Scaling and Multiscaling in Financial Series: a Simple Model

Francesco Caravenna

Università degli Studi di Milano-Bicocca

Joint work with Alessandro Andreoli (Padova),
Paolo Dai Pra (Padova) and Gustavo Posta (Politecnico di Milano).

Additional results by Paolo Pigato and Mario Bonino.

Modena $\sim$ March 12, 2012

Outline

Outline

Black & Scholes model

Black & Scholes model for the price S_t of a financial asset:

$$\mathrm{d}S_t = S_t \left(r \, \mathrm{d}t + \sigma \, \mathrm{d}B_t \right)$$

- σ (the volatility) and r (the interest rate) are constant
- $(B_t)_{t \geq 0}$ is a standard Brownian motion.

Black & Scholes model

Black & Scholes model for the price S_t of a financial asset:

$$\mathrm{d}S_t = S_t\,(r\,\mathrm{d}t + \sigma\,\mathrm{d}B_t)$$

- σ (the volatility) and r (the interest rate) are constant
- $(B_t)_{t\geq 0}$ is a standard Brownian motion.

Therefore $(S_t)_{t\geq 0}$ is a geometric Brownian motion, i.e., the detrended log-price $X_t := \log S_t - r' t$ (with $r' := r - \sigma^2/2$) is BM:

$$\mathrm{d}X_t = \sigma\,\mathrm{d}B_t \qquad \Longrightarrow \qquad X_t = X_0 + \sigma B_t\,.$$

Black & Scholes model

Black & Scholes model for the price S_t of a financial asset:

$$\mathrm{d}S_t = S_t\,(r\,\mathrm{d}t + \sigma\,\mathrm{d}B_t)$$

- σ (the volatility) and r (the interest rate) are constant
- $(B_t)_{t\geq 0}$ is a standard Brownian motion.

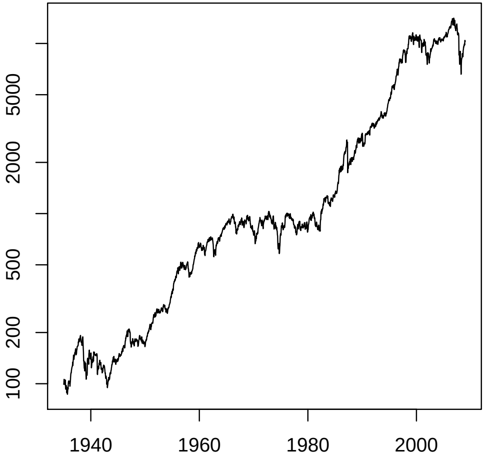
Therefore $(S_t)_{t\geq 0}$ is a geometric Brownian motion, i.e., the detrended log-price $X_t := \log S_t - r'\,t$ (with $r' := r - \sigma^2/2$) is BM:

$$\mathrm{d}X_t = \sigma\,\mathrm{d}B_t \qquad \Longrightarrow \qquad X_t = X_0 + \sigma\,B_t\,.$$

Basic example: Dow Jones Industrial Average (DJIA).

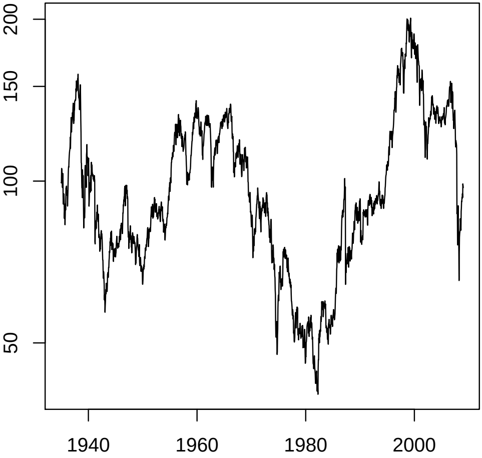
DJIA time series (1935-2009)

Exponential growth of the DJIA [log plot]:



DJIA time series (1935-2009)

DJIA after linear detrend [log plot]:



Beyond the Black & Scholes model

Despite its success, this model is not consistent with a number of stylized facts that are empirically detected in many real time series.

Beyond the Black & Scholes model

Despite its success, this model is not consistent with a number of stylized facts that are empirically detected in many real time series.

Detrended log-price $X_t := \log S_t - \overline{d}_t$ B&S: $\mathrm{d}X_t = \sigma \, \mathrm{d}B_t$

Beyond the Black & Scholes model

Despite its success, this model is not consistent with a number of stylized facts that are empirically detected in many real time series.

Detrended log-price $X_t := \log S_t - \overline{d}_t$ B&S: $\mathrm{d}X_t = \sigma \, \mathrm{d}B_t$

- The volatility σ is not constant: it may have high peaks ("shocks" in the market).

Beyond the Black & Scholes model

Despite its success, this model is not consistent with a number of stylized facts that are empirically detected in many real time series.

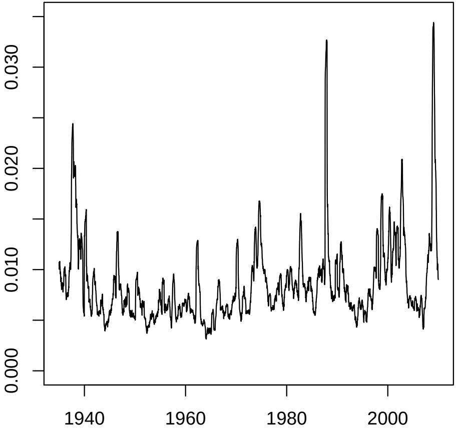
Detrended log-price $X_t := \log S_t - \overline{d}_t$ B&S: $\mathrm{d}X_t = \sigma \, \mathrm{d}B_t$

- The volatility σ is not constant: it may have high peaks ("shocks" in the market).

$$\text{Empirical volatility:} \quad \overline{\sigma}_t := \frac{1}{100} \sum_{i=t-99}^{t} (x_i - x_{i-1})^2$$

DJIA time series (1935-2009)

Empirical volatility $[\overline{\sigma}_t \text{ vs. } t]$



Local standard deviation of log-returns in a window of 100 days

Beyond the Black & Scholes model

Despite its success, this model is not consistent with a number of stylized facts that are empirically detected in many real time series.

Detrended log-price $X_t := \log S_t - \overline{d}_t$ B&S: $\mathrm{d}X_t = \sigma \, \mathrm{d}B_t$

- The volatility σ is not constant: it may have high peaks ("shocks" in the market).
- The increments $(X_{t+h} - X_t)$, called log-returns, for small h have a distribution with heavy (power-law) tails.

Beyond the Black & Scholes model

Despite its success, this model is not consistent with a number of stylized facts that are empirically detected in many real time series.

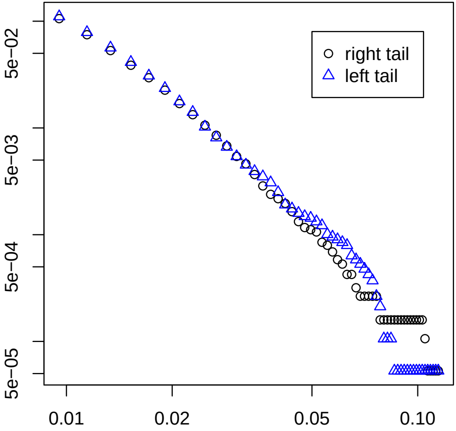
Detrended log-price $X_t := \log S_t - \overline{d}_t$ B&S: $\mathrm{d}X_t = \sigma \, \mathrm{d}B_t$

- The volatility σ is not constant: it may have high peaks ("shocks" in the market).
- The increments $(X_{t+h} - X_t)$, called log-returns, for small h have a distribution with heavy (power-law) tails.

Empirical daily $(h = 1)$ tail: $\overline{q}(y) := \dfrac{1}{T_1 - T_0} \displaystyle\sum_{i=T_0+1}^{T_1} \mathbf{1}_{\{x_i - x_{i-1} > y\}}$

DJIA time series (1935-2009)

Daily log-return tail [$\log \overline{q}(y)$ vs. $\log y$]



Daily log-return standard deviation ≈ 0.01 $\longrightarrow$ Range: 1 to 12 st. dev.

Beyond the Black & Scholes model

Despite its success, this model is not consistent with a number of stylized facts that are empirically detected in many real time series.

Detrended log-price $X_t := \log S_t - \overline{d}_t$ B&S: $\mathrm{d}X_t = \sigma \, \mathrm{d}B_t$

- The volatility σ is not constant: it may have high peaks ("shocks" in the market).
- The increments $(X_{t+h} - X_t)$, called log-returns, for small h have a distribution with heavy (power-law) tails.
- Log-returns corresponding to disjoint time intervals are uncorrelated...

Beyond the Black & Scholes model

Despite its success, this model is not consistent with a number of stylized facts that are empirically detected in many real time series.

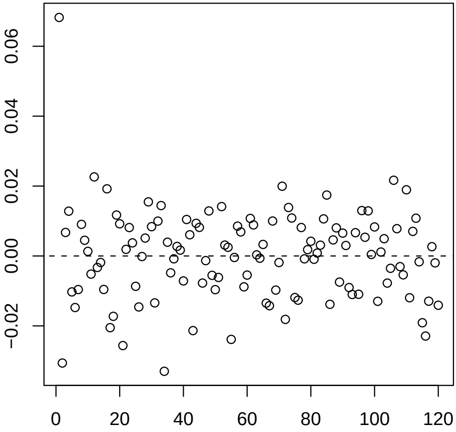
Detrended log-price $X_t := \log S_t - \overline{d}_t$ B&S: $\mathrm{d}X_t = \sigma \, \mathrm{d}B_t$

- The volatility σ is not constant: it may have high peaks ("shocks" in the market).
- The increments $(X_{t+h} - X_t)$, called log-returns, for small h have a distribution with heavy (power-law) tails.
- Log-returns corresponding to disjoint time intervals are uncorrelated...

$$\overline{\rho}(t) := \frac{1}{T_1 - T_0 - t} \sum_{i=T_0+1}^{T_1-t} \frac{(x_i - x_{i-1})(x_{i+t} - x_{i+t-1})}{\overline{s}_x^2} .$$

DJIA time series (1935-2009)

Decorrelation of daily log-returns [$\overline{\rho}(t)$ vs. t]



Beyond the Black & Scholes model

Despite its success, this model is not consistent with a number of stylized facts that are empirically detected in many real time series.

Detrended log-price $X_t := \log S_t - \overline{d}_t$ B&S: $\mathrm{d}X_t = \sigma \, \mathrm{d}B_t$

- The volatility σ is not constant: it may have high peaks ("shocks" in the market).
- The increments $(X_{t+h} - X_t)$, called log-returns, for small h have a distribution with heavy (power-law) tails.
- Log-returns corresponding to disjoint time intervals are uncorrelated... but not independent!

Beyond the Black & Scholes model

Despite its success, this model is not consistent with a number of stylized facts that are empirically detected in many real time series.

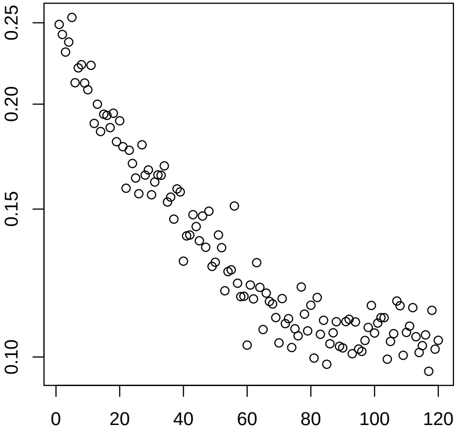
Detrended log-price $X_t := \log S_t - \overline{d}_t$ B&S: $\mathrm{d}X_t = \sigma\,\mathrm{d}B_t$

- The volatility σ is not constant: it may have high peaks ("shocks" in the market).
- The increments $(X_{t+h} - X_t)$, called log-returns, for small h have a distribution with heavy (power-law) tails.
- Log-returns corresponding to disjoint time intervals are uncorrelated... but not independent!

  The correlation between $|X_{t+h} - X_t|$ and $|X_{s+h} - X_s|$, called volatility autocorrelation, has a slow decay in $|t - s|$, up to moderate values of $|t - s|$ (clustering of volatility).

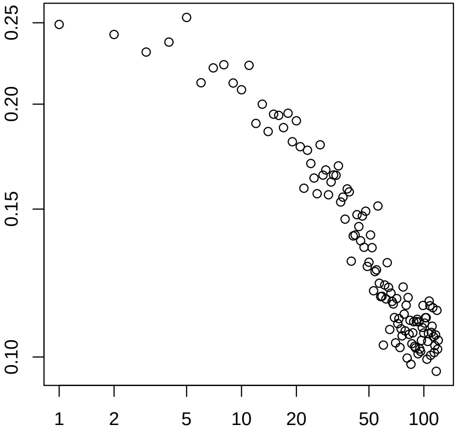
DJIA time series (1935-2009)

Volatility autocorrelation over 1–120 days [log plot]



DJIA time series (1935-2009)

Volatility autocorrelation over 1–120 days [log-log plot]



Further properties: diffusive scaling

Let us look more closely at the empirical log-return distribution over h days, for an observed time series $(x_t)_{1 \le t \le T}$:

$$\hat{p}_h(\cdot) := \frac{1}{T-h} \sum_{t=1}^{T-h} \delta_{x_{t+h}-x_t}(\cdot) \,,$$

where $\delta_x(\cdot)$ denotes the Dirac measure at $x \in \mathbb{R}$

Further properties: diffusive scaling

Let us look more closely at the empirical log-return distribution over h days, for an observed time series $(x_t)_{1 \le t \le T}$:

$$\hat{p}_h(\cdot) := \frac{1}{T-h} \sum_{t=1}^{T-h} \delta_{x_{t+h}-x_t}(\cdot)\,,$$

where $\delta_x(\cdot)$ denotes the Dirac measure at $x \in \mathbb{R}$

For h small (up to a few days) $\hat{p}_h(\cdot)$ has power-law tails.

Further properties: diffusive scaling

Let us look more closely at the empirical log-return distribution over h days, for an observed time series $(x_t)_{1 \le t \le T}$:

$$\hat{p}_h(\cdot) := \frac{1}{T-h} \sum_{t=1}^{T-h} \delta_{x_{t+h}-x_t}(\cdot) \,,$$

where $\delta_x(\cdot)$ denotes the Dirac measure at $x \in \mathbb{R}$

For h small (up to a few days) $\hat{p}_h(\cdot)$ has power-law tails.

It turns out that $\hat{p}_h$ obeys approximately a diffusive scaling relation:

$$X_{t+h} - X_t \overset{d}{\approx} \sqrt{h}\,(X_{t+1} - X_t)$$

Further properties: diffusive scaling

Let us look more closely at the empirical log-return distribution over h days, for an observed time series $(x_t)_{1 \le t \le T}$:

$$\hat{p}_h(\cdot) := \frac{1}{T-h} \sum_{t=1}^{T-h} \delta_{x_{t+h}-x_t}(\cdot) \,,$$

where $\delta_x(\cdot)$ denotes the Dirac measure at $x \in \mathbb{R}$

For h small (up to a few days) $\hat{p}_h(\cdot)$ has power-law tails.

