0$ so that

$$Z_N^\omega(\sqrt{N}\, x) \xrightarrow[N\to\infty]{d} \mathcal{Z}(x) \qquad \text{(random field on } \mathbb{R}^d\text{)}$$

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Case $d = 1$

For $d = 1$ the right scaling is $\beta_N = \dfrac{\hat{\beta}}{N^{1/4}}$

Theorem [Alberts, Khanin, Quastel (AOP '14)]

- Convergence in distribution

$$\boldsymbol{Z}^{\omega}_{Nt}(\sqrt{N}x) \xrightarrow[N\to\infty]{d} \mathcal{Z}_t(x)$$

- $\mathcal{Z}_t(x)$ is solution of 1d Stochastic Heat Equation (SHE)

$$\begin{cases} \partial_t \mathcal{Z} = \frac{1}{2}\Delta_x \mathcal{Z} + \hat{\beta}\, \dot{W}\, \mathcal{Z} \\ \mathcal{Z}_0 \equiv 1 \end{cases}$$

$W =$ Gaussian white noise on $[0, \infty) \times \mathbb{R}$

Case $d = 1$

Non-trivial limiting field: $\mathscr{Z}_t(x) > 0$ for every $\hat{\beta} \in (0, \infty)$

Corollary

Strong disorder emerges smoothly on the scale $\beta \propto \frac{1}{N^{1/4}}$

$$\boldsymbol{Z}_N^\omega \xrightarrow[N\to\infty]{d} \begin{cases} 1 & \text{if } \beta \ll \frac{1}{N^{1/4}} \\ \mathscr{Z} > 0 & \text{if } \beta \sim \frac{\hat{\beta}}{N^{1/4}} \\ 0 & \text{if } \beta \gg \frac{1}{N^{1/4}} \end{cases}$$

$\mathscr{Z}_t(x) \rightsquigarrow$ Brownian Directed Polymer in Random Environment

[Alberts, Khanin, Quastel (JSP '14)]

Case $d = 2$: marginal relevance

Henceforth we focus on $d = 2$

The right scaling is $\beta_N \sim \sqrt{\frac{\pi}{\log N}}\,\hat{\beta}$ [Lacoin '10] [Berger, Lacoin '15]

Logarithmic replica overlap

$$R_N := \mathbf{E}^{\mathrm{rw}}\left[\sum_{n=1}^{N} \mathbb{1}_{\{S_n = S'_n\}}\right] \sim \frac{1}{\pi}\log N$$

We look again for $Z_N^{\omega}(\sqrt{N}x) \xrightarrow[N\to\infty]{d} \mathcal{Z}(x)$

Unlike the case $d = 1$, there is a phase transition in $\hat{\beta}$

Phase transition

Theorem [C., Sun, Zygouras (AAP to appear)]

- For every fixed $x \in \mathbb{R}^2$

$$\boldsymbol{Z}_N^\omega(\sqrt{N}x) \xrightarrow[N\to\infty]{d} \tilde{\mathscr{Z}}(x) \begin{cases} > 0 \text{ a.s.} & \text{if } \hat{\beta} < 1 \\ = 0 \text{ a.s.} & \text{if } \hat{\beta} \geq 1 \end{cases}$$

- $(\hat{\beta} < 1)$ Log-normal marginals with $\mathbb{E}[\tilde{\mathscr{Z}}(x)] \equiv 1$

$$\tilde{\mathscr{Z}}(x) \stackrel{d}{=} \exp\left\{ N(0,\sigma^2) - \tfrac{1}{2}\sigma^2 \right\} \qquad \text{with} \quad \sigma^2 = \log \frac{1}{1-\hat{\beta}^2}$$

- $(\hat{\beta} < 1)$ Joint distributions: for any $x \neq x'$

$$\tilde{\mathscr{Z}}(x) \text{ and } \tilde{\mathscr{Z}}(x') \text{ are independent (!)}$$

[Dependence in $\boldsymbol{Z}_N^\omega(z)$, $\boldsymbol{Z}_N^\omega(z')$ at all scales $|z - z'| = o(\sqrt{N})$]

A different viewpoint

Recall that $\beta_N \sim \dfrac{\sqrt{\pi}\,\hat{\beta}}{\sqrt{\log N}}$

- ($\hat{\beta} < 1$) Disorder has weak effects ($\tilde{\mathcal{Z}}(x)$ indep. of $\tilde{\mathcal{Z}}(x')$)
- ($\hat{\beta} \geq 1$) Trivial limit $\tilde{\mathcal{Z}}(x) \equiv 0$

