Polynomial Chaos and Scaling Limits of Disordered Systems

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Coworkers

Joint work with Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

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(Inspired by recent work of Alberts, Quastel and Khanin on DPRE)

Outline

General Framework

- Lattice $\Omega \subseteq \mathbb{R}^d \quad \rightsquigarrow \quad$ "spins" $\sigma = (\sigma_x)_{x \in \Omega} \quad \sigma_x = \pm 1$ or $\sigma_x \in \{0, 1\}$

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$$Z^\omega_{\Omega,\lambda,h} = \mathbf{E}_\Omega^{\mathrm{ref}}[e^{\sum_{x \in \Omega} (\lambda \omega_x + h) \sigma_x}] \qquad \text{(Partition function)}$$

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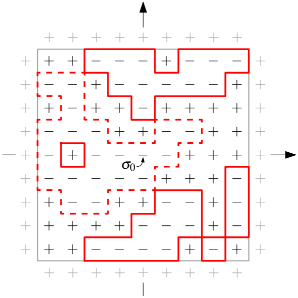

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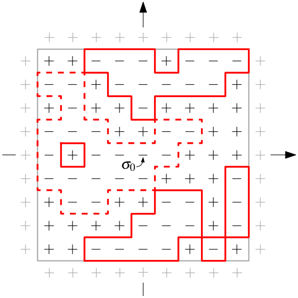

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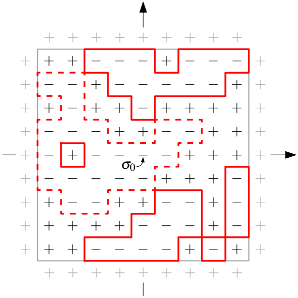

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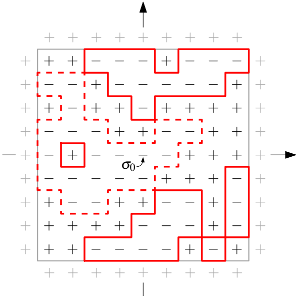

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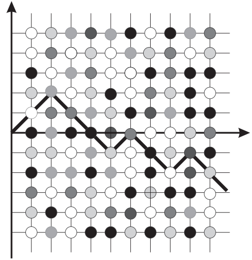

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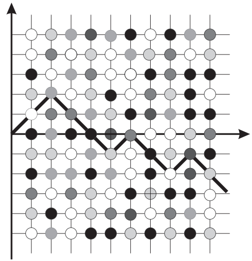

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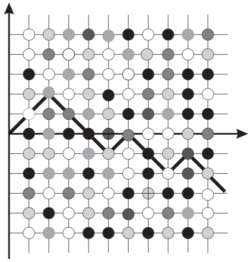

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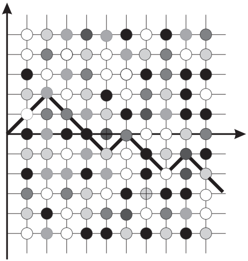

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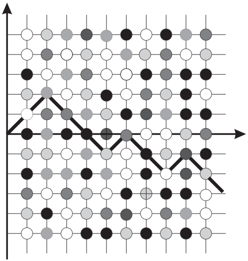

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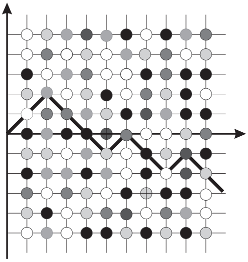

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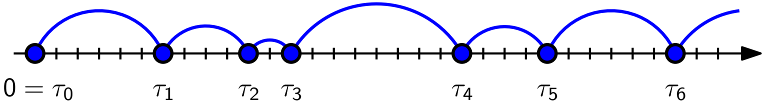

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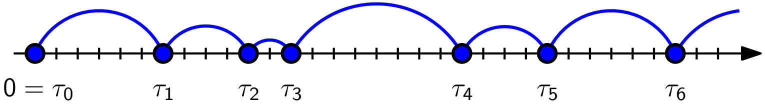