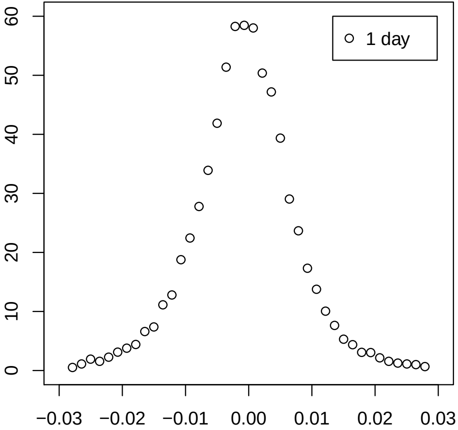
It turns out that $\hat{p}_h$ obeys approximately a diffusive scaling relation:

$$X_{t+h} - X_t \overset{d}{\approx} \sqrt{h}\,(X_{t+1} - X_t) \quad \to \quad \hat{p}_h(\mathrm{d}r) \simeq \frac{1}{\sqrt{h}}\, g\left(\frac{r}{\sqrt{h}}\right) \mathrm{d}r$$

where g is a non-Gaussian density.

DJIA time series (1935-2009)

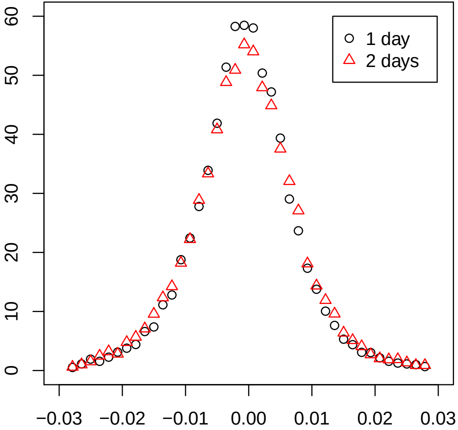
Rescaled empirical density of log-returns (1 day)



Daily log-return standard deviation ≈ 0.01 $\longrightarrow$ Range: -3 to +3 st. dev.

DJIA time series (1935-2009)

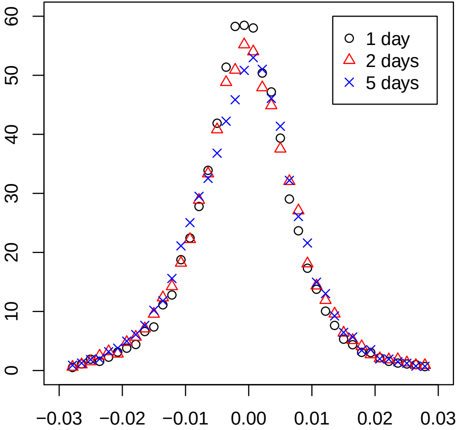
Rescaled empirical density of log-returns (1-2 days)



Daily log-return standard deviation ≈ 0.01 $\longrightarrow$ Range: -3 to +3 st. dev.

DJIA time series (1935-2009)

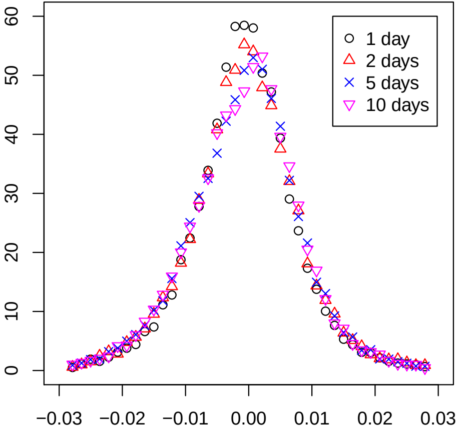
Rescaled empirical density of log-returns (1-2-5 days)



Daily log-return standard deviation ≈ 0.01 $\longrightarrow$ Range: -3 to +3 st. dev.

DJIA time series (1935-2009)

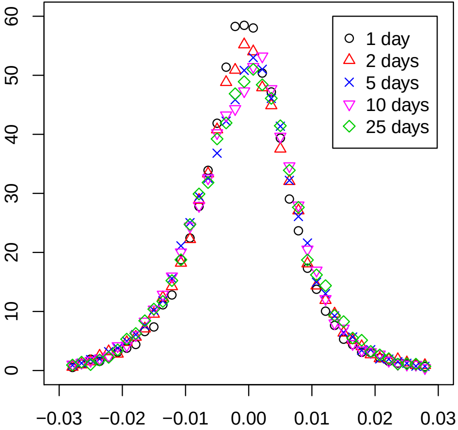
Rescaled empirical density of log-returns (1-2-5-10 days)



Daily log-return standard deviation ≈ 0.01 $\longrightarrow$ Range: -3 to +3 st. dev.

DJIA time series (1935-2009)

Rescaled empirical density of log-returns (1-2-5-10-25 days)



Daily log-return standard deviation ≈ 0.01 $\longrightarrow$ Range: -3 to +3 st. dev.

Further properties: multiscaling of moments

Consider the empirical *q-th moment* of the log-return over h days:

$$\hat{m}_q(h) := \frac{1}{T-h} \sum_{i=1}^{T-h} |x_{i+h} - x_i|^q = \int |r|^q \, \hat{p}_h(\mathrm{d}r)$$

Further properties: multiscaling of moments

Consider the empirical q-th moment of the log-return over h days:

$$\hat{m}_q(h) := \frac{1}{T-h} \sum_{i=1}^{T-h} |x_{i+h} - x_i|^q = \int |r|^q \, \hat{p}_h(\mathrm{d}r)$$

From the diffusive scaling $X_{t+h} - X_t \overset{d}{\approx} \sqrt{h}\,(X_{t+1} - X_t)$ it is natural to guess

$$\hat{m}_q(h) \approx h^{q/2} \qquad \text{for } h \text{ small}\,.$$

Further properties: multiscaling of moments

Consider the empirical q-th moment of the log-return over h days:

$$\hat{m}_q(h) := \frac{1}{T-h} \sum_{i=1}^{T-h} |x_{i+h} - x_i|^q = \int |r|^q \, \hat{p}_h(\mathrm{d}r)$$

From the diffusive scaling $X_{t+h} - X_t \overset{d}{\approx} \sqrt{h}\,(X_{t+1} - X_t)$ it is natural to guess

$$\hat{m}_q(h) \approx h^{q/2} \qquad \text{for } h \text{ small}\,.$$

This is true only if $q \le q^*$ (with $q^* \simeq 3$ for the DJIA).

Further properties: multiscaling of moments

Consider the empirical q-th moment of the log-return over h days:

$$\hat{m}_q(h) := \frac{1}{T-h} \sum_{i=1}^{T-h} |x_{i+h} - x_i|^q = \int |r|^q \, \hat{p}_h(\mathrm{d}r)$$

From the diffusive scaling $X_{t+h} - X_t \overset{d}{\approx} \sqrt{h}\,(X_{t+1} - X_t)$ it is natural to guess

$$\hat{m}_q(h) \approx h^{q/2} \qquad \text{for } h \text{ small}\,.$$

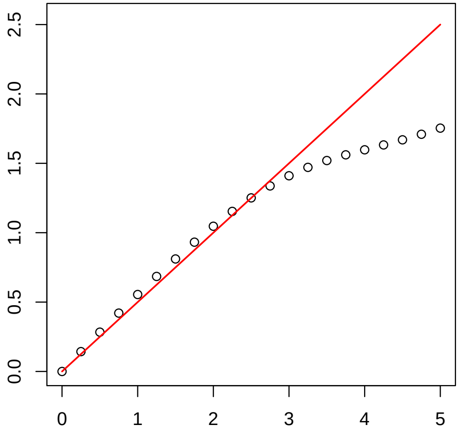
This is true only if $q \le q^*$ (with $q^* \simeq 3$ for the DJIA).

If $q > q^*$ we have the anomalous scaling (or multiscaling)

$$\hat{m}_q(h) \approx h^{A(q)} \qquad \text{with} \quad A(q) < \frac{q}{2}\,.$$

DJIA time series (1935-2009)

Scaling exponent $A(q)$ (linear regression of $\log \hat{m}_q(h)$ vs. $\log h$)



Some comments

- The mentioned stylized facts are common to the main financial indexes (DJIA, S&P 500, FTSE 100, NIKKEI 225).

Some comments

- The mentioned stylized facts are common to the main financial indexes (DJIA, S&P 500, FTSE 100, NIKKEI 225).
- Scaling features (esp. multiscaling of moments) have been strongly stressed in the econophysics literature.

In particular, the interactions we had with F. Baldovin and A. Stella motivated our original interest.

Some comments

- The mentioned stylized facts are common to the main financial indexes (DJIA, S&P 500, FTSE 100, NIKKEI 225).
- Scaling features (esp. multiscaling of moments) have been strongly stressed in the econophysics literature.

In particular, the interactions we had with F. Baldovin and A. Stella motivated our original interest.

Goal: identify a model (as simple as possible) that fits well all mentioned stylized facts.

Some comments

- The mentioned stylized facts are common to the main financial indexes (DJIA, S&P 500, FTSE 100, NIKKEI 225).
- Scaling features (esp. multiscaling of moments) have been strongly stressed in the econophysics literature.

In particular, the interactions we had with F. Baldovin and A. Stella motivated our original interest.

Goal: identify a model (as simple as possible) that fits well all mentioned stylized facts.

Baldovin & Stella's standpoint: the scaling properties should primarily guide the construction of the model.

Alternative models: stochastic volatility

Stochastic volatility processes: the constant σ is replaced by a stochastic process $(\sigma_t)_{t \geq 0}$, usually independent of the BM B:

$$\mathrm{d}X_t = \sigma_t \, \mathrm{d}B_t$$

This defines a wide class of models (including ours!).

Alternative models: stochastic volatility

Stochastic volatility processes: the constant σ is replaced by a stochastic process $(\sigma_t)_{t \geq 0}$, usually independent of the BM B:

$$\mathrm{d}X_t = \sigma_t \, \mathrm{d}B_t$$

This defines a wide class of models (including ours!).

Much studied is the case of a generalized Ornstein-Uhlenbeck (O-U) processes (Barndorff-Nielsen & Shephard)

$$\mathrm{d}\sigma_t^2 = -\alpha \, \sigma_t^2 \, \mathrm{d}t + \mathrm{d}L_t,$$

where L_t is a subordinator (increasing Lévy process).

Alternative models: stochastic volatility

Stochastic volatility processes: the constant σ is replaced by a stochastic process $(\sigma_t)_{t\geq 0}$, usually independent of the BM B:

$$\mathrm{d}X_t = \sigma_t \, \mathrm{d}B_t$$

This defines a wide class of models (including ours!).

Much studied is the case of a generalized Ornstein-Uhlenbeck (O-U) processes (Barndorff-Nielsen & Shephard)

$$\mathrm{d}\sigma_t^2 = -\alpha \, \sigma_t^2 \, \mathrm{d}t + \mathrm{d}L_t,$$

where L_t is a subordinator (increasing Lévy process).

Multiscaling of moments?

Alternative models: multifractal models

Multiscaling of moments (as well as many other features) is properly reproduced by the so-called multifractal models [Mandelbrot, Calvet, Fisher].

Alternative models: multifractal models

Multiscaling of moments (as well as many other features) is properly reproduced by the so-called multifractal models [Mandelbrot, Calvet, Fisher].

These are suitable (independent) random time-changes of BM:

$$X_t := W_{I_t}$$

- $W = (W_s)_{s \geq 0}$ is a Brownian motion
- $I = (I_s)_{s \geq 0}$ is a continuous, increasing process (usually independent of W) displaying multifractal features.

Alternative models: multifractal models

Multiscaling of moments (as well as many other features) is properly reproduced by the so-called multifractal models [Mandelbrot, Calvet, Fisher].

These are suitable (independent) random time-changes of BM:

$$X_t := W_{I_t}$$

- $W = (W_s)_{s \geq 0}$ is a Brownian motion
- $I = (I_s)_{s \geq 0}$ is a continuous, increasing process (usually independent of W) displaying multifractal features.

The paths of $I = (I_s)_{s \geq 0}$ are a.s. non absolutely continuous.

Stochastic volatility and random time-change

Fact: every stochastic volatility process is an independent random time change of a (different) Brownian motion:

$$\mathrm{d}X_t = \sigma_t \, \mathrm{d}B_t \qquad \Longrightarrow \qquad X_t = W_{I_t} \, ,$$

Stochastic volatility and random time-change

Fact: every stochastic volatility process is an independent random time change of a (different) Brownian motion:

$$\mathrm{d}X_t = \sigma_t \, \mathrm{d}B_t \qquad \Longrightarrow \qquad X_t = W_{I_t} \, ,$$

where $$I_t := \langle X \rangle_t = \int_0^t \sigma_s^2 \, \mathrm{d}s \qquad (W_t := X_{I^{-1}(t)}) \, .$$

[Every continuous martingale is a time-changed BM (Dambis/Dubins-Schwarz)]

Stochastic volatility and random time-change

Fact: every stochastic volatility process is an independent random time change of a (different) Brownian motion:

$$\mathrm{d}X_t = \sigma_t \, \mathrm{d}B_t \qquad \Longrightarrow \qquad X_t = W_{I_t} \, ,$$

where $$I_t := \langle X \rangle_t = \int_0^t \sigma_s^2 \, \mathrm{d}s \qquad (W_t := X_{I^{-1}(t)}) \, .$$

[Every continuous martingale is a time-changed BM (Dambis/Dubins-Schwarz)]

Viceversa, every independent random time change of BM $X_t = W_{I_t}$ is a stochastic volatility process $\mathrm{d}X_t = \sigma_t \, \mathrm{d}B_t$ if (and only if) the time-change process I_t has absolutely continuous paths.

Stochastic volatility and random time-change

Fact: every stochastic volatility process is an independent random time change of a (different) Brownian motion:

$$\mathrm{d}X_t = \sigma_t \, \mathrm{d}B_t \qquad \Longrightarrow \qquad X_t = W_{I_t} \,,$$

where $$I_t := \langle X \rangle_t = \int_0^t \sigma_s^2 \, \mathrm{d}s \qquad (W_t := X_{I^{-1}(t)}) \,.$$

[Every continuous martingale is a time-changed BM (Dambis/Dubins-Schwarz)]

Viceversa, every independent random time change of BM $X_t = W_{I_t}$ is a stochastic volatility process $\mathrm{d}X_t = \sigma_t \, \mathrm{d}B_t$ if (and only if) the time-change process I_t has absolutely continuous paths.

Our goal: define a simple stochastic volatility process that fits all the above-mentioned stylized facts.

Outline

Our model

Our parameters are $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in (0, \infty)$.

More generally, σ can be taken as a probability on $(0, \infty)$

Our model

Our parameters are $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in (0, \infty)$.

More generally, σ can be taken as a probability on $(0, \infty)$

We need three independent sources $(B, \mathcal{T}, \Sigma)$ of randomness:

- a standard Brownian motion $B = (B_t)_{t \geq 0}$;

Our model

Our parameters are $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in (0, \infty)$.

More generally, σ can be taken as a probability on $(0, \infty)$

We need three independent sources $(B, \mathcal{T}, \Sigma)$ of randomness:

- a standard Brownian motion $B = (B_t)_{t \geq 0}$;
- a Poisson point process $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ on $\mathbb{R}$ with intensity λ;

Our model

Our parameters are $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in (0, \infty)$.

More generally, σ can be taken as a probability on $(0, \infty)$

We need three independent sources $(B, \mathcal{T}, \Sigma)$ of randomness:

- a standard Brownian motion $B = (B_t)_{t \geq 0}$;
- a Poisson point process $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ on $\mathbb{R}$ with intensity λ;
- an i.i.d. sequence of r.v.s $\Sigma = (\sigma_n)_{n \geq 0}$ with marginal law σ.

Our model

Our parameters are $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in (0, \infty)$.