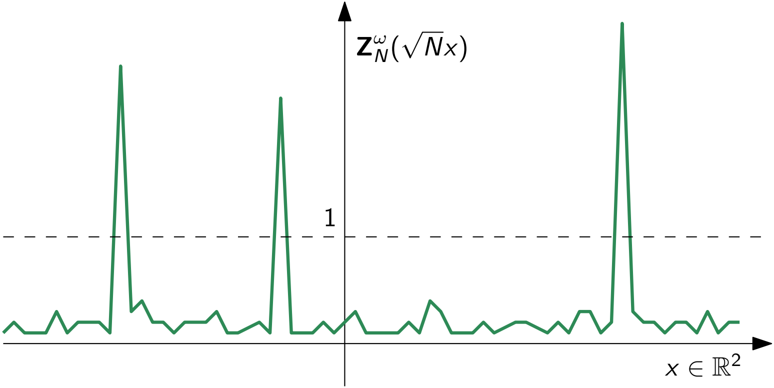
Can we obtain an interesting limit $\mathcal{Z}(x) \not\equiv 0$ for $\hat{\beta} \geq 1$?

$\boldsymbol{Z}_N^\omega(\sqrt{N}x)$ is an *irregular function* of $x \in \mathbb{R}^2$

$\rightsquigarrow$ We should look for a limit in the space of (Schwartz) distributions !

(Instead of distributions we can focus on measures, because $\boldsymbol{Z}_N^\omega \geq 0$)

Heuristic picture



Averaged partition function

Henceforth we look at $Z_N^\omega(\sqrt{N}x)$ as a random measure on $\mathbb{R}^2$

For positive continuous $\phi : \mathbb{R}^2 \to \mathbb{R}^+$ we define

$$\langle Z_N^\omega , \phi \rangle := \int_{\mathbb{R}^2} Z_N^\omega(\sqrt{N}x)\, \phi(x)\, \mathrm{d}x$$

We can revisit our results for $\hat{\beta} < 1$

Proposition

For $\hat{\beta} < 1$ we have $Z_N^\omega(\sqrt{N}x) \xrightarrow{d} \mathcal{Z}(x) \equiv 1$

$$\langle Z_N^\omega , \phi \rangle \xrightarrow[N\to\infty]{d} \langle 1 , \phi \rangle = \int_{\mathbb{R}^2} \phi(x)\, \mathrm{d}x$$

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What happens for $\hat{\beta} = 1$?

We now set $\hat{\beta} = 1$. More generally, we explore the critical window

$$\beta_N = \sqrt{\frac{\pi}{\log N}\left(1 + \frac{\vartheta}{\log N}\right)} \qquad \text{with } \vartheta \in \mathbb{R}$$

For fixed $x \in \mathbb{R}^2$ we already know that $\boldsymbol{Z}_N^\omega(\sqrt{N}x) \xrightarrow{d} 0$

We now look at $\boldsymbol{Z}_N^\omega(\sqrt{N}x)$ as a random measure

Conjecture

$\boldsymbol{Z}_N^\omega(\sqrt{N}x)$ converges to a generalized random field $\mathscr{Z}(x)$ on $\mathbb{R}^2$

$$\langle \boldsymbol{Z}_N^\omega , \phi \rangle \xrightarrow[N\to\infty]{d} \langle \mathscr{Z} , \phi \rangle \qquad \text{for every } \phi$$

$\mathscr{Z}$ is a random measure on $\mathbb{R}^2$ (expected to be singular wrt Lebesgue)

Second moment in the critical window

What is known [Bertini, Cancrini '95 (on 2d SHE)]

Tightness via second moment bounds

$$\mathbb{E}\big[\langle \boldsymbol{Z}_N^\omega , \phi\rangle\big] \equiv \big\langle 1, \phi \big\rangle \qquad \sup_{N\in\mathbb{N}} \mathbb{E}\big[\langle \boldsymbol{Z}_N^\omega , \phi\rangle^2\big] < \infty$$

More precisely $$\mathbb{V}\mathrm{ar}\big[\langle \boldsymbol{Z}_N^\omega , \phi\rangle\big] \xrightarrow[N\to\infty]{} \big\langle \phi, K\phi \big\rangle < \infty$$

Explicit $$K\big(x,x'\big) \sim C \log \frac{1}{|x-x'|} \qquad \text{as } |x-x'| \to 0$$

Corollary

Existence of subsequential limits $\langle \boldsymbol{Z}_N^\omega , \phi\rangle \xrightarrow[N\to\infty]{d} \langle \mathcal{Z} , \phi\rangle$