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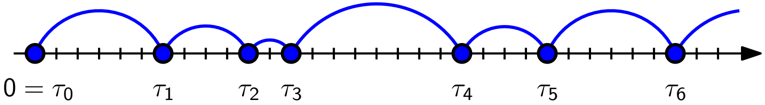

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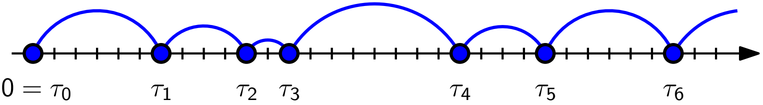

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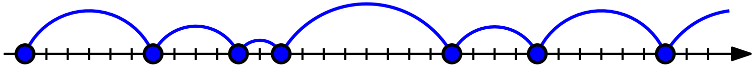

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Continuum limit?

Fix $\Omega \subset \mathbb{R}^d$ bounded open with smooth boundary, and consider the lattice

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Common feature of the examples

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Does the disordered model $\mathbf{P}^{\omega}_{\Omega_\delta,\lambda,h}$ admit a non-trivial continuum limit?

A naive approach

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$$\mathbf{P}^{\omega}_{\Omega_\delta,\lambda,h}(\mathrm{d}\sigma) \;\propto\; \exp\left(\sum_{x\in\Omega_\delta} (\lambda\omega_x + h)\sigma_x \right) \mathbf{P}^{\mathrm{ref}}_{\Omega_\delta}(\mathrm{d}\sigma)$$

- Replace discrete spins $\mathbf{P}^{\mathrm{ref}}_{\Omega_\delta}(\mathrm{d}\sigma)$ with continuum spins $\mathscr{P}^{\mathrm{ref}}_{\Omega}(\mathrm{d}\boldsymbol{\sigma})$
- Replace discrete disorder $(\omega_x)_{x\in\Omega_\delta}$ by White noise $(\mathrm{d}W_x)_{x\in\Omega}$

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Difficulty is substantial: $\mathscr{P}^{\omega}_{\Omega,\lambda,h}$ can be singular w.r.t. $\mathscr{P}^{\mathrm{ref}}_{\Omega}$!

A way out: the partition function

Forget the random probability $\mathbf{P}^{\omega}_{\Omega_\delta,\lambda,h}$ and focus on the partition function

$$Z^{\omega}_{\Omega_\delta,\lambda,h} = \mathbf{E}^{\mathrm{ref}}\left[\exp\left(\sum_{x\in\Omega_\delta}(\lambda\omega_x + h)\sigma_x\right)\right]$$

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Main question: scaling limit of $Z^{\omega}_{\Omega_\delta,\lambda,h}$

Does $Z^{\omega}_{\Omega_\delta,\lambda,h}$ have a (non-trivial) limit in distribution as $\delta\downarrow 0$, letting $\lambda, h \to 0$ at suitable rates? (Continuum and weak disorder regime)

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  The "f.d.d." of the law $\mathbf{P}^{\omega}_{\Omega_\delta,\lambda,h}$ can be reconstructed from $Z^{\omega}_{\Omega_\delta,\lambda,h}$

Outline

Assumptions on the reference law

k-point function $\mathbf{E}^{\text{ref}}_{\Omega_\delta}[\sigma_{x_1} \cdots \sigma_{x_k}]$ defined on $(\Omega_\delta)^k$ $\rightsquigarrow$ extended on Ω^k

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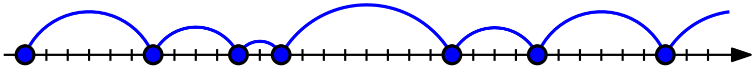
L^2 convergence requires $\boxed{\gamma < \frac{d}{2}}$

An example: Pinning



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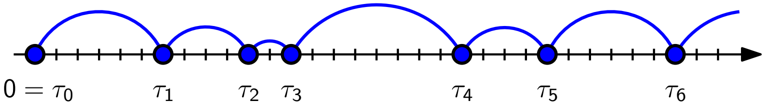

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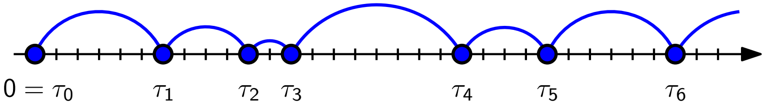

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To have L^2 conv., $\alpha \in (\frac{1}{2}, 1)$ $\rightsquigarrow$ Harris criterion for disorder relevance!