More generally, σ can be taken as a probability on $(0, \infty)$

We need three independent sources $(B, \mathcal{T}, \Sigma)$ of randomness:

- a standard Brownian motion $B = (B_t)_{t \geq 0}$;
- a Poisson point process $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ on $\mathbb{R}$ with intensity λ;
- an i.i.d. sequence of r.v.s $\Sigma = (\sigma_n)_{n \geq 0}$ with marginal law σ.

(The parameter D enters later.)

Our model

Our parameters are $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in (0, \infty)$.

More generally, σ can be taken as a probability on $(0, \infty)$

We need three independent sources $(B, \mathcal{T}, \Sigma)$ of randomness:

- a standard Brownian motion $B = (B_t)_{t \geq 0}$;
- a Poisson point process $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ on $\mathbb{R}$ with intensity λ;
- an i.i.d. sequence of r.v.s $\Sigma = (\sigma_n)_{n \geq 0}$ with marginal law σ.

(The parameter D enters later.)

Our model $X = (X_t)_{t \geq 0}$ for the log-price of an index is

$$\mathrm{d}X_t = v_t \, \mathrm{d}B_t$$

where $\{v_t = v_t(\mathcal{T}, \Sigma)\}_{t \geq 0}$ is defined in a moment (and is independent of B).

Our model

We label $\tau_0 < 0 < \tau_1 < \ldots$ and for $t \geq 0$ we set

$$i(t) := \sup\{n \geq 0 : \tau_n \leq t\} = \#(\mathcal{T} \cap [0,t]) \qquad (\sim \mathrm{Po}(\lambda t)),$$

so that $\tau_{i(t)}$ is the last point in $\mathcal{T}$ before t.

Our model

We label $\tau_0 < 0 < \tau_1 < \ldots$ and for $t \geq 0$ we set

$$i(t) := \sup\{n \geq 0 : \tau_n \leq t\} = \#(\mathcal{T} \cap [0,t]) \qquad (\sim \mathrm{Po}(\lambda t)),$$

so that $\tau_{i(t)}$ is the last point in $\mathcal{T}$ before t.

The starting point is the generalized O-U equation driven by $i(t)$:

$$\mathrm{d}v_t^2 = -\alpha\, v_t^2\, \mathrm{d}t + \beta\, \mathrm{d}i(t), \qquad \alpha, \beta > 0.$$

Random jumps of size β are followed by exponential damping.

Our model

We label $\tau_0 < 0 < \tau_1 < \ldots$ and for $t \geq 0$ we set

$$i(t) := \sup\{n \geq 0 : \tau_n \leq t\} = \#(\mathcal{T} \cap [0,t]) \qquad (\sim \mathrm{Po}(\lambda t)),$$

so that $\tau_{i(t)}$ is the last point in $\mathcal{T}$ before t.

The starting point is the generalized O-U equation driven by $i(t)$:

$$\mathrm{d}v_t^2 = -\alpha\, v_t^2\, \mathrm{d}t + \beta\, \mathrm{d}i(t), \qquad \alpha, \beta > 0.$$

Random jumps of size β are followed by exponential damping.

We want to let $\beta \to \infty$ (very high volatility peaks). How to get a non-degenerate limiting equation?

Our model

We label $\tau_0 < 0 < \tau_1 < \dots$ and for $t \geq 0$ we set

$$i(t) := \sup\{n \geq 0 : \tau_n \leq t\} = \#(\mathcal{T} \cap [0,t]) \qquad (\sim \mathrm{Po}(\lambda t)),$$

so that $\tau_{i(t)}$ is the last point in $\mathcal{T}$ before t.

The starting point is the generalized O-U equation driven by $i(t)$:

$$\mathrm{d}v_t^2 = -\alpha\, v_t^2\, \mathrm{d}t + \beta\, \mathrm{d}i(t), \qquad \alpha, \beta > 0.$$

Random jumps of size β are followed by exponential damping.

We want to let $\beta \to \infty$ (very high volatility peaks). How to get a non-degenerate limiting equation? $\alpha \to \infty$ does not work.

Our model

A natural solution is to take a superlinear drift term, for fixed α:

$$\mathrm{d}v_t^2 \;=\; -\alpha\,(v_t^2)^{\gamma}\,\mathrm{d}t \,+\, \infty\,\mathrm{d}i(t)\,, \qquad \alpha > 0,\ \gamma > 1\,.$$

Our model

A natural solution is to take a superlinear drift term, for fixed α:

$$\mathrm{d}v_t^2 \;=\; -\alpha\,(v_t^2)^{\gamma}\,\mathrm{d}t \,+\, \infty\,\mathrm{d}i(t)\,, \qquad \alpha > 0,\; \gamma > 1\,.$$

The pathwise solution is well-defined: for $t \in (\tau_n, \tau_{n+1})$

$$v_t^2 \;=\; const.(\alpha,\gamma)\,\frac{1}{(t-\tau_n)^{1/(\gamma-1)}}\,.$$

Our model

A natural solution is to take a superlinear drift term, for fixed α:

$$\mathrm{d}v_t^2 \;=\; -\alpha\,(v_t^2)^{\gamma}\,\mathrm{d}t \,+\, \infty\,\mathrm{d}i(t)\,, \qquad \alpha > 0,\; \gamma > 1\,.$$

The pathwise solution is well-defined: for $t \in (\tau_n, \tau_{n+1})$

$$v_t^2 \;=\; const.(\alpha,\gamma)\,\frac{1}{(t-\tau_n)^{1/(\gamma-1)}}\,.$$

[In order for the SDE $\mathrm{d}X_t = v_t\,\mathrm{d}B_t$ to make sense, the trajectories $t \mapsto v_t^2$ must be locally integrable $\longrightarrow$ we must impose $\gamma > 2$.]

Our model

A natural solution is to take a superlinear drift term, for fixed α:

$$\mathrm{d}v_t^2 \;=\; -\alpha\,(v_t^2)^{\gamma}\,\mathrm{d}t \,+\, \infty\,\mathrm{d}i(t)\,, \qquad \alpha > 0,\ \gamma > 1\,.$$

The pathwise solution is well-defined: for $t \in (\tau_n, \tau_{n+1})$

$$v_t^2 \;=\; const.(\alpha,\gamma)\,\frac{1}{(t-\tau_n)^{1/(\gamma-1)}}\,.$$

[In order for the SDE $\mathrm{d}X_t = v_t\,\mathrm{d}B_t$ to make sense, the trajectories $t \mapsto v_t^2$ must be locally integrable $\longrightarrow$ we must impose $\gamma > 2$.]

We can now complete the definition of our process, expressing α and γ in terms of our parameters $D \in (0, \frac{1}{2})$ and $\sigma \in (0,\infty)$.

Definition of our model

We define $\gamma = \gamma(D) \in (2, \infty)$ and $\alpha = \alpha(\sigma, D) \in (0, \infty)$ by

$$\gamma = 2 + \frac{2D}{1-2D}, \qquad \alpha = \frac{1-2D}{(2D)^{1/(1-2D)}} \frac{1}{\sigma^{1/(1-2D)}}.$$

Definition of our model

We define $\gamma = \gamma(D) \in (2, \infty)$ and $\alpha = \alpha(\sigma, D) \in (0, \infty)$ by

$$\gamma = 2 + \frac{2D}{1-2D}, \qquad \alpha = \frac{1-2D}{(2D)^{1/(1-2D)}} \frac{1}{\sigma^{1/(1-2D)}}.$$

Definition

Our process $X = (X_t)_{t \geq 0}$ is the solution to the (Wiener) SDE

$$\mathrm{d}X_t = v_t \, \mathrm{d}B_t, \qquad X_0 := 0 \text{ (say)}.$$

Definition of our model

We define $\gamma = \gamma(D) \in (2, \infty)$ and $\alpha = \alpha(\sigma, D) \in (0, \infty)$ by

$$\gamma = 2 + \frac{2D}{1-2D}, \qquad \alpha = \frac{1-2D}{(2D)^{1/(1-2D)}} \frac{1}{\sigma^{1/(1-2D)}}.$$

Definition

Our process $X = (X_t)_{t \geq 0}$ is the solution to the (Wiener) SDE

$$\mathrm{d}X_t = v_t \, \mathrm{d}B_t, \qquad X_0 := 0 \text{ (say)}.$$

The volatility process $\{v_t\}_{t \geq 0}$ is the solution to the (S)DE

$$\mathrm{d}v_t^2 = -\alpha \, (v_t^2)^{\gamma} \, \mathrm{d}t + \infty \, \mathrm{d}i(t).$$

Definition of our model

We define $\gamma = \gamma(D) \in (2, \infty)$ and $\alpha = \alpha(\sigma, D) \in (0, \infty)$ by

$$\gamma = 2 + \frac{2D}{1-2D}, \qquad \alpha = \frac{1-2D}{(2D)^{1/(1-2D)}} \frac{1}{\sigma^{1/(1-2D)}}.$$

Definition

Our process $X = (X_t)_{t \geq 0}$ is the solution to the (Wiener) SDE

$$\mathrm{d}X_t = v_t \, \mathrm{d}B_t, \qquad X_0 := 0 \text{ (say)}.$$

The volatility process $\{v_t\}_{t \geq 0}$ is the solution to the (S)DE

$$\mathrm{d}v_t^2 = -\alpha \, (v_t^2)^\gamma \, \mathrm{d}t + \infty \, \mathrm{d}i(t).$$

More generally:

$$\mathrm{d}v_t^2 = -\alpha(\sigma_{i(t)}) \, (v_t^2)^\gamma \, \mathrm{d}t + \infty \, \mathrm{d}i(t).$$

The value of the constant α is renewed at each jump of $i(t)$.

An alternative description

Recall: every stochastic volatility process is an independent random time change of a (different) Brownian motion $W = (W_t)_{t \geq 0}$.

$$\mathrm{d}X_t = v_t \, \mathrm{d}B_t \qquad \Longrightarrow \qquad X_t = W_{I_t} \,,$$

An alternative description

Recall: every stochastic volatility process is an independent random time change of a (different) Brownian motion $W = (W_t)_{t \geq 0}$.

$$\mathrm{d}X_t = v_t \, \mathrm{d}B_t \qquad \Longrightarrow \qquad X_t = W_{I_t} \,,$$

where $$I_t := \langle X \rangle_t = \int_0^t v_s^2 \, \mathrm{d}s \qquad (\text{and } \; W_t := X_{I^{-1}(t)}) \,.$$

An alternative description

Recall: every stochastic volatility process is an independent random time change of a (different) Brownian motion $W = (W_t)_{t \geq 0}$.

$$\mathrm{d}X_t = v_t \, \mathrm{d}B_t \qquad \Longrightarrow \qquad X_t = W_{I_t} \,,$$

where $\qquad I_t := \langle X \rangle_t = \int_0^t v_s^2 \, \mathrm{d}s \qquad (\text{and } \; W_t := X_{I^{-1}(t)}) \; .$

- $I = (I_t)_{t \geq 0}$ increasing process with absol. continuous paths;
- $W = (W_t)_{t \geq 0}$ standard Brownian motion;
- $I = (I_t)_{t \geq 0}$ and $W = (W_t)_{t \geq 0}$ are independent.

An alternative description

Recall: every stochastic volatility process is an independent random time change of a (different) Brownian motion $W = (W_t)_{t \ge 0}$.

$$\mathrm{d}X_t = v_t \, \mathrm{d}B_t \qquad \Longrightarrow \qquad X_t = W_{I_t} ,$$

where $\quad I_t := \langle X \rangle_t = \int_0^t v_s^2 \, \mathrm{d}s \qquad (\text{and } \; W_t := X_{I^{-1}(t)}) \; .$

- $I = (I_t)_{t \ge 0}$ increasing process with absol. continuous paths;
- $W = (W_t)_{t \ge 0}$ standard Brownian motion;
- $I = (I_t)_{t \ge 0}$ and $W = (W_t)_{t \ge 0}$ are independent.

Henceforth we work with $(W, \mathcal{T}, \Sigma)$ instead of $(B, \mathcal{T}, \Sigma)$.

An alternative description

Recall: every stochastic volatility process is an independent random time change of a (different) Brownian motion $W = (W_t)_{t \geq 0}$.

$$\mathrm{d}X_t = v_t \, \mathrm{d}B_t \qquad \Longrightarrow \qquad X_t = W_{I_t} \, ,$$

where $\qquad I_t := \langle X \rangle_t = \int_0^t v_s^2 \, \mathrm{d}s \qquad$ (and $W_t := X_{I^{-1}(t)}$) .

- $I = (I_t)_{t \geq 0}$ increasing process with absol. continuous paths;
- $W = (W_t)_{t \geq 0}$ standard Brownian motion;
- $I = (I_t)_{t \geq 0}$ and $W = (W_t)_{t \geq 0}$ are independent.

Henceforth we work with $(W, \mathcal{T}, \Sigma)$ instead of $(B, \mathcal{T}, \Sigma)$.

Remark: explicit formula for $v_t^2 \implies$ explicit formula for I_t

Alternative definition of our model

Recall $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in \mathcal{M}_1((0, \infty))$ and $(W, \mathcal{T}, \Sigma)$

Alternative definition of our model

Recall $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in \mathcal{M}_1((0, \infty))$ and $(W, \mathcal{T}, \Sigma)$

- $W = (W_t)_{t \geq 0}$ standard Brownian motion;

Alternative definition of our model

Recall $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in \mathcal{M}_1((0, \infty))$ and $(W, \mathcal{T}, \Sigma)$

- $W = (W_t)_{t \geq 0}$ standard Brownian motion;
- $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ Poisson point process on $\mathbb{R}$ with intensity λ;

Alternative definition of our model

Recall $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in \mathcal{M}_1((0, \infty))$ and $(W, \mathcal{T}, \Sigma)$

- $W = (W_t)_{t \geq 0}$ standard Brownian motion;
- $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ Poisson point process on $\mathbb{R}$ with intensity λ;
- $\Sigma = (\sigma_n)_{n \geq 0}$ i.i.d. sequence of r.v.s with marginal law σ.

Alternative definition of our model

Recall $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in \mathcal{M}_1((0, \infty))$ and $(W, \mathcal{T}, \Sigma)$

- $W = (W_t)_{t \geq 0}$ standard Brownian motion;
- $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ Poisson point process on $\mathbb{R}$ with intensity λ;
- $\Sigma = (\sigma_n)_{n \geq 0}$ i.i.d. sequence of r.v.s with marginal law σ.

Our process $X = (X_t)_{t \geq 0}$ is defined by $\boxed{X_t = W_{I_t}}$

Alternative definition of our model

Recall $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in \mathcal{M}_1((0, \infty))$ and $(W, \mathcal{T}, \Sigma)$

- $W = (W_t)_{t \geq 0}$ standard Brownian motion;
- $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ Poisson point process on $\mathbb{R}$ with intensity λ;
- $\Sigma = (\sigma_n)_{n \geq 0}$ i.i.d. sequence of r.v.s with marginal law σ.

Our process $X = (X_t)_{t \geq 0}$ is defined by $\boxed{X_t = W_{I_t}}$ where $I_t = I_t(\mathcal{T}, \Sigma)$ is explicit function of $\mathcal{T}, \Sigma$ (hence indep. of W).

Alternative definition of our model

Recall $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in \mathcal{M}_1((0, \infty))$ and $(W, \mathcal{T}, \Sigma)$

- $W = (W_t)_{t \geq 0}$ standard Brownian motion;
- $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ Poisson point process on $\mathbb{R}$ with intensity λ;
- $\Sigma = (\sigma_n)_{n \geq 0}$ i.i.d. sequence of r.v.s with marginal law σ.