New results: third moment

Theorem [C., Sun, Zygouras '17+]

$$\lim_{N\to\infty} \mathbb{E}\Big[\langle \boldsymbol{Z}_N^\omega \,,\, \phi\rangle^3\Big] = C(\phi) < \infty$$

Corollary

Any subsequential limit $\mathscr{Z}$ has the same covariance kernel $K(x,x')$

$$\rightsquigarrow \quad \mathscr{Z} \not\equiv 1 \;\text{ is non-degenerate!}$$

- Explicit expression for $C(\phi)$ as a series of multiple integrals

Work in progress

- Uniqueness of subsequential limit $\mathscr{Z}$ via coarse-graining arguments

  $\rightsquigarrow$ Existence of the limit $\boldsymbol{Z}_N^\omega \xrightarrow[N\to\infty]{d} \mathscr{Z}$

- Investigate properties of the limiting random measure $\mathscr{Z}$

  (it looks not so close to Gaussian Multiplicative Chaos)

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Partition function and polynomial chaos

$$
\begin{aligned}
Z_N^{\omega} &= \mathbf{E}^{\mathrm{rw}}\Big[e^{H_N(\omega,S)}\Big] = \mathbf{E}^{\mathrm{rw}}\Big[e^{\sum_{1\le n\le N}\sum_{x\in\mathbb{Z}^2}(\beta\omega(n,x)-\lambda(\beta))\,\mathbb{1}_{\{S_n=x\}}}\Big] \\
&= \mathbf{E}^{\mathrm{rw}}\Bigg[\prod_{1\le n\le N}\prod_{x\in\mathbb{Z}^2} e^{(\beta\omega(n,x)-\lambda(\beta))\,\mathbb{1}_{\{S_n=x\}}}\Bigg] \\
&= \mathbf{E}^{\mathrm{rw}}\Bigg[\prod_{1\le n\le N}\prod_{x\in\mathbb{Z}^2}\big(1+X_{n,x}\,\mathbb{1}_{\{S_n=x\}}\big)\Bigg] \\
&= 1 + \sum_{\substack{1\le n\le N\\ x\in\mathbb{Z}^2}} \mathbf{P}^{\mathrm{rw}}(S_n=x)\,X_{n,x} \\
&\qquad + \sum_{\substack{1\le n<m\le N\\ x,y\in\mathbb{Z}^2}} \mathbf{P}^{\mathrm{rw}}(S_n=x,\,S_m=y)\,X_{n,x}\,X_{m,y} + \ldots
\end{aligned}
$$

Z_N^{ω} multi-linear polynomial of new RVs $X_{n,x} := e^{\beta\omega(n,x)-\lambda(\beta)} - 1$

Polynomial chaos

$$\mathbb{E}[X_{n,x}] = 0 \qquad \mathbb{V}\mathrm{ar}[X_{n,x}] \sim \beta^2$$

Let us pretend $X_{n,x} = \beta\, Y_{n,x}$ with $\big(Y_{n,x}\big)_{n,x}$ i.i.d. $\mathcal{N}(0,1)$

Then $Z_N^\omega \simeq 1 + \sum_{k=1}^{\infty} \beta^k Z_N^{(k)}$

$$Z_N^{(1)} := \sum_{\substack{1\le n\le N \\ x\in\mathbb{Z}^2}} \mathbf{P}^{\mathrm{rw}}(S_n = x)\, Y_{n,x}$$

$$Z_N^{(2)} := \sum_{\substack{1\le n\le m\le N \\ x,y\in\mathbb{Z}^2}} \mathbf{P}^{\mathrm{rw}}(S_n = x,\, S_m = y)\, Y_{n,x}\, Y_{m,y}$$

The choice of β

$Z_N^{(1)}$ is Gaussian with variance given by the replica overlap R_N:

$$\mathbb{V}\mathrm{ar}[Z_N^{(1)}] = \sum_{1\le n\le N} \sum_{x\in\mathbb{Z}^2} \mathbf{P}^{\mathrm{rw}}(S_n = x)^2 = \sum_{1\le n\le N} \mathbf{P}^{\mathrm{rw}}(S_n = S_n')$$

$$\sim \frac{1}{\pi} \sum_{1\le n\le N} \frac{1}{n} \sim \frac{\log N}{\pi}$$

To normalize $\beta\, Z_N^{(1)}$ we choose $\beta = \dfrac{\hat\beta}{\sqrt{\frac{\log N}{\pi}}}$

Similarly $$\mathbb{V}\mathrm{ar}[Z_N^{(2)}] \sim \frac{1}{\pi^2} \sum_{1\le n<m\le N} \frac{1}{n}\,\frac{1}{m-n} \lesssim \left(\frac{\log N}{\pi}\right)^2$$