Main result (I): partition function

Theorem [C., Sun, Zygouras '13]

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Wiener chaos expansion (converging in L^{2-})

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Case $\sigma_x \in \{-1,1\}$: minor modifications

Some considerations

- Assumptions satisfied by motivating models (Ising, DPRE, Pinning) under restrictions on the parameters ($\gamma < \frac{d}{2}$) $\rightsquigarrow$ disorder relevance!

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  Not a random signed measure... but integrals are well-defined.

  (For $d = 1$ $\rightsquigarrow$ Ito integrals w.r.t. Brownian motion)

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The law of a multilinear polynomial is insensitive toward the distribution of the ϵ_x (keeping same mean and variance)

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Choosing $\lambda = \hat{\lambda} \, \delta^{\frac{d}{2} - \gamma}$ and $h' = \hat{h} \, \delta^{d - \gamma}$ the δ's disappear. □

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Rescaling $\lambda = \frac{\hat{\lambda}}{N^{\alpha-1/2}}$, $h = \frac{\hat{h}}{N^{\alpha}} - \frac{\lambda^2}{2}$, partition function $Z^{\omega}_{N,\lambda,h}$ converges

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The same happens for $(1+1)$-dim. DPRE with Cauchy tails ($\alpha = 1$) and for $(1+2)$-dim. DPRE with finite variance (e.g. SRW)

Outline

Logarithmic overlap

Recall the 1-point function $\begin{cases} \mathbf{E}^{\mathrm{ref}}[\sigma_x] = \mathbf{P}^{\mathrm{ref}}(\tau \text{ visits } n) & \text{(Pinning)} \\ \mathbf{E}^{\mathrm{ref}}[\sigma_{(n,x)}] = \mathbf{P}^{\mathrm{ref}}(X_n = x) & \text{(DPRE)} \end{cases}$

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What happens if $\hat{\lambda} = 1$? Heuristically, $u_\epsilon^W(t, x)$ should converge as $\epsilon \to 0$ to a distribution-valued random field with log-correlations.

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We can (re)prove the following result [Bertini-Cancrini '98]: defining

$$\langle u_\epsilon^W, \phi \rangle := \epsilon^2 \sum_{x \in \mathbb{Z}^2} \phi(\epsilon x)\; u_\epsilon^W(t,x)$$

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$$\lim_{\epsilon \to 0} \mathbb{C}\mathrm{ov}\big[\langle u_\epsilon^W, \phi \rangle, \langle u_\epsilon^W, \psi \rangle\big] = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \phi(x)\, \psi(y)\, f(x-y)\, \mathrm{d}x\, \mathrm{d}y \,,$$

with (explicit) f such that $f(t) \sim C \log \frac{1}{t}$ as $t \to 0$.

Sketch of the proof

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Z^{\omega}_{N,\lambda,-\frac{\lambda^2}{2}} &\simeq 1 + \lambda \sum_{n=1}^{N} \frac{\omega_n}{\sqrt{n}} + \lambda^2 \sum_{1 \le m < n \le N} \frac{\omega_m \, \omega_n}{\sqrt{m}\sqrt{n-m}} + \ldots \\
&= 1 + \quad I_1 \quad + \quad I_2 \quad + \ldots
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The term I_2 is trickier. Using integrals instead of sums, we can write

$$\int_{s \in [1,N]} \frac{W(\mathrm{d}s)}{\sqrt{s}} \int_{u \in [s+1,N]} \frac{W(\mathrm{d}u)}{\sqrt{u-s}} \simeq \int_{s \in [1,N]} \frac{W(\mathrm{d}s)}{\sqrt{s}} \int_{t \in [1,N]} \frac{W(s+\mathrm{d}t)}{\sqrt{t}}$$