Our process $X = (X_t)_{t \geq 0}$ is defined by $\boxed{X_t = W_{I_t}}$ where $I_t = I_t(\mathcal{T}, \Sigma)$ is explicit function of $\mathcal{T}, \Sigma$ (hence indep. of W).

$t \mapsto I_t$ contin., $\quad I_0 = 0, \quad$ for $h \in [\tau_n, \tau_{n+1}]$: $\quad I_{\tau_n + h} = I_{\tau_n} + \sigma_n^2 \, h^{2D}$

Alternative definition of our model

Recall $D \in (0, 1/2)$, $\lambda \in (0, \infty)$, $\sigma \in \mathcal{M}_1((0, \infty))$ and $(W, \mathcal{T}, \Sigma)$

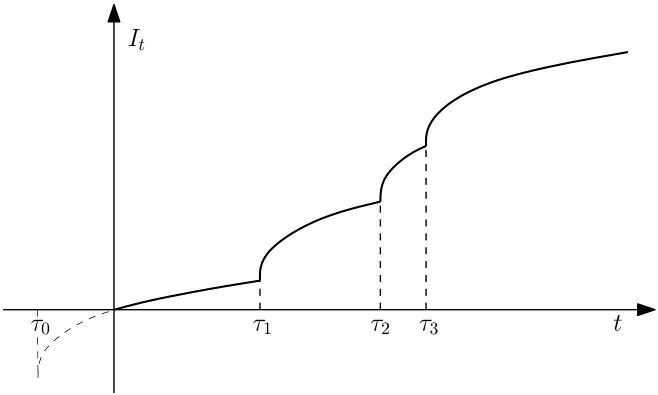
- $W = (W_t)_{t \geq 0}$ standard Brownian motion;
- $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ Poisson point process on $\mathbb{R}$ with intensity λ;
- $\Sigma = (\sigma_n)_{n \geq 0}$ i.i.d. sequence of r.v.s with marginal law σ.

Our process $X = (X_t)_{t \geq 0}$ is defined by $\boxed{X_t = W_{I_t}}$ where $I_t = I_t(\mathcal{T}, \Sigma)$ is explicit function of $\mathcal{T}, \Sigma$ (hence indep. of W).

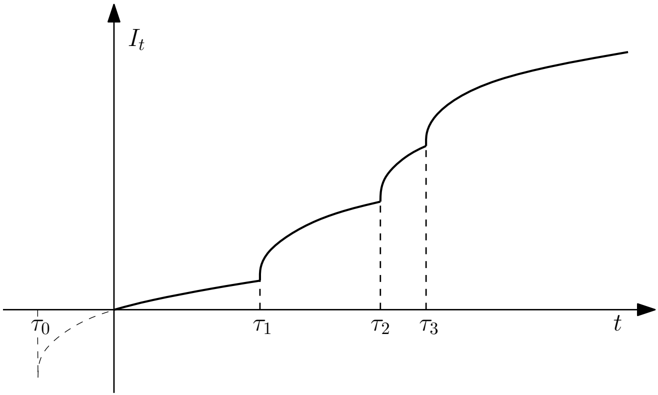
$t \mapsto I_t$ contin., $\quad I_0 = 0, \quad$ for $h \in [\tau_n, \tau_{n+1}]$: $\quad I_{\tau_n + h} = I_{\tau_n} + \sigma_n^2 \, h^{2D}$

$$I_t := \sigma_{i(t)}^2 \left(t - \tau_{i(t)}\right)^{2D} + \sum_{k=1}^{i(t)} \sigma_{k-1}^2 \left(\tau_k - \tau_{k-1}\right)^{2D} - \sigma_0^2 \left(-\tau_0\right)^{2D}$$

The process $(I_t)_{t \geq 0}$



The process $(I_t)_{t\geq 0}$



$v_t^2 = \frac{\mathrm{d}}{\mathrm{d}t} I_t = (2D)\sigma_{i(t)}^2 \left(t - \tau_{i(t)}\right)^{2D-1}$ singularities $\leftrightarrow$ shocks

Basic properties of our model

- The process X has stationary mixing increments.

Basic properties of our model

- The process X has stationary mixing increments.
- The process X is a stochastic volatility process:

$$\mathrm{d}X_t = v_t \, \mathrm{d}B_t \,,$$

where

$$B_t := \int_0^{I_t} \frac{1}{\sqrt{I'(I^{-1}(u))}} \, \mathrm{d}W_u \,, \qquad v_t := \sqrt{I'(t)} = \frac{\sqrt{2D}\, \sigma_{i(t)}}{\left(t - \tau_{i(t)}\right)^{\frac{1}{2}-D}} \,,$$

and $(B_t)_{t \geq 0}$ is a standard Brownian motion.

Basic properties of our model

- The process X has stationary mixing increments.
- The process X is a stochastic volatility process:

$$\mathrm{d}X_t = v_t \, \mathrm{d}B_t \,,$$

where

$$B_t := \int_0^{I_t} \frac{1}{\sqrt{I'(I^{-1}(u))}} \, \mathrm{d}W_u \,, \qquad v_t := \sqrt{I'(t)} = \frac{\sqrt{2D}\, \sigma_{i(t)}}{\left(t - \tau_{i(t)}\right)^{\frac{1}{2}-D}} \,,$$

and $(B_t)_{t \geq 0}$ is a standard Brownian motion.

- The process X is a zero-mean, square-integrable martingale, provided $E(\sigma^2) = \int \sigma^2 \nu(d\sigma) < \infty$.

Basic properties of our model

- The process X has stationary mixing increments.
- The process X is a stochastic volatility process:

$$\mathrm{d}X_t = v_t \, \mathrm{d}B_t \,,$$

where

$$B_t := \int_0^{I_t} \frac{1}{\sqrt{I'(I^{-1}(u))}} \, \mathrm{d}W_u \,, \qquad v_t := \sqrt{I'(t)} = \frac{\sqrt{2D}\,\sigma_{i(t)}}{\left(t - \tau_{i(t)}\right)^{\frac{1}{2}-D}} \,,$$

and $(B_t)_{t\geq 0}$ is a standard Brownian motion.

- The process X is a zero-mean, square-integrable martingale, provided $E(\sigma^2) = \int \sigma^2 \nu(d\sigma) < \infty$.
- $E\left[|X_t|^q\right] < +\infty$ iff $E\left(\sigma^q\right) < +\infty$. Heavy tails???

Outline

Approximate Diffusive Scaling

Theorem

- [large time] *If* $E(\sigma^2) < \infty$ *(typical), as* $h \uparrow \infty$ *we have the convergence in distribution*

$$\frac{(X_{t+h} - X_t)}{\sqrt{h}} \xrightarrow[h\uparrow\infty]{d} \mathcal{N}(0, c^2) \qquad c^2 = \lambda^{1-2D}\, E(\sigma^2)\, \Gamma(2D+1)\,.$$

Approximate Diffusive Scaling

Theorem

- [large time] *If* $E(\sigma^2) < \infty$ *(typical), as* $h \uparrow \infty$ *we have the convergence in distribution*

$$\frac{(X_{t+h} - X_t)}{\sqrt{h}} \xrightarrow[h\uparrow\infty]{d} \mathcal{N}(0, c^2) \qquad c^2 = \lambda^{1-2D}\, E(\sigma^2)\, \Gamma(2D+1)\,.$$

- [small time] *As* $h \downarrow 0$ *we have the convergence in distribution*

$$\frac{(X_{t+h} - X_t)}{\sqrt{h}} \xrightarrow[h\downarrow 0]{d} f(x)\, dx\,,$$

where $f(\cdot)$ *is the density of the random variable*

$$\sqrt{2D}\,\sigma\,\tau_1^{D-1/2}\,W_1\,.$$

Approximate Diffusive Scaling

The small-time asymptotic density $f(\cdot)$ is an explicit mixture of centered Gaussian densities:

$$f(x)\,\mathrm{d}x \overset{d}{=} \sqrt{2D}\,\sigma\,\tau_1^{D-1/2}\,W_1$$

Approximate Diffusive Scaling

The small-time asymptotic density $f(\cdot)$ is an explicit mixture of centered Gaussian densities:

$$f(x)\,\mathrm{d}x \overset{d}{=} \sqrt{2D}\,\sigma\,\tau_1^{D-1/2}\,W_1$$

Note that $f(x)$ has always polynomial tails:

$$\int |x|^q f(x) dx < \infty \qquad \Longleftrightarrow \qquad q < q^* := \frac{1}{\frac{1}{2} - D}$$

Approximate Diffusive Scaling

The small-time asymptotic density $f(\cdot)$ is an explicit mixture of centered Gaussian densities:

$$f(x)\,\mathrm{d}x \overset{d}{=} \sqrt{2D}\,\sigma\,\tau_1^{D-1/2}\,W_1$$

Note that $f(x)$ has always polynomial tails:

$$\int |x|^q f(x) dx < \infty \qquad \iff \qquad q < q^* := \frac{1}{\frac{1}{2} - D}$$

There is a crossover phenomenon in the log-return distribution, from power-law (small time) to Gaussian (large time).

Approximate Diffusive Scaling

The small-time asymptotic density $f(\cdot)$ is an explicit mixture of centered Gaussian densities:

$$f(x)\,\mathrm{d}x \overset{d}{=} \sqrt{2D}\,\sigma\,\tau_1^{D-1/2}\,W_1$$

Note that $f(x)$ has always polynomial tails:

$$\int |x|^q f(x)dx < \infty \qquad \Longleftrightarrow \qquad q < q^* := \frac{1}{\frac{1}{2} - D}$$

There is a crossover phenomenon in the log-return distribution, from power-law (small time) to Gaussian (large time).

Although $E\left[|X_t|^q\right] < +\infty\ \forall q$ when $E\left(\sigma^q\right) < +\infty\ \forall q$, for small t the empirical distribution of X_t does display power-law tails up to several standard deviations! ($X_t \approx \sqrt{t}f(\sqrt{t}x)$, see below.)

Multiscaling of Moments

Theorem

Assume $E(\sigma^q) < +\infty$. *The moment* $m_q(h) := E(|X_{t+h} - X_t|^q)$ *is finite and has the following asymptotic behavior as* $h \downarrow 0$*:*

$$m_q(h) \sim \begin{cases} C_q\, h^{\frac{q}{2}} & \text{if } q < q^* \\ C_q\, h^{\frac{q}{2}} \log(\frac{1}{h}) & \text{if } q = q^* \\ C_q\, h^{Dq+1} & \text{if } q > q^* \end{cases}, \qquad \text{where} \quad q^* := \frac{1}{(\frac{1}{2} - D)} .$$

Multiscaling of Moments

Theorem

Assume $E(\sigma^q) < +\infty$. The moment $m_q(h) := E(|X_{t+h} - X_t|^q)$ is finite and has the following asymptotic behavior as $h \downarrow 0$:

$$m_q(h) \sim \begin{cases} C_q \, h^{\frac{q}{2}} & \text{if } q < q^* \\ C_q \, h^{\frac{q}{2}} \log(\frac{1}{h}) & \text{if } q = q^* \\ C_q \, h^{Dq+1} & \text{if } q > q^* \end{cases}, \qquad \text{where} \quad q^* := \frac{1}{(\frac{1}{2} - D)} .$$

- We can write $m_q(h) \approx h^{A(q)}$ with scaling exponent $A(q)$

$$A(q) := \lim_{h \downarrow 0} \frac{\log m_q(h)}{\log h} = \begin{cases} q/2 & \text{if } q \le q^* \\ Dq + 1 & \text{if } q \ge q^* \end{cases} .$$

Multiscaling of Moments

Theorem

Assume $E(\sigma^q) < +\infty$. *The moment* $m_q(h) := E(|X_{t+h} - X_t|^q)$ *is finite and has the following asymptotic behavior as* $h \downarrow 0$*:*

$$m_q(h) \sim \begin{cases} C_q\, h^{\frac{q}{2}} & \text{if } q < q^* \\ C_q\, h^{\frac{q}{2}} \log(\frac{1}{h}) & \text{if } q = q^* \\ C_q\, h^{Dq+1} & \text{if } q > q^* \end{cases}, \qquad \text{where} \quad q^* := \frac{1}{(\frac{1}{2} - D)}.$$

- We can write $m_q(h) \approx h^{A(q)}$ with scaling exponent $A(q)$

$$A(q) := \lim_{h \downarrow 0} \frac{\log m_q(h)}{\log h} = \begin{cases} q/2 & \text{if } q \le q^* \\ Dq + 1 & \text{if } q \ge q^* \end{cases}.$$

- C_q explicit function of D, λ and $E(\sigma^q)$ (used in estimation)

Decay of Correlations

Theorem

The correlation of the absolute values of the increments of the process X has the following asymptotic behavior as $h \downarrow 0$:

$$\lim_{h \downarrow 0} \rho(|X_{s+h} - X_s|, |X_{t+h} - X_t|)$$
$$=: \rho(t-s) = \frac{2}{\pi \, Var(\sigma \, |W_1| \, S^{D-1/2})} \, e^{-\lambda|t-s|} \, \phi(\lambda|t-s|) \, .$$

where

$$\phi(x) := Cov\left(\sigma \, S^{D-1/2} \, , \, \sigma \, (S+x)^{D-1/2}\right)$$

and σ, $S \sim Exp(1)$ are independent and independent of W.

Decay of Correlations

Theorem

The correlation of the absolute values of the increments of the process X has the following asymptotic behavior as $h \downarrow 0$:

$$\lim_{h \downarrow 0} \rho(|X_{s+h} - X_s|, |X_{t+h} - X_t|)$$

$$=: \rho(t-s) = \frac{2}{\pi \, Var(\sigma \, |W_1| \, S^{D-1/2})} \, e^{-\lambda|t-s|} \, \phi(\lambda|t-s|) \, .$$

where

$$\phi(x) := Cov\big(\sigma \, S^{D-1/2} \, , \, \sigma \, (S+x)^{D-1/2}\big)$$

and σ, $S \sim Exp(1)$ are independent and independent of W.

- The function $\phi(\cdot)$ has a slower than exponential decay.

Outline

Estimation of the Parameters

The parameters of our model are D, λ and the law of σ, that we want to estimate on the DJIA time series (1935–2009).

Estimation of the Parameters

The parameters of our model are D, λ and the law of σ, that we want to estimate on the DJIA time series (1935–2009).

We focus on 4 real parameters: D, λ, $E(\sigma)$ and $E(\sigma^2)$, that we estimate using the quantities $A(q)$, C_1, C_2, $\rho(t)$.

Estimation of the Parameters

The parameters of our model are D, λ and the law of σ, that we want to estimate on the DJIA time series (1935–2009).

We focus on 4 real parameters: D, λ, $E(\sigma)$ and $E(\sigma^2)$, that we estimate using the quantities $A(q)$, C_1, C_2, $\rho(t)$.

[Recall the multiscaling of moments $m_q(h) = E(|X_h|^q) \sim C_q\, h^{A(q)}$]

Estimation of the Parameters

The parameters of our model are D, λ and the law of σ, that we want to estimate on the DJIA time series (1935–2009).

We focus on 4 real parameters: D, λ, $E(\sigma)$ and $E(\sigma^2)$, that we estimate using the quantities $A(q)$, C_1, C_2, $\rho(t)$.

[Recall the multiscaling of moments $m_q(h) = E(|X_h|^q) \sim C_q \, h^{A(q)}$]

1. Scaling exponent $A(q)$ function of D:

$$A(q) = \begin{cases} q/2 & \text{if } q \leq q^* \\ Dq + 1 & \text{if } q \geq q^* \end{cases} .$$

Estimation of the Parameters

The parameters of our model are D, λ and the law of σ, that we want to estimate on the DJIA time series (1935–2009).