Variance bounds for $\hat{\beta} < 1$

More generally

$$\mathbb{V}\mathrm{ar}\big[Z_N^{(k)}\big] \lesssim \left(\frac{\log N}{\pi}\right)^k \qquad (\star)$$

For $\hat{\beta} < 1$

$$\mathbb{V}\mathrm{ar}\big[\boldsymbol{Z}_N^{\omega}\big] \lesssim \sum_{k=1}^{\infty} (\beta^2)^k \, \mathbb{V}\mathrm{ar}\big[Z_N^{(k)}\big]$$

$$\lesssim \sum_{k=1}^{\infty} \left(\frac{\hat{\beta}^2}{\frac{\log N}{\pi}}\right)^k \left(\tfrac{\log N}{\pi}\right)^k \lesssim \sum_{k=1}^{\infty} \hat{\beta}^{2k} < \infty$$

To deal with $\hat{\beta} = 1$ we need to refine $(\star)$

Sharp asymptotics

Lemma

$$\mathbb{V}\mathrm{ar}\left[Z_N^{(k)}\right] \sim \sum_{0<n_1<\ldots<n_k\le N} \frac{1}{n_1}\,\frac{1}{n_2-n_1}\cdots\frac{1}{n_k-n_{k-1}}$$

$$\sim \left(\frac{\log N}{\pi}\right)^k \mathbf{P}\left(\boldsymbol{T}_{\frac{k}{\log N}} \le 1\right)$$

- $(\boldsymbol{T}_s)_{s\ge 0}$ increasing Lévy process (subordinator) with Lévy measure

$$\nu(\mathrm{d}t) = \frac{1}{t}\,\mathbb{1}_{(0,1)}(t)$$

- One can compute $\mathbf{P}(\boldsymbol{T}_s \le 1) = \dfrac{e^{-\gamma s}}{\Gamma(1+s)}$

Variance and covariances in the critical window

Variance

For $\hat{\beta} = 1$
$$\mathbb{V}\mathrm{ar}\left[Z_N^\omega\right] \sim \sum_{k=1}^{\infty} \mathbf{P}\left(T_{\frac{k}{\log N}} \le 1\right) \sim C \log N$$

where
$$C := \int_0^\infty \mathbf{P}(T_s \le 1)\, \mathrm{d}s = \int_0^\infty \frac{e^{-\gamma s}}{\Gamma(1+s)}\, \mathrm{d}s$$

Covariances

$$\mathbb{C}\mathrm{ov}\left[Z_N^\omega(x), Z_N^\omega(x')\right] \sim K(x, x')$$
$$= \int_0^1 \frac{e^{-\frac{|x'-x|^2}{2t}}}{2t} \left(\int_0^\infty \frac{e^{-\gamma s}(1-t)^s}{\Gamma(1+s)}\, \mathrm{d}s \right) \mathrm{d}t$$

Third moment in the critical window

$\langle Z_N^\omega, \phi \rangle$ is multilinear polynomial (sum of products) of i.i.d. RVs $X_{n,x}$

$$\langle Z_N^\omega, \phi \rangle = \sum_{I \subseteq \{1,\dots,N\} \times \mathbb{Z}^2} c(I) \prod_{(n,x) \in I} X_{n,x}$$

for suitable $c(I) = c(I, N, \phi)$

- Expand $\mathbb{E}[\langle Z_N^\omega, \phi \rangle^3]$ in 3 sums
- X's from different sums match in pairs or triples (by $\mathbb{E}[X_{n,x}] = 0$)
- Triple matchings give negligible contribution

Pairwise matching of the X's $\rightsquigarrow$ highly non-trivial, yet manageable combinatorial structure $\rightsquigarrow$ sharp asymptotics for $\mathbb{E}[\langle Z_N^\omega, \phi \rangle^3]$

Thanks

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Multi-scale correlations for $\hat{\beta} < 1$

Theorem

Fix $\hat{\beta} < 1$ and

$$|z - z'| \asymp N^{\zeta} \qquad \zeta \in [0, \tfrac{1}{2}]$$

Then

$$\left(Z_N^{\omega}(z) \, , \, Z_N^{\omega}(z') \right) \xrightarrow[N \to \infty]{d} \left(e^{Y - \frac{1}{2}\mathbb{V}\mathrm{ar}[Y]} \, , \, e^{Y' - \frac{1}{2}\mathbb{V}\mathrm{ar}[Y']} \right)$$