Sketch of the proof

We now change variables $s = N^a$ and $t = N^b$, so that

$$I_2 \simeq \left(\frac{\hat{\lambda}}{\sqrt{\log N}} \right)^2 \int_{s \in [1,N]} \frac{W(\mathrm{d}s)}{\sqrt{s}} \int_{t \in [1,N]} \frac{W(s + \mathrm{d}t)}{\sqrt{t}}$$

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$$= \hat{\lambda}^2 \int_{a\in[0,1]} \widetilde{W}(\mathrm{d}a) \int_{b\in[0,1]} \widetilde{W}_a(\mathrm{d}b)$$

Note that $\widetilde{W}(\cdot)$ is white noise; for *fixed* a, $\widetilde{W}_a(\cdot)$ is also white noise.

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$$I_2 \simeq \left(\frac{\hat{\lambda}}{\sqrt{\log N}} \right)^2 \int_{s \in [1,N]} \frac{W(\mathrm{d}s)}{\sqrt{s}} \int_{t \in [1,N]} \frac{W(s + \mathrm{d}t)}{\sqrt{t}}$$

$$= \hat{\lambda}^2 \int_{a \in [0,1]} \frac{W(\mathrm{d}N^a)}{\sqrt{N^a \log N}} \int_{b \in [0,1]} \frac{W(N^a + \mathrm{d}N^b)}{\sqrt{N^b \log N}}$$

$$= \hat{\lambda}^2 \int_{a \in [0,1]} \widetilde{W}(\mathrm{d}a) \int_{b \in [0,1]} \widetilde{W}_a(\mathrm{d}b)$$

Note that $\widetilde{W}(\cdot)$ is white noise; for *fixed* a, $\widetilde{W}_a(\cdot)$ is also white noise.

- If $a > b$, then $N^a + \mathrm{d}N^b$ is a tiny window close to N^a, hence $\widetilde{W}_a(\cdot)$ is asymptotically independent of $\widetilde{W}(\cdot)$ and $\widetilde{W}(\mathrm{d}a)\widetilde{W}_a(\mathrm{d}b)$ becomes asymptotically a $2d$ white noise $\widetilde{W}^{(2)}(\mathrm{d}a, \mathrm{d}b)$.

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- If $a < b$, then $N^a + \mathrm{d}N^b$ is essentially $\mathrm{d}N^b$, hence $\widetilde{W}_a(\cdot)$ asymptotically coincides with $\widetilde{W}(\cdot)$.

Sketch of the proof

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$$I_2 \simeq \hat{\lambda}^2 \left(\int_{0 \le b < a \le 1} \widetilde{W}^{(2)}(\mathrm{d}a, \mathrm{d}b) + \int_{0 \le a < b \le 1} \widetilde{W}(\mathrm{d}a) \widetilde{W}(\mathrm{d}b) \right)$$

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One can perform analogous computations at every order k, splitting the domain of integration into subdomains, that give rise to independent white noise of all dimensions.

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Collecting all the terms and reorganizing the sum, we reconstruct the explicit Wiener chaos series of a log-normal random variable.

Thanks

Outline

1. Disordered Systems and their Scaling Limits

2. Partition Function

3. The marginal regime

4. Further Developments

Motivating models: Ising

Pointwise convergence of k-point function, with exponent $\gamma = \frac{1}{8}$, toward

$$\psi_{\Omega}^{(k)}(x_1, \ldots, x_k) \quad \text{conformally covariant,}$$

was proved in [Chelkak, Hongler, Izyurov '12].