We focus on 4 real parameters: D, λ, $E(\sigma)$ and $E(\sigma^2)$, that we estimate using the quantities $A(q)$, C_1, C_2, $\rho(t)$.

[Recall the multiscaling of moments $m_q(h) = E(|X_h|^q) \sim C_q\, h^{A(q)}$]

1. Scaling exponent $A(q)$ function of D:

$$A(q) = \begin{cases} q/2 & \text{if } q \leq q^* \\ Dq + 1 & \text{if } q \geq q^* \end{cases} .$$

2. Constants C_1 and C_2 functions of D, λ, $E(\sigma)$ and $E(\sigma^2)$:

$$C_1 = \frac{2}{\sqrt{\pi}} \sqrt{D}\, \Gamma(\tfrac{1}{2} + D)\, E(\sigma)\, \lambda^{1/2-D} \qquad C_2 = 2D\, \Gamma(2D)\, E(\sigma^2)\, \lambda^{1-2D} .$$

Estimation of the Parameters

3. Volatility autocorrelation $\rho(t)$ function of D, λ, $E(\sigma)$, $E(\sigma^2)$:

$$\rho(t) = \frac{2}{\pi \, Var(\sigma \, |W_1| \, S^{D-1/2})} \, e^{-\lambda t} \, \phi(\lambda t)$$

with $\phi(\cdot)$ (quite) easily computable.

Estimation of the Parameters

3. Volatility autocorrelation $\rho(t)$ function of D, λ, $E(\sigma)$, $E(\sigma^2)$:

$$\rho(t) = \frac{2}{\pi \, Var(\sigma \, |W_1| \, S^{D-1/2})} \, e^{-\lambda t} \, \phi(\lambda t)$$

with $\phi(\cdot)$ (quite) easily computable.

We evaluate the corresponding statistics $\widehat{A}(q)$, $\widehat{C}_1$, $\widehat{C}_2$, $\widehat{\rho}(t)$ on the (detrended log-)DJIA time series $(x_i)_{1 \le i \le T = 18849}$

Estimation of the Parameters

3. Volatility autocorrelation $\rho(t)$ function of D, λ, $E(\sigma)$, $E(\sigma^2)$:

$$\rho(t) = \frac{2}{\pi \, Var(\sigma \, |W_1| \, S^{D-1/2})} \, e^{-\lambda t} \, \phi(\lambda t)$$

with $\phi(\cdot)$ (quite) easily computable.

We evaluate the corresponding statistics $\widehat{A}(q)$, $\widehat{C}_1$, $\widehat{C}_2$, $\widehat{\rho}(t)$ on the (detrended log-)DJIA time series $(x_i)_{1 \le i \le T = 18849}$

$$\log \widehat{m}_q(h) \sim \widehat{A}(q) \, (\log h) + \log \widehat{C}_q \qquad \widehat{m}_q(h) := \frac{1}{T-h} \sum_{i=1}^{T-h} |x_{i+h} - x_i|^q$$

Estimation of the Parameters

3. Volatility autocorrelation $\rho(t)$ function of D, λ, $E(\sigma)$, $E(\sigma^2)$:

$$\rho(t) = \frac{2}{\pi \, Var(\sigma \, |W_1| \, S^{D-1/2})} \, e^{-\lambda t} \, \phi(\lambda t)$$

with $\phi(\cdot)$ (quite) easily computable.

We evaluate the corresponding statistics $\widehat{A}(q)$, $\widehat{C}_1$, $\widehat{C}_2$, $\widehat{\rho}(t)$ on the (detrended log-)DJIA time series $(x_i)_{1 \le i \le T = 18849}$

$$\log \widehat{m}_q(h) \sim \widehat{A}(q) \, (\log h) + \log \widehat{C}_q \qquad \widehat{m}_q(h) := \frac{1}{T-h} \sum_{i=1}^{T-h} |x_{i+h} - x_i|^q$$

$$\widehat{\rho}(t) := Corr\Big((x_{i+1} - x_i)_{1 \le i \le T-1-t} \, , \, (x_{i+t+1} - x_{i+t})_{1 \le i \le T-1-t} \Big)$$

Estimation of the Parameters

Loss function: $(T = 40)$

$$L(D, \lambda, E(\sigma), E(\sigma^2)) = \frac{1}{2}\left\{\left(\frac{\widehat{C}_1}{C_1} - 1\right)^2 + \left(\frac{\widehat{C}_2}{C_2} - 1\right)^2\right\} + \int_0^5 \left(\frac{\widehat{A}(q)}{A(q)} - 1\right)^2 \frac{\mathrm{d}q}{5} + \sum_{t=1}^{400} \frac{e^{-t/T}}{\sum_{s=1}^{400} e^{-s/T}} \left(\frac{\widehat{\rho}(t)}{\rho(t)} - 1\right)^2$$

Estimation of the Parameters

Loss function: $(T = 40)$

$$L(D, \lambda, E(\sigma), E(\sigma^2)) = \frac{1}{2} \left\{ \left(\frac{\widehat{C}_1}{C_1} - 1 \right)^2 + \left(\frac{\widehat{C}_2}{C_2} - 1 \right)^2 \right\} + \int_0^5 \left(\frac{\widehat{A}(q)}{A(q)} - 1 \right)^2 \frac{\mathrm{d}q}{5} + \sum_{t=1}^{400} \frac{e^{-t/T}}{\sum_{s=1}^{400} e^{-s/T}} \left(\frac{\widehat{\rho}(t)}{\rho(t)} - 1 \right)^2$$

Estimator: minimization constrained on $E(\sigma^2) \geq E(\sigma)^2$.

$$\left(\widehat{D}\,,\ \widehat{\lambda}\,,\ \widehat{E(\sigma)}\,,\ \widehat{E(\sigma^2)} \right) = \arg\min L(D, \lambda, E(\sigma), E(\sigma^2))$$

Estimation of the Parameters

Loss function: $(T = 40)$

$$L(D, \lambda, E(\sigma), E(\sigma^2)) = \frac{1}{2}\left\{\left(\frac{\widehat{C}_1}{C_1} - 1\right)^2 + \left(\frac{\widehat{C}_2}{C_2} - 1\right)^2\right\}$$

$$+ \int_0^5 \left(\frac{\widehat{A}(q)}{A(q)} - 1\right)^2 \frac{\mathrm{d}q}{5} + \sum_{t=1}^{400} \frac{e^{-t/T}}{\sum_{s=1}^{400} e^{-s/T}} \left(\frac{\widehat{\rho}(t)}{\rho(t)} - 1\right)^2$$

Estimator: minimization constrained on $E(\sigma^2) \geq E(\sigma)^2$.

$$(\widehat{D}\,, \widehat{\lambda}\,, \widehat{E(\sigma)}\,, \widehat{E(\sigma^2)}) = \arg\min L(D, \lambda, E(\sigma), E(\sigma^2))$$

$\widehat{D} \simeq 0.16$ $\widehat{\lambda} \simeq 0.00097$ $\widehat{E(\sigma)} \simeq 0.108$ $\widehat{E(\sigma^2)} \simeq \left(\widehat{E(\sigma)}\right)^2$

Estimation of the Parameters

Loss function: $(T = 40)$

$$L(D, \lambda, E(\sigma), E(\sigma^2)) = \frac{1}{2}\left\{\left(\frac{\widehat{C}_1}{C_1} - 1\right)^2 + \left(\frac{\widehat{C}_2}{C_2} - 1\right)^2\right\} + \int_0^5 \left(\frac{\widehat{A}(q)}{A(q)} - 1\right)^2 \frac{\mathrm{d}q}{5} + \sum_{t=1}^{400} \frac{e^{-t/T}}{\sum_{s=1}^{400} e^{-s/T}} \left(\frac{\widehat{\rho}(t)}{\rho(t)} - 1\right)^2$$

Estimator: minimization constrained on $E(\sigma^2) \geq E(\sigma)^2$.

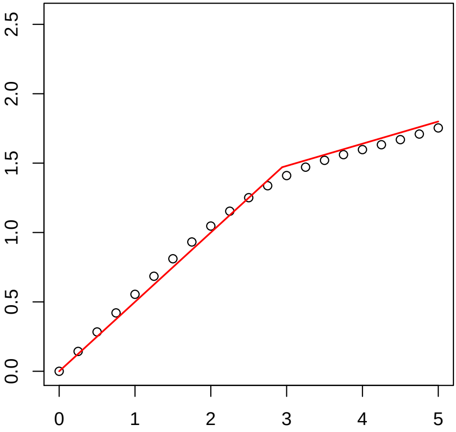
$$(\widehat{D}\,,\,\widehat{\lambda}\,,\,\widehat{E(\sigma)}\,,\,\widehat{E(\sigma^2)}) = \arg\min L(D, \lambda, E(\sigma), E(\sigma^2))$$

$$\widehat{D} \simeq 0.16 \qquad \widehat{\lambda} \simeq 0.00097 \qquad \widehat{E(\sigma)} \simeq 0.108 \qquad \widehat{E(\sigma^2)} \simeq \left(\widehat{E(\sigma)}\right)^2$$

The fit turns out to be very satisfactory, as we now show.

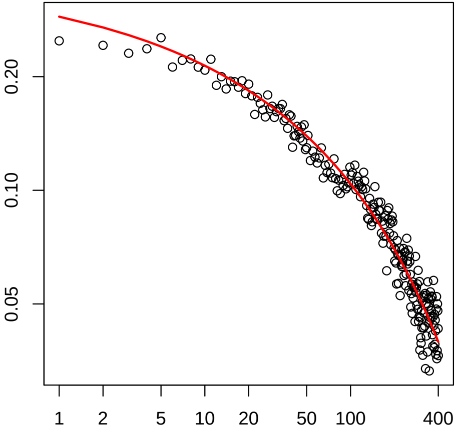
DJIA Time Series (1935-2009)

Empirical (circles) and theoretical (line) scaling exponent $A(q)$



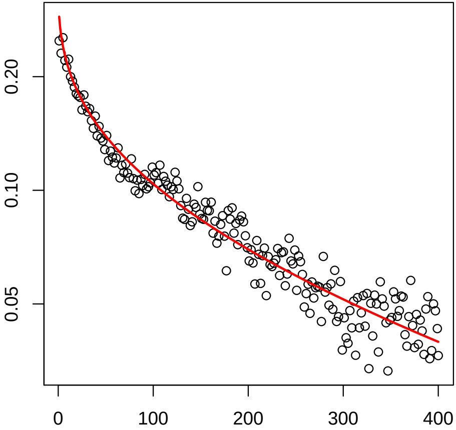
DJIA Time Series (1935-2009)

Empirical (circles) and theoretical (line) volatility autocorrelation [log plot]



DJIA Time Series (1935-2009)

Empirical (circles) and theoretical (line) volatility autocorrelation [log-log plot]



Estimation of the Law of σ

The estimated values give $E(\sigma^2) \simeq E(\sigma)^2 \ \rightarrow \ Var(\sigma) \simeq 0$

Estimation of the Law of σ

The estimated values give $E(\sigma^2) \simeq E(\sigma)^2 \ \rightarrow \ Var(\sigma) \simeq 0$

The law of σ (hence the model) is therefore completely specified.

Estimation of the Law of σ

The estimated values give $E(\sigma^2) \simeq E(\sigma)^2 \;\rightarrow\; Var(\sigma) \simeq 0$

The law of σ (hence the model) is therefore completely specified.

We then compare the law of X_1 (daily log-return) predicted by our model with the empirical one evaluated on the DJIA time series.

Estimation of the Law of σ

The estimated values give $E(\sigma^2) \simeq E(\sigma)^2 \;\rightarrow\; Var(\sigma) \simeq 0$

The law of σ (hence the model) is therefore completely specified.

We then compare the law of X_1 (daily log-return) predicted by our model with the empirical one evaluated on the DJIA time series. No further parameter has to be estimated!

Estimation of the Law of σ

The estimated values give $E(\sigma^2) \simeq E(\sigma)^2 \;\rightarrow\; Var(\sigma) \simeq 0$

The law of σ (hence the model) is therefore completely specified.

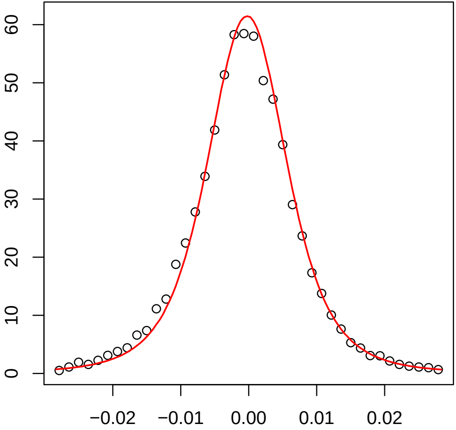
We then compare the law of X_1 (daily log-return) predicted by our model with the empirical one evaluated on the DJIA time series. No further parameter has to be estimated!

The agreement is remarkably good (both bulk and tails).

In particular, (apparent) power-law tails are visible up to several standard deviations from the mean.

DJIA Time Series (1935-2009)

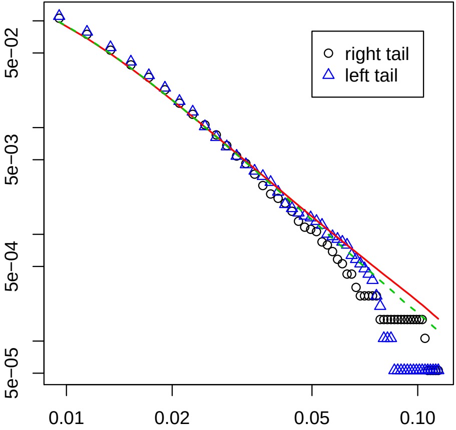
Empirical (circles) and theoretical (line) distribution of daily log return



Daily log-return standard deviation ≈ 0.01 $\longrightarrow$ Range: -3 to 3 st. dev.

DJIA Time Series (1935-2009)

Empirical and theoretical tails of daily log return [log-log plot]



Daily log-return standard deviation ≈ 0.01 $\longrightarrow$ Range: 1 to 12 st. dev.

On the Law of σ

Estimating the law of σ might appear a difficult task in general: what if we had not found $Var(\sigma) \simeq 0$?

On the Law of σ

Estimating the law of σ might appear a difficult task in general: what if we had not found $Var(\sigma) \simeq 0$?

Even when $Var(\sigma) > 0$, the details of the law of σ beyond $E(\sigma)$ and $E(\sigma^2)$ would not be relevant.

On the Law of σ

Estimating the law of σ might appear a difficult task in general: what if we had not found $Var(\sigma) \simeq 0$?

Even when $Var(\sigma) > 0$, the details of the law of σ beyond $E(\sigma)$ and $E(\sigma^2)$ would not be relevant.

In fact $1/\lambda \simeq 1000$ working days $\longrightarrow$ in 75 years we sample only $18849/1000 \simeq 18$ different variables σ_k.

On the Law of σ

Estimating the law of σ might appear a difficult task in general: what if we had not found $Var(\sigma) \simeq 0$?

Even when $Var(\sigma) > 0$, the details of the law of σ beyond $E(\sigma)$ and $E(\sigma^2)$ would not be relevant.

In fact $1/\lambda \simeq 1000$ working days $\longrightarrow$ in 75 years we sample only $18849/1000 \simeq 18$ different variables σ_k.

This is not enough to see the details of the law of σ.

On the Law of σ

Estimating the law of σ might appear a difficult task in general: what if we had not found $Var(\sigma) \simeq 0$?

Even when $Var(\sigma) > 0$, the details of the law of σ beyond $E(\sigma)$ and $E(\sigma^2)$ would not be relevant.