- Y, Y' jointly Normal with variance $\sigma^2 = \log \frac{1}{1 - \hat{\beta}^2}$
- $\mathbb{C}\mathrm{ov}\left[Y, Y' \right] = \log \frac{1 - (2\zeta)\,\hat{\beta}^2}{1 - \hat{\beta}^2}$

Path diffusivity for $\hat{\beta} < 1$

Diffusivity

- Central Limit Theorem

$$\mathbf{P}_N^{\omega}\left(\frac{S_N}{\sqrt{N}} \in \cdot\right) \xrightarrow[N\to\infty]{d} N(0,1) \qquad \text{in } \mathbb{P}(\mathrm{d}\omega)\text{-probability}$$

- Local Limit Theorem with random corrections

$$(\sqrt{N})^2\, \mathbf{P}_N^{\omega}\left(S_N = \lfloor x\sqrt{N}\rfloor\right) \xrightarrow[N\to\infty]{d} e^{Y_x - \frac{1}{2}\mathbb{V}\mathrm{ar}[Y_x]}\, \frac{e^{-|x|^2/2}}{2\pi}$$

Partition function fluctuations for $\hat{\beta} < 1$

For $\hat{\beta} < 1$ $\quad Z_N^\omega(\sqrt{N}x) \xrightarrow[N\to\infty]{\mathbb{P}} 1$ (as a Schwartz distribution on $\mathbb{R}^2$)

This can be viewed as a LLN. Here is the corresponding CLT.

Theorem [C., Sun, Zygouras (AAP to appear)]

$$Z_N^\omega(\sqrt{N}x) \overset{d}{\approx} 1 + \frac{1}{\sqrt{\log N}}\, G(x) \qquad \text{in } \mathcal{S}'$$

where $G(x)$ is a generalized Gaussian field on $\mathbb{R}^2$ with

$$\mathbb{C}\mathrm{ov}\left[G(x), G(x')\right] \sim C \log \frac{1}{|x - x'|}$$

More precisely

$$\left\langle \sqrt{\log N}\,\left(Z_N^\omega(\sqrt{N}\,\cdot\,) - 1\right),\, \phi \right\rangle \xrightarrow[N\to\infty]{d} \langle G, \phi \rangle \qquad \forall\, \phi \in C_0(\mathbb{R}^2)$$

Second moment in the critical window

Theorem (variance vs. covariances)

- $\mathbb{V}\mathrm{ar}[Z_N^\omega(\sqrt{N}x)] \simeq \log N \to \infty$
- $\mathbb{C}\mathrm{ov}[Z_N^\omega(\sqrt{N}x), Z_N^\omega(\sqrt{N}x')] \xrightarrow[N\to\infty]{} K(x,x') < \infty$

$$K(x,x') \sim C \log \tfrac{1}{|x-x'|} \qquad \text{as } |x-x'| \to 0$$

Corollary

$$\mathbb{V}\mathrm{ar}\big[\langle Z_N^\omega , \phi \rangle\big] \xrightarrow[N\to\infty]{} \langle \phi, K\phi \rangle < \infty$$

Explicit kernel: $K(x,x') = \int_0^1 \frac{1}{2t}\, e^{-\frac{|x'-x|^2}{2t}} \left(\int_0^\infty \frac{e^{(\pi\vartheta-\gamma)s}\,(1-t)^s}{\Gamma(1+s)}\, \mathrm{d}s \right) \mathrm{d}t$

The 2d Stochastic Heat Equation

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x) + \beta\, \dot{W}(t,x)\, u(t,x) \\ u(0,x) \equiv 1 \end{cases}$$

where $W(\mathrm{d}t, \mathrm{d}x)$ is Gaussian white noise on $[0,\infty) \times \mathbb{R}^2$

Mollified noise: $$W_\delta(\mathrm{d}t, x) := \int_{y\in\mathbb{R}^2} \tfrac{1}{\delta}\, j\big(\tfrac{x-y}{\sqrt{\delta}}\big)\, W(\mathrm{d}t, \mathrm{d}y)$$

Mollified solution $u_\delta(t,x) \overset{d}{\approx} Z^\omega_{Nt}(\sqrt{N}x)$ for $N = \dfrac{1}{\delta}$

Generalized Feynman-Kac Formula [Bertini, Cancrini '95]

$$u_\delta(t,x) \overset{d}{=} \mathbf{E}^{\mathrm{BM}}\left[\exp\left\{ \int_0^{\frac{t}{\delta}} \left(\beta\, W_1(\mathrm{d}s, B_s) \;-\; \tfrac{1}{2}\beta^2\, \mathrm{d}s\right)\right\} \,\middle|\, B_{\frac{t}{\delta}} = \frac{x}{\sqrt{\delta}} \right]$$