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Recall that we consider random field $2d$ Ising model at the critical point, with external field $(\lambda\omega_x + h)_{x \in \Omega_\delta}$

We fix continuous functions $\hat{\lambda} : \overline{\Omega} \to (0, \infty)$ and $\hat{h} : \overline{\Omega} \to \mathbb{R}$ and set

$$\lambda = \hat{\lambda}(x)\, \delta^{7/8} \qquad h = \hat{h}(x)\, \delta^{15/8}$$

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Theorem [C., Sun, Zygouras '13]

As $\delta \downarrow 0$ one has the convergence in law

$$e^{-\frac{1}{2}\|\hat{\lambda}\|_2^2\,\delta^{-1/4}}\, Z^{\omega}_{\Omega_\delta,\lambda,h} \implies \mathbf{Z}^{W}_{\Omega;\hat{\lambda},\hat{h}}$$

where $W(\mathrm{d}x)$ is white noise on $\mathbb{R}^d$ and

$$\mathbf{Z}^{W}_{\Omega;\hat{\lambda},\hat{h}} := \sum_{k=0}^{\infty} \frac{1}{k!} \int \cdots \int_{\Omega^k} \psi_\Omega^{(k)}(x_1,\ldots,x_k) \prod_{i=1}^{k} \big(\hat{\lambda}(x_i)\, W(\mathrm{d}x_i) + \hat{h}(x_i)\,\mathrm{d}x_i\big)$$

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Conformal covariance: if $\phi : \tilde{\Omega} \to \Omega$ is a conformal map,

$$\mathbf{Z}^{W}_{\Omega;\hat{\lambda},\hat{h}} \overset{dist.}{=} \mathbf{Z}^{W}_{\tilde{\Omega};\tilde{\lambda},\tilde{h}}$$

where $\tilde{\lambda}(x) := |\phi'(x)|^{7/8}\hat{\lambda}(\phi(x))$ and $\tilde{h}(x) := |\phi'(x)|^{15/8}\hat{h}(\phi(x))$

Continuum free energy and critical exponents

Continuum partition function $\mathbf{Z}^{W}_{\Omega,\hat{\lambda},\hat{h}}$ $\rightsquigarrow$ continuum free energy

$$\mathbf{F}(\hat{\lambda}, \hat{h}) := \lim_{\Omega \uparrow \mathbb{R}^d} \frac{1}{Leb(\Omega)} \log \mathbf{Z}^{W}_{\Omega,\hat{\lambda},\hat{h}}$$

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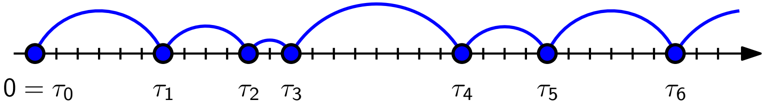
Conjecture

$$\lim_{h\downarrow 0} \frac{\langle\sigma_0\rangle_{\hat{\lambda}\,h^{\frac{7}{15}},h}}{h^{\frac{1}{15}}} = \frac{\partial \mathbf{F}}{\partial h}(\hat{\lambda},1)$$ refining [Camia, Garban, Newman '12]

Back to pinning models



$\tau = \{\tau_0 < \tau_1 < \tau_2 < \ldots\}$ random element of $E := \{$closed subsets of $\mathbb{R}\}$

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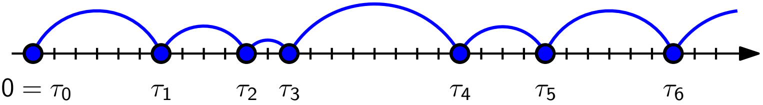

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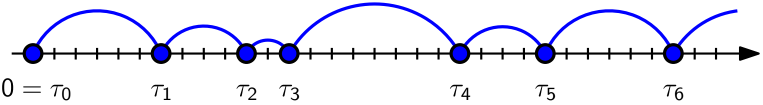

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What happens for the disordered model $\mathbf{P}^{\omega}_{\Omega_\delta,\lambda,h}$? $(\Omega = (0,1))$

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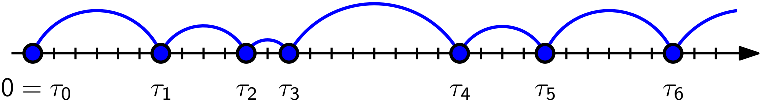