In fact $1/\lambda \simeq 1000$ working days $\longrightarrow$ in 75 years we sample only $18849/1000 \simeq 18$ different variables σ_k.

This is not enough to see the details of the law of σ.

Different laws for σ with the same $E(\sigma)$ and $E(\sigma^2)$ give very similar results.

The law of the log-returns (in the rage of interest) is effectively determined by the t^{2D} time scaling at the points of $\mathcal{T}$.

Outline

More than one index

Ho can we deal with more than one index at the same time?

More than one index

Ho can we deal with more than one index at the same time?

Paolo Pigato has worked on a bivariate model for the time series of DJIA and FTSE 100 indexes in the period 1984-2011 (6822 data).

More than one index

Ho can we deal with more than one index at the same time?

Paolo Pigato has worked on a bivariate model for the time series of DJIA and FTSE 100 indexes in the period 1984-2011 (6822 data).

Joint process $\{(X_t, Y_t)\}_{t\geq 0}$ such that $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are distributed according to our model.

More than one index

Ho can we deal with more than one index at the same time?

Paolo Pigato has worked on a bivariate model for the time series of DJIA and FTSE 100 indexes in the period 1984-2011 (6822 data).

Joint process $\{(X_t, Y_t)\}_{t\geq 0}$ such that $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are distributed according to our model.

Marginal parameters $(D^X, \lambda^X, \sigma^X)$, $(D^Y, \lambda^Y, \sigma^Y)$

Marginal randomness $(W^X, \mathcal{T}^X, \Sigma^X)$, $(W^Y, \mathcal{T}^Y, \Sigma^Y)$

More than one index

Ho can we deal with more than one index at the same time?

Paolo Pigato has worked on a bivariate model for the time series of DJIA and FTSE 100 indexes in the period 1984-2011 (6822 data).

Joint process $\{(X_t, Y_t)\}_{t\geq 0}$ such that $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are distributed according to our model.

Marginal parameters $(D^X, \lambda^X, \sigma^X)$, $(D^Y, \lambda^Y, \sigma^Y)$

Marginal randomness $(W^X, \mathcal{T}^X, \Sigma^X)$, $(W^Y, \mathcal{T}^Y, \Sigma^Y)$

$$X_t = W^X_{I^X_t}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} I^X_t := 2D^X \, \sigma^2_{i^X(t)} \left(t - \tau^X_{i^X(t)} \right)^{2D^X - 1},$$

$$Y_t = W^Y_{I^Y_t}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} I^Y_t := 2D^Y \, \sigma^2_{i^Y(t)} \left(t - \tau^Y_{i^Y(t)} \right)^{2D^Y - 1}.$$

More than one index

Ho can we deal with more than one index at the same time?

Paolo Pigato has worked on a bivariate model for the time series of DJIA and FTSE 100 indexes in the period 1984-2011 (6822 data).

Joint process $\{(X_t, Y_t)\}_{t \geq 0}$ such that $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ are distributed according to our model.

Marginal parameters $(D^X, \lambda^X, \sigma^X)$, $(D^Y, \lambda^Y, \sigma^Y)$

Marginal randomness $(W^X, \mathcal{T}^X, \Sigma^X)$, $(W^Y, \mathcal{T}^Y, \Sigma^Y)$

$$X_t = W^X_{I^X_t}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} I^X_t := 2D^X \, \sigma^2_{i^X(t)} \left(t - \tau^X_{i^X(t)} \right)^{2D^X - 1},$$

$$Y_t = W^Y_{I^Y_t}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} I^Y_t := 2D^Y \, \sigma^2_{i^Y(t)} \left(t - \tau^Y_{i^Y(t)} \right)^{2D^Y - 1}.$$

Which joint distribution for $(W^X, \mathcal{T}^X, \Sigma^X)$, $(W^Y, \mathcal{T}^Y, \Sigma^Y)$?

More than one index

The simplest (natural) idea is to correlate only $\mathcal{T}^X$ and $\mathcal{T}^Y$.

More than one index

The simplest (natural) idea is to correlate only $\mathcal{T}^X$ and $\mathcal{T}^Y$.

$$\mathcal{T}^X = \mathcal{T}^{(1)} \cup \mathcal{T}^{(3)} \qquad \mathcal{T}^Y = \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$$

More than one index

The simplest (natural) idea is to correlate only $\mathcal{T}^X$ and $\mathcal{T}^Y$.

$$\mathcal{T}^X = \mathcal{T}^{(1)} \cup \mathcal{T}^{(3)} \qquad \mathcal{T}^Y = \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$$

$\mathcal{T}^{(1)}, \mathcal{T}^{(2)}, \mathcal{T}^{(3)}$ are independent PPP with rates λ_1, λ_2, λ_3

$$\lambda^X = \lambda_1 + \lambda_3 \qquad \lambda^Y = \lambda_2 + \lambda_3$$

More than one index

The simplest (natural) idea is to correlate only $\mathcal{T}^X$ and $\mathcal{T}^Y$.

$$\mathcal{T}^X = \mathcal{T}^{(1)} \cup \mathcal{T}^{(3)} \qquad \mathcal{T}^Y = \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$$

$\mathcal{T}^{(1)}, \mathcal{T}^{(2)}, \mathcal{T}^{(3)}$ are independent PPP with rates λ_1, λ_2, λ_3

$$\lambda^X = \lambda_1 + \lambda_3 \qquad \lambda^Y = \lambda_2 + \lambda_3$$

$(W^X, W^Y, \mathcal{T}^{(1)}, \mathcal{T}^{(2)}, \mathcal{T}^{(3)}, \Sigma^X, \Sigma^Y)$ are independent processes

More than one index

The simplest (natural) idea is to correlate only $\mathcal{T}^X$ and $\mathcal{T}^Y$.

$$\mathcal{T}^X = \mathcal{T}^{(1)} \cup \mathcal{T}^{(3)} \qquad \mathcal{T}^Y = \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$$

$\mathcal{T}^{(1)}, \mathcal{T}^{(2)}, \mathcal{T}^{(3)}$ are independent PPP with rates λ_1, λ_2, λ_3

$$\lambda^X = \lambda_1 + \lambda_3 \qquad \lambda^Y = \lambda_2 + \lambda_3$$

$(W^X, W^Y, \mathcal{T}^{(1)}, \mathcal{T}^{(2)}, \mathcal{T}^{(3)}, \Sigma^X, \Sigma^Y)$ are independent processes

How do cross correlations behave for such a model?

$$\rho^{X,Y}(s,t) := \lim_{h \downarrow 0} \rho(|X_{s+h} - X_s|, |Y_{t+h} - Y_t|)$$

Cross correlations

Theorem

The cross correlations have the following asymptotic behavior:

For $t > s$: $\rho^{X,Y}(s,t) = C\, e^{-\lambda^Y (t-s)}\, \phi^{X,Y}(\lambda^Y (t-s))$

Cross correlations

Theorem

The cross correlations have the following asymptotic behavior:

For $t > s$: $$\rho^{X,Y}(s,t) \;=\; C\, e^{-\lambda^Y(t-s)}\, \phi^{X,Y}(\lambda^Y(t-s))$$

$$C = \frac{2}{\pi \sqrt{Var(\sigma^X\,|W_1|\,S^{D^X-1/2}) Var(\sigma^Y\,|W_1|\,S^{D^Y-1/2})}}$$

$$\phi^{X,Y}(u) := Cov\big(\sigma^X\,(S^X)^{D^X-1/2},\, \sigma^Y\,\big(S^Y+u\big)^{D^Y-1/2}\big)$$

where $S^X, S^Y \sim Exp(1)$ are correlated (like τ_1^X and τ_1^Y).

Cross correlations

Theorem

The cross correlations have the following asymptotic behavior:

For $t > s$: $$\rho^{X,Y}(s,t) = C\, e^{-\lambda^Y (t-s)}\, \phi^{X,Y}(\lambda^Y (t-s))$$

$$C = \frac{2}{\pi \sqrt{Var(\sigma^X\, |W_1|\, S^{D^X - 1/2}) Var(\sigma^Y\, |W_1|\, S^{D^Y - 1/2})}}$$

$$\phi^{X,Y}(u) := Cov\Big(\sigma^X\, (S^X)^{D^X - 1/2},\, \sigma^Y\, \big(S^Y + u\big)^{D^Y - 1/2}\Big)$$

where $S^X, S^Y \sim Exp(1)$ are correlated (like τ_1^X and τ_1^Y).

- The cross correlations $\rho^{X,Y}(t)$ behave very similarly to the autocorrelations $\rho^X(t)$, $\rho^Y(t)$. They coincide in the limiting case $\mathcal{T}^X = \mathcal{T}^Y$ (i.e. $\mathcal{T}^{(3)} = \emptyset$), $D^X = D^Y$, $\sigma^X = \sigma^Y = cst$.

Cross correlations

Theorem

The cross correlations have the following asymptotic behavior:

For $t > s$: $$\rho^{X,Y}(s,t) = C\, e^{-\lambda^Y (t-s)}\, \phi^{X,Y}(\lambda^Y (t-s))$$

$$C = \frac{2}{\pi \sqrt{Var(\sigma^X\, |W_1|\, S^{D^X - 1/2}) Var(\sigma^Y\, |W_1|\, S^{D^Y - 1/2})}}$$

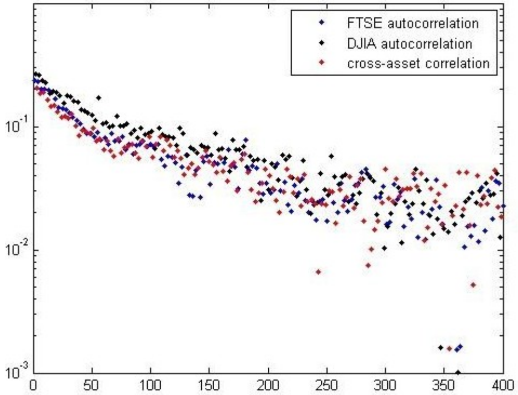
$$\phi^{X,Y}(u) := Cov\Big(\sigma^X\, (S^X)^{D^X - 1/2},\, \sigma^Y\, \big(S^Y + u\big)^{D^Y - 1/2}\Big)$$

where $S^X, S^Y \sim Exp(1)$ are correlated (like τ_1^X and τ_1^Y).

- The cross correlations $\rho^{X,Y}(t)$ behave very similarly to the autocorrelations $\rho^X(t)$, $\rho^Y(t)$. They coincide in the limiting case $\mathcal{T}^X = \mathcal{T}^Y$ (i.e. $\mathcal{T}^{(3)} = \emptyset$), $D^X = D^Y$, $\sigma^X = \sigma^Y = cst$.
- This is indeed what one observes! (Not obvious a priori.)

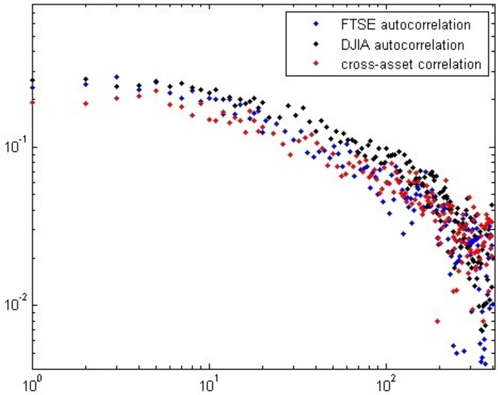
DJIA and FTSE Time Series (1984-2011)

Empirical autocorrelations ρ^X, ρ^Y and cross correlations $\rho^{X,Y}$: log plot



DJIA and FTSE Time Series (1984-2011)

Empirical autocorrelations ρ^X, ρ^Y and cross correlations $\rho^{X,Y}$: log-log plot



Numerical analysis

Do this model fits well the joint time series of DJIA and FTSE?

Numerical analysis

Do this model fits well the joint time series of DJIA and FTSE?

First step: look separately at the individual time series.

[DJIA was already done, but for a different (longer) time period.]

Numerical analysis

Do this model fits well the joint time series of DJIA and FTSE?

First step: look separately at the individual time series.

[DJIA was already done, but for a different (longer) time period.]

Estimated parameters (X = DJIA, Y = FTSE)

$$D^X \simeq 0.14, \qquad \lambda^X \simeq 0.0013, \qquad \sigma^X \simeq 0.135 \simeq const.$$

Numerical analysis

Do this model fits well the joint time series of DJIA and FTSE?

First step: look separately at the individual time series.

[DJIA was already done, but for a different (longer) time period.]

Estimated parameters (X = DJIA, Y = FTSE)

$$D^X \simeq 0.14, \qquad \lambda^X \simeq 0.0013, \qquad \sigma^X \simeq 0.135 \simeq \mathit{const}.$$

$$D^Y \simeq 0.16, \qquad \lambda^Y \simeq 0.0018, \qquad \sigma^Y \simeq 0.11 \simeq \mathit{const}.$$

Numerical analysis

Do this model fits well the joint time series of DJIA and FTSE?

First step: look separately at the individual time series.

[DJIA was already done, but for a different (longer) time period.]

Estimated parameters (X = DJIA, Y = FTSE)

$$D^X \simeq 0.14, \qquad \lambda^X \simeq 0.0013, \qquad \sigma^X \simeq 0.135 \simeq \mathit{const.}$$

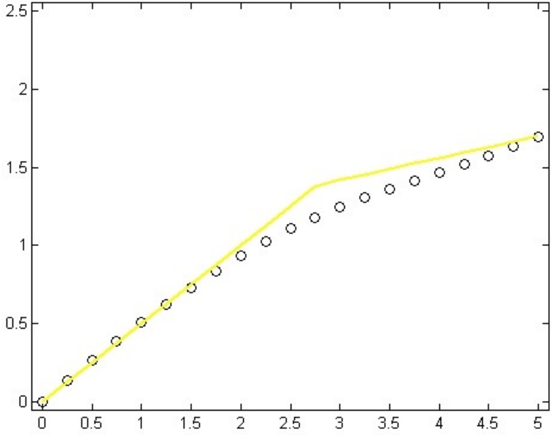
$$D^Y \simeq 0.16, \qquad \lambda^Y \simeq 0.0018, \qquad \sigma^Y \simeq 0.11 \simeq \mathit{const.}$$

For both indexes, the agreement is very satisfactory.

Again, the fit of the law of the log-returns is very good, even with no explicit calibration on it.

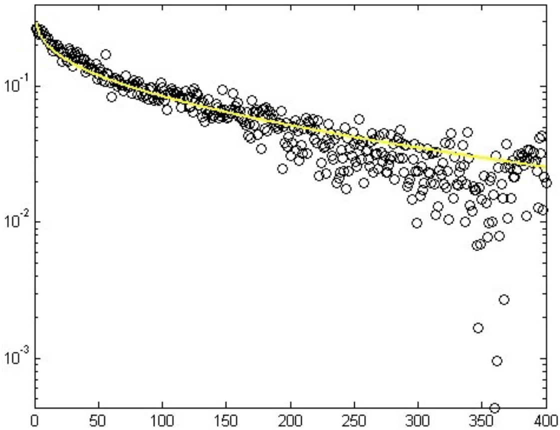
DJIA Time Series (1984-2011)

Empirical (circles) and theoretical (line) scaling exponent $A(q)$



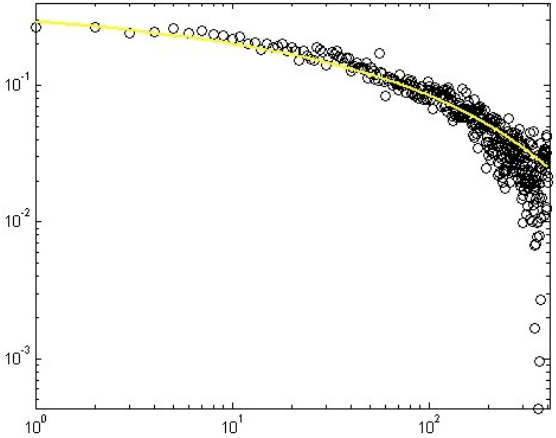
DJIA Time Series (1984-2011)

Empirical (circles) and theoretical (line) volatility autocorrelation [log plot]



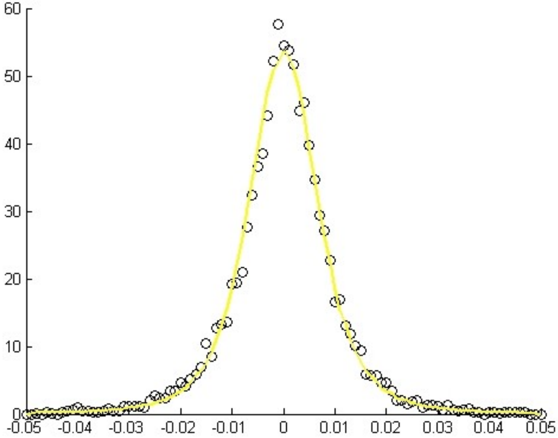
DJIA Time Series (1984-2011)

Empirical (circles) and theoretical (line) volatility autocorrelation [log-log plot]



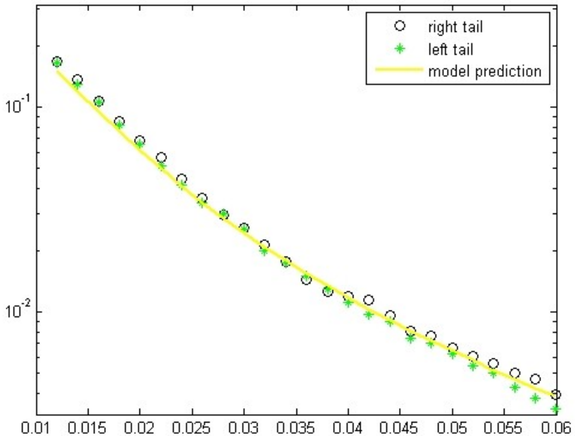
DJIA Time Series (1984-2011)

Empirical (circles) and theoretical (line) distribution of daily log return



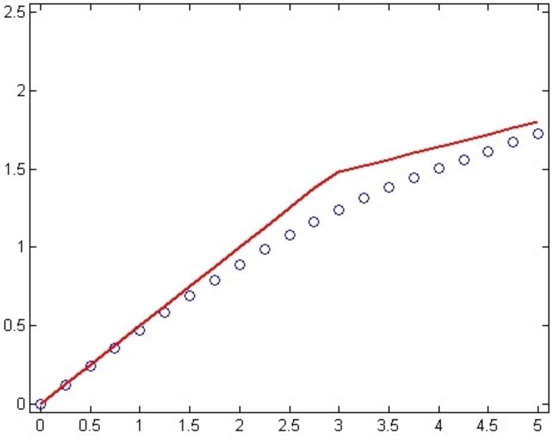
DJIA Time Series (1984-2011)

Empirical and theoretical tails of daily log return [log plot]



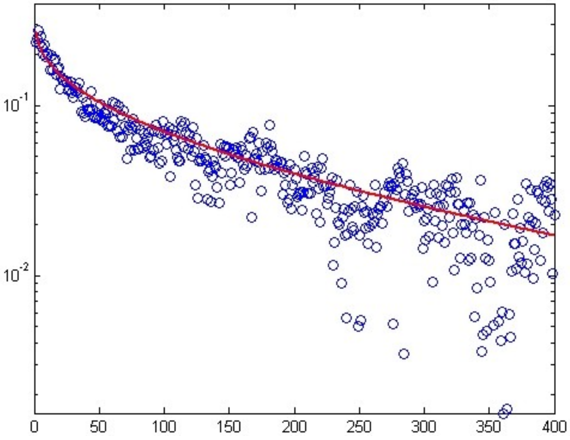
FTSE Time Series (1984-2011)

Empirical (circles) and theoretical (line) scaling exponent $A(q)$



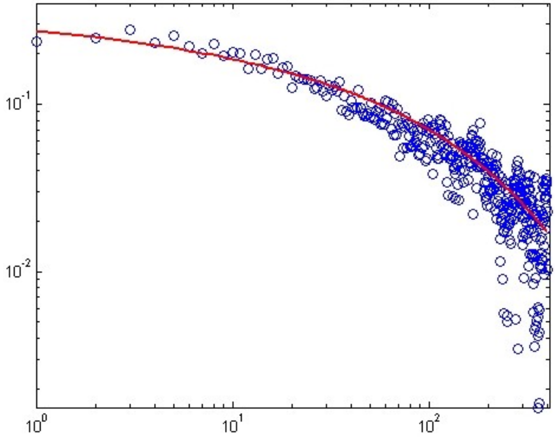
FTSE Time Series (1984-2011)

Empirical (circles) and theoretical (line) volatility autocorrelation [log plot]



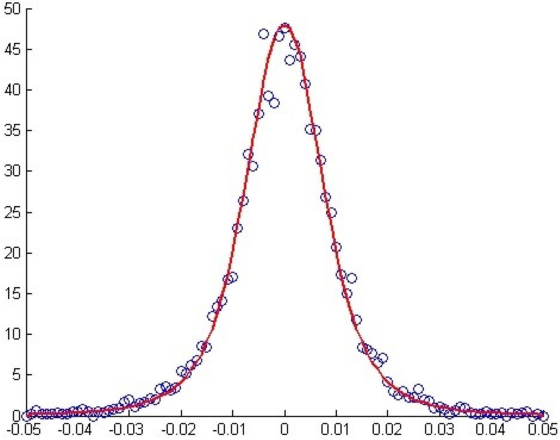
FTSE Time Series (1984-2011)

Empirical (circles) and theoretical (line) volatility autocorrelation [log-log plot]



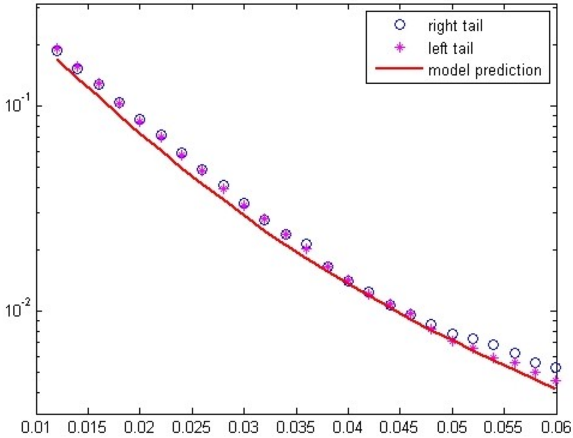
FTSE Time Series (1984-2011)

Empirical (circles) and theoretical (line) distribution of daily log return



FTSE Time Series (1984-2011)

Empirical and theoretical tails of daily log return [log plot]



Joint behavior

Finally we focus on the joint behavior of the indexes, in particular on their cross correlations.

Joint behavior

Finally we focus on the joint behavior of the indexes, in particular on their cross correlations.

Recall that $\mathcal{T}^X = \mathcal{T}^{(1)} \cup \mathcal{T}^{(3)}$ and $\mathcal{T}^Y = \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$, therefore $\lambda^X = \lambda_1 + \lambda_3$ and $\lambda^Y = \lambda_2 + \lambda_3$.

Joint behavior

Finally we focus on the joint behavior of the indexes, in particular on their cross correlations.

Recall that $\mathcal{T}^X = \mathcal{T}^{(1)} \cup \mathcal{T}^{(3)}$ and $\mathcal{T}^Y = \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$, therefore $\lambda^X = \lambda_1 + \lambda_3$ and $\lambda^Y = \lambda_2 + \lambda_3$.

Problem

How do we estimate the rate λ_3 of the common part $\mathcal{T}^{(3)}$?

Joint behavior

Finally we focus on the joint behavior of the indexes, in particular on their cross correlations.

Recall that $\mathcal{T}^X = \mathcal{T}^{(1)} \cup \mathcal{T}^{(3)}$ and $\mathcal{T}^Y = \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$, therefore $\lambda^X = \lambda_1 + \lambda_3$ and $\lambda^Y = \lambda_2 + \lambda_3$.

Problem

How do we estimate the rate λ_3 of the common part $\mathcal{T}^{(3)}$?

The best would be to estimate the random sets $\mathcal{T}^X$ and $\mathcal{T}^Y$ on the two time series of DJIA and FTSE and see how much they overlap.

Joint behavior

Finally we focus on the joint behavior of the indexes, in particular on their cross correlations.

Recall that $\mathcal{T}^X = \mathcal{T}^{(1)} \cup \mathcal{T}^{(3)}$ and $\mathcal{T}^Y = \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$, therefore $\lambda^X = \lambda_1 + \lambda_3$ and $\lambda^Y = \lambda_2 + \lambda_3$.

Problem

How do we estimate the rate λ_3 of the common part $\mathcal{T}^{(3)}$?

The best would be to estimate the random sets $\mathcal{T}^X$ and $\mathcal{T}^Y$ on the two time series of DJIA and FTSE and see how much they overlap.

However, $\mathcal{T}^X$ and $\mathcal{T}^Y$ are the location of the shock times, which are not easily and directly observable.

Joint behavior

Finally we focus on the joint behavior of the indexes, in particular on their cross correlations.

Recall that $\mathcal{T}^X = \mathcal{T}^{(1)} \cup \mathcal{T}^{(3)}$ and $\mathcal{T}^Y = \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$, therefore $\lambda^X = \lambda_1 + \lambda_3$ and $\lambda^Y = \lambda_2 + \lambda_3$.

Problem

How do we estimate the rate λ_3 of the common part $\mathcal{T}^{(3)}$?

The best would be to estimate the random sets $\mathcal{T}^X$ and $\mathcal{T}^Y$ on the two time series of DJIA and FTSE and see how much they overlap.

However, $\mathcal{T}^X$ and $\mathcal{T}^Y$ are the location of the shock times, which are not easily and directly observable. They may only be an idealization of our model... or are they real?

Locating the shock times

Mario Bonino has devised a smart algorithm to locate the random set of shock points $\mathcal{T}$.

Locating the shock times

Mario Bonino has devised a smart algorithm to locate the random set of shock points $\mathcal{T}$.

Recall that $\mathrm{d}X_t = v_t \, \mathrm{d}B_t$ with $v_t^2 = I'(t) \propto (t - \tau_{i(t)})^{2D-1}$.

Basic observation: the volatility v_t^2 diverges precisely on the set $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ of shock times.

Locating the shock times

Mario Bonino has devised a smart algorithm to locate the random set of shock points $\mathcal{T}$.

Recall that $\mathrm{d}X_t = v_t \, \mathrm{d}B_t$ with $v_t^2 = I'(t) \propto (t - \tau_{i(t)})^{2D-1}$.

Basic observation: the volatility v_t^2 diverges precisely on the set $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ of shock times.

Given a time series $(x_i)_{1 \le i \le T}$, consider the quantity

$$V_T(t) := \frac{1}{T-t} \sum_{i=t+1}^{T} (x_{i+1} - x_i)^2$$

Locating the shock times

Mario Bonino has devised a smart algorithm to locate the random set of shock points $\mathcal{T}$.

Recall that $\mathrm{d}X_t = v_t \, \mathrm{d}B_t$ with $v_t^2 = I'(t) \propto (t - \tau_{i(t)})^{2D-1}$.

Basic observation: the volatility v_t^2 diverges precisely on the set $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ of shock times.

Given a time series $(x_i)_{1 \le i \le T}$, consider the quantity

$$V_T(t) := \frac{1}{T-t} \sum_{i=t+1}^{T} (x_{i+1} - x_i)^2 \overset{\text{for } T-t \gg 1}{\simeq} \frac{1}{T-t} \int_t^T v_s^2 \, \mathrm{d}s$$

Therefore, for fixed $T > 0$, $V_T(t)$ should attain its (first) maximum at the location $t = \bar{t} = i(T)$ of the most recent shock time.

Locating the shock times

Mario Bonino has devised a smart algorithm to locate the random set of shock points $\mathcal{T}$.

Recall that $\mathrm{d}X_t = v_t \, \mathrm{d}B_t$ with $v_t^2 = I'(t) \propto (t - \tau_{i(t)})^{2D-1}$.

Basic observation: the volatility v_t^2 diverges precisely on the set $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ of shock times.

Given a time series $(x_i)_{1 \le i \le T}$, consider the quantity

$$V_T(t) := \frac{1}{T-t} \sum_{i=t+1}^{T} (x_{i+1} - x_i)^2 \overset{\text{for } T-t \gg 1}{\simeq} \frac{1}{T-t} \int_t^T v_s^2 \, \mathrm{d}s$$

Therefore, for fixed $T > 0$, $V_T(t)$ should attain its (first) maximum at the location $t = \bar{t} = i(T)$ of the most recent shock time.

Unfortunately, due to fluctuations, there may be several local maxima... How to locate the right one?

Locating the shock times

Idea: compare locations of the maxima for different values of T.

If $\bar{t}$ is a "true" shock time, it should be detected as a maximum of $V_T(t)$ for (almost) every fixed value of $T > \bar{t}$.

Locating the shock times

Idea: compare locations of the maxima for different values of T.

If $\bar{t}$ is a "true" shock time, it should be detected as a maximum of $V_T(t)$ for (almost) every fixed value of $T > \bar{t}$.

Concretely, given the time series $(x_i)_{1 \le i \le T}$, we set

$$g_T := \operatorname{argmax}\big\{ V_T(t) : \ T - 2000 \le t \le T - 22 \big\}$$

$$g_{T-1} := \operatorname{argmax}\big\{ V_T(t) : \ (T-1) - 2000 \le t \le (T-1) - 22 \big\}$$

$$\ldots$$

Locating the shock times

Idea: compare locations of the maxima for different values of T.

If $\bar{t}$ is a "true" shock time, it should be detected as a maximum of $V_T(t)$ for (almost) every fixed value of $T > \bar{t}$.

Concretely, given the time series $(x_i)_{1 \le i \le T}$, we set

$$g_T := \operatorname{argmax}\big\{ V_T(t) : \ T - 2000 \le t \le T - 22 \big\}$$

$$g_{T-1} := \operatorname{argmax}\big\{ V_T(t) : \ (T-1) - 2000 \le t \le (T-1) - 22 \big\}$$

$$\ldots$$

If our predictions are right, the set $\{g_T, g_{T-1}, g_{T-2}, \ldots\}$ should consist of only few distinct values (each attained by several g_i's) corresponding to the shock points, i.e. the points of $\mathcal{T}$.

Locating the shock times

Idea: compare locations of the maxima for different values of T.

If $\bar{t}$ is a "true" shock time, it should be detected as a maximum of $V_T(t)$ for (almost) every fixed value of $T > \bar{t}$.