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What happens for the disordered model $\mathbf{P}^{\omega}_{\Omega_\delta,\lambda,h}$? $(\Omega = (0,1))$

Restrict $\alpha \in (\frac{1}{2}, 1)$. Fix $\hat{\lambda} > 0$, $\hat{h} \in \mathbb{R}$ and set

$$\lambda := \hat{\lambda}\, \delta^{\alpha - \frac{1}{2}} \qquad h := \hat{h}\, \delta^{\alpha} - \tfrac{1}{2}\lambda^2$$

Continuum Disordered Pinning Model [C., Sun, Zygouras '14]

$E := \{\text{closed subsets of } \mathbb{R}\}$ equipped with the Hausdorff distance

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Theorem (existence and universality of the CDPM)

As $\delta \downarrow 0$, the rescaled discrete set $(\delta\tau, \mathbf{P}^{\omega}_{\Omega_\delta,\lambda,h})$ converges in distribution on E to a universal random closed set $(\boldsymbol{\tau}, \mathscr{P}^{W}_{\Omega,\hat{\lambda},\hat{h}})$, called CDPM

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The CDPM has any a.s. property of the α-stable regenerative set $\mathscr{P}^{\mathrm{ref}}$

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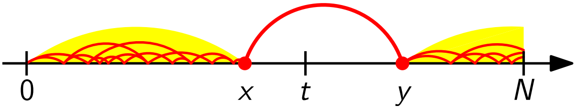
Theorem (singularity)

The CDPM $\mathscr{P}^{W}_{\Omega,\hat{\lambda},\hat{h}}$ law is singular w.r.t. $\mathscr{P}^{\mathrm{ref}}$ for $\mathbb{P}$-a.e. W

Construction strategy

Macroscopic observables (finite-dimensional distributions) expressed using partition functions with suitable boundary conditions



$$\mathbf{P}^{\omega}_{\Omega_\delta,\lambda,h}(\ldots) = \frac{Z^{\mathrm{cond}}_{0,x} \, \frac{C}{(y-x)^{1+\alpha}} \, Z_{y,N}}{Z_{0,N}}$$

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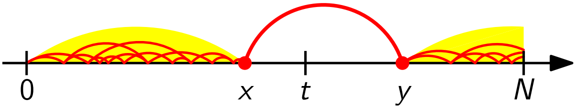

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Scaling limit (at the process level) of $(Z^{\mathrm{cond}}_{x,y}, Z_{x,y})_{0 \le x < y \le N}$ $\rightsquigarrow$ Definition of CDPM via "finite-dimensional distributions"

The same can be done for DPRE, cf. [Alberts, Khanin, Quastel '12]

Continuum random field Ising model?

Analogous procedure for Ising?

Need joint scaling limit of partition functions for “many” domains and boundary conditions

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Possible alternative approach: define continuum disordered law $\mathscr{P}^{W}_{\Omega,\hat{\lambda},\hat{h}}$ assigning its k-point function $\mathscr{E}^{W}_{\Omega,\hat{\lambda},\hat{h}}\big[\sigma_{x_1}\cdots\sigma_{x_k}\big]$?

A generalization of our theorem about the scaling limit of partition functions yields the corresponding scaling limit of correlations:

$$\mathrm{E}^{\omega}_{\Omega_\delta,\lambda,h}\big[\sigma_{x_1}\cdots\sigma_{x_k}\big] \xrightarrow[\delta\downarrow 0]{d} \mathscr{E}^{W}_{\Omega,\hat{\lambda},\hat{h}}\big[\sigma_{x_1}\cdots\sigma_{x_k}\big] := \text{Wiener chaos expansion}$$

Disorder relevance vs. irrelevance

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- The regime $\alpha < \frac{1}{2}$ is disorder-irrelevant for pinning models

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Our restriction involving L^2 convergence of k-point function ($\gamma < \frac{d}{2}$) matches with Harris criterion $\nu < \frac{2}{d}$ for disorder relevance

(ν correlation length exponent $\rightsquigarrow$ $\nu = \frac{1}{d-\gamma}$)