Concretely, given the time series $(x_i)_{1 \le i \le T}$, we set

$$g_T := \operatorname{argmax}\big\{ V_T(t) : \ T - 2000 \le t \le T - 22 \big\}$$

$$g_{T-1} := \operatorname{argmax}\big\{ V_T(t) : \ (T-1) - 2000 \le t \le (T-1) - 22 \big\}$$

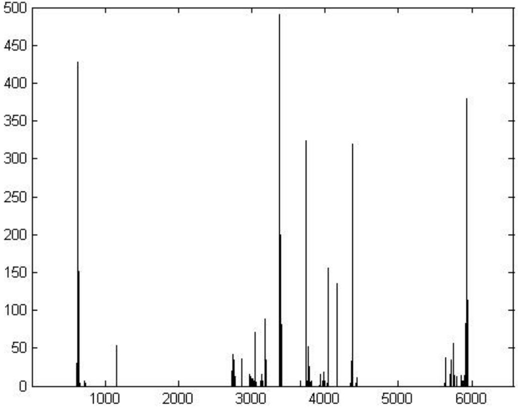
$$\ldots$$

If our predictions are right, the set $\{g_T, g_{T-1}, g_{T-2}, \ldots\}$ should consist of only few distinct values (each attained by several g_i's) corresponding to the shock points, i.e. the points of $\mathcal{T}$.

This is indeed (almost) the case! We just need to identify couples of very close (< 20 days) shock points.

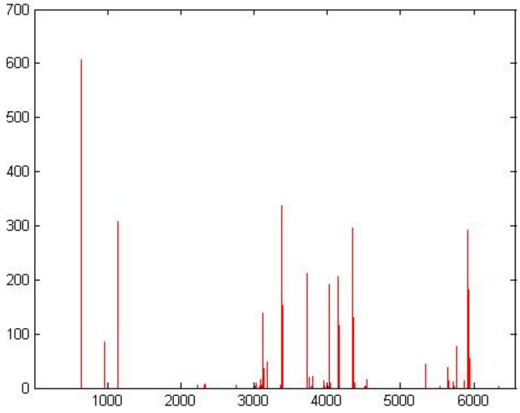
DJIA Time Series (1984-2011)

Shock times $\mathcal{T}^X$ for the DJIA



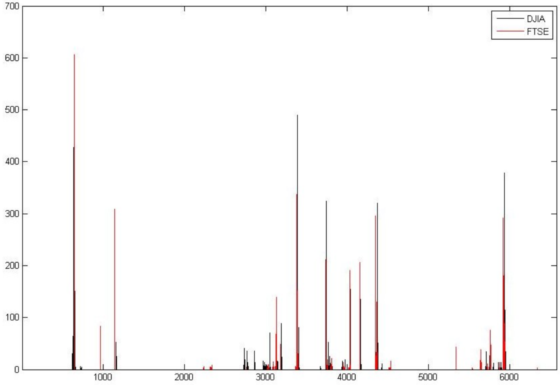
FTSE Time Series (1984-2011)

Shock times $\mathcal{T}^Y$ for the FTSE



DJIA and FTSE Time Series (1984-2011)

Shock times $\mathcal{T}^X$ and $\mathcal{T}^Y$ for the DJIA and FTSE



Locating the shock times

There is a considerable overlap between the (empirical) sets $\mathcal{T}^X$ and $\mathcal{T}^Y$ of shock times of the DJIA and FTSE time series.

Locating the shock times

There is a considerable overlap between the (empirical) sets $\mathcal{T}^X$ and $\mathcal{T}^Y$ of shock times of the DJIA and FTSE time series.

Recall the estimated values

$$\lambda^X \simeq 0.0013\,, \qquad \lambda^Y \simeq 0.0018\,,$$

and we want to find λ_3 such that $\lambda^X = \lambda_1 + \lambda_3$, $\lambda^Y = \lambda_2 + \lambda_3$.

Locating the shock times

There is a considerable overlap between the (empirical) sets $\mathcal{T}^X$ and $\mathcal{T}^Y$ of shock times of the DJIA and FTSE time series.

Recall the estimated values

$$\lambda^X \simeq 0.0013\,, \qquad \lambda^Y \simeq 0.0018\,,$$

and we want to find λ_3 such that $\lambda^X = \lambda_1 + \lambda_3$, $\lambda^Y = \lambda_2 + \lambda_3$.

Guess: large value of λ_3. More quantitatively, the cross correlation

$$\rho^{X,Y}(t-s) := \lim_{h \downarrow 0} \rho(|X_{s+h} - X_s|, |Y_{t+h} - Y_t|)$$

depends on $\lambda_1, \lambda_2, \lambda_3$. By comparison with the empirical cross correlation, we can choose the best value of λ_3.

Locating the shock times

There is a considerable overlap between the (empirical) sets $\mathcal{T}^X$ and $\mathcal{T}^Y$ of shock times of the DJIA and FTSE time series.

Recall the estimated values

$$\lambda^X \simeq 0.0013\,, \qquad \lambda^Y \simeq 0.0018\,,$$

and we want to find λ_3 such that $\lambda^X = \lambda_1 + \lambda_3$, $\lambda^Y = \lambda_2 + \lambda_3$.

Guess: large value of λ_3. More quantitatively, the cross correlation

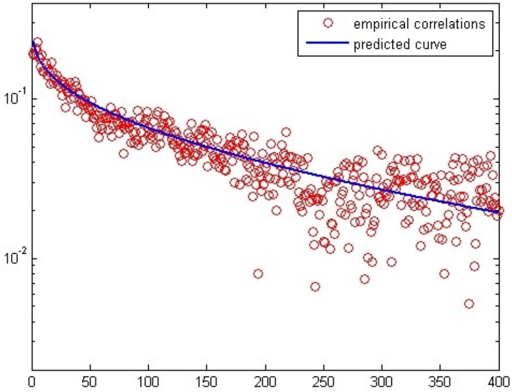
$$\rho^{X,Y}(t-s) := \lim_{h \downarrow 0} \rho(|X_{s+h} - X_s|, |Y_{t+h} - Y_t|)$$

depends on $\lambda_1, \lambda_2, \lambda_3$. By comparison with the empirical cross correlation, we can choose the best value of λ_3.

Result: $\lambda_1 \simeq 0.0001$, $\lambda_2 \simeq 0.0006$, $\lambda_3 \simeq 0.0012$

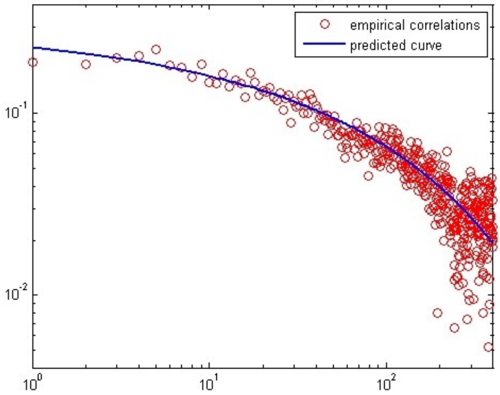
DJIA and FTSE Time Series (1984-2011)

Empirical (circles) and theoretical (lines) cross correlations: log plot



DJIA and FTSE Time Series (1984-2011)

Empirical (circles) and theoretical (lines) cross correlations: log-log plot



Locating the shock times

The agreement is again very good.

Locating the shock times

The agreement is again very good.

Actually, it would be good even with $\lambda_1 = 0$, i.e. if every shock time of DJIA were a shock time of FTSE.

Outline

1. Black & Scholes and beyond
2. The Model
3. Main Results
4. Estimation and Simulations
5. Bivariate Model
6. Conclusions

Conclusions

We have proposed a model with the following features:

- It is analytically tractable. In particular, sharp asymptotics for scaling relations and correlations are obtained.

Conclusions

We have proposed a model with the following features:

- It is analytically tractable. In particular, sharp asymptotics for scaling relations and correlations are obtained.
- It is easy to simulate.

Conclusions

We have proposed a model with the following features:

- It is analytically tractable. In particular, sharp asymptotics for scaling relations and correlations are obtained.
- It is easy to simulate.
- Despite of the few parameters, it accounts for various phenomena observed in real time series.

Conclusions

We have proposed a model with the following features:

- It is analytically tractable. In particular, sharp asymptotics for scaling relations and correlations are obtained.
- It is easy to simulate.
- Despite of the few parameters, it accounts for various phenomena observed in real time series.
- Several generalizations can be considered (correlations between Σ, $\mathcal{T}$ and W, modified time-change $t \mapsto t^{2D}$, ...).

Conclusions

We have proposed a model with the following features:

- It is analytically tractable. In particular, sharp asymptotics for scaling relations and correlations are obtained.
- It is easy to simulate.
- Despite of the few parameters, it accounts for various phenomena observed in real time series.
- Several generalizations can be considered (correlations between Σ, $\mathcal{T}$ and W, modified time-change $t \mapsto t^{2D}$, ...).

Next steps [under investigation]:

- Enrich the model ($\rightarrow$ multifractal volatility)

Conclusions

We have proposed a model with the following features:

- It is analytically tractable. In particular, sharp asymptotics for scaling relations and correlations are obtained.
- It is easy to simulate.
- Despite of the few parameters, it accounts for various phenomena observed in real time series.
- Several generalizations can be considered (correlations between Σ, $\mathcal{T}$ and W, modified time-change $t \mapsto t^{2D}$, ...).

Next steps [under investigation]:

- Enrich the model ($\rightarrow$ multifractal volatility)
- Solve specific financial problems: pricing of options, portfolio management, ...

Thanks.

Variability in subperiods

A natural question on the DJIA time series is the amount of variability of the data set in subperiods. Is the period 1935-2009 long enough to be close to the ergodic limit?

More concretely: are the statistics of the DJIA time series in (large) subperiods close to those of the whole period 1935-2009?

Variability in subperiods

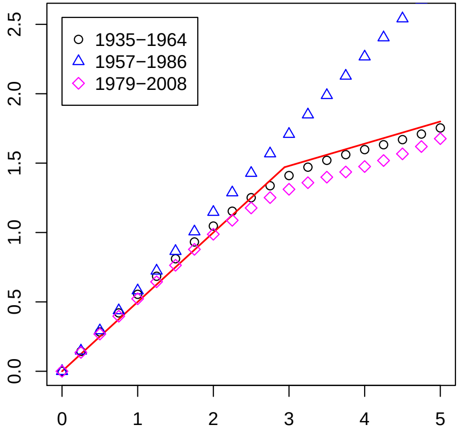
A natural question on the DJIA time series is the amount of variability of the data set in subperiods. Is the period 1935-2009 long enough to be close to the ergodic limit?

More concretely: are the statistics of the DJIA time series in (large) subperiods close to those of the whole period 1935-2009?

It turns out that a considerable variability is present for all the quantities we observe (multiscaling of moments, decay of correlations and empirical distribution) if one takes different (suitably chosen) large time windows of 30 years.

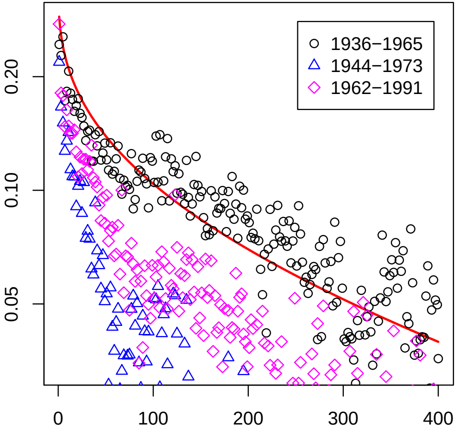
DJIA Time Series (1935-2009)

Empirical scaling exponent $A(q)$ over sub-periods of 30 years.



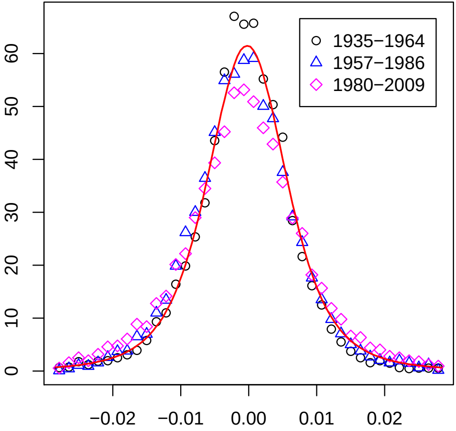
DJIA Time Series (1935-2009)

Volatility autocorrelation over sub-periods of 30 years [log plot]



DJIA Time Series (1935-2009)

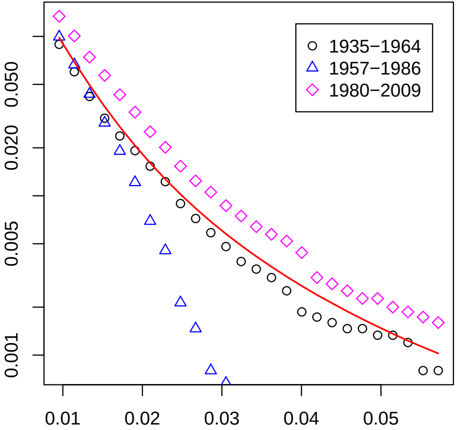
Variability of the distribution in sub-periods of 30 years



Daily log-return standard deviation ≈ 0.01 $\longrightarrow$ Range: -3 to 3 st. dev.

DJIA Time Series (1935-2009)

Variability of the left tail in sub-periods of 30 years



Daily log-return standard deviation ≈ 0.01 $\longrightarrow$ Range: 1 to 6 st. dev.

Variability of estimators

These plots show that the DJIA time series in the period 1935-2009 is not so close to the ergodic limit: empirical averages over subperiods of 30 years exhibit non negligible fluctuations.

Variability of estimators

These plots show that the DJIA time series in the period 1935-2009 is not so close to the ergodic limit: empirical averages over subperiods of 30 years exhibit non negligible fluctuations.

It is relevant to show that this is also consistent with our model.

Variability of estimators

These plots show that the DJIA time series in the period 1935-2009 is not so close to the ergodic limit: empirical averages over subperiods of 30 years exhibit non negligible fluctuations.

It is relevant to show that this is also consistent with our model.

We have therefore simulated 75 years of data from our model and evaluated the quantities of interest (multiscaling of moments, decay of correlations and empirical distribution) in different subperiods of 30 years.

Variability of estimators

These plots show that the DJIA time series in the period 1935-2009 is not so close to the ergodic limit: empirical averages over subperiods of 30 years exhibit non negligible fluctuations.

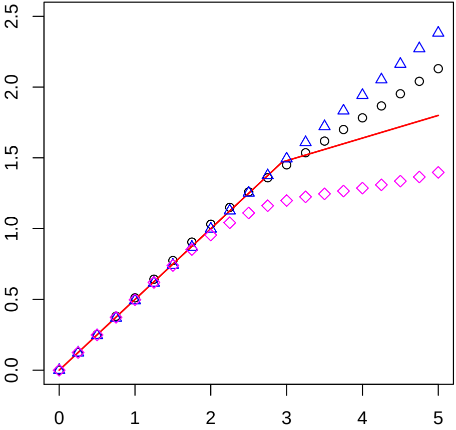
It is relevant to show that this is also consistent with our model.

We have therefore simulated 75 years of data from our model and evaluated the quantities of interest (multiscaling of moments, decay of correlations and empirical distribution) in different subperiods of 30 years.

A significant, comparable variability is present also in our model.

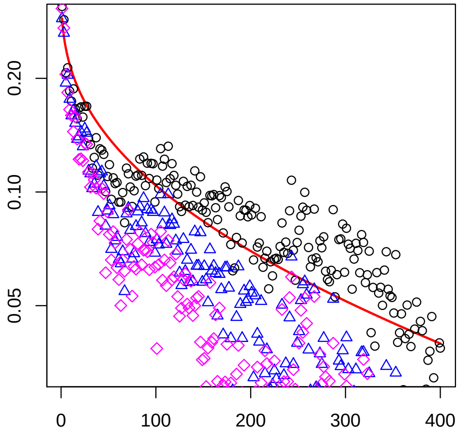
Simulated Data (75 years)

Simulated scaling exponent of our model over sub-periods of 30 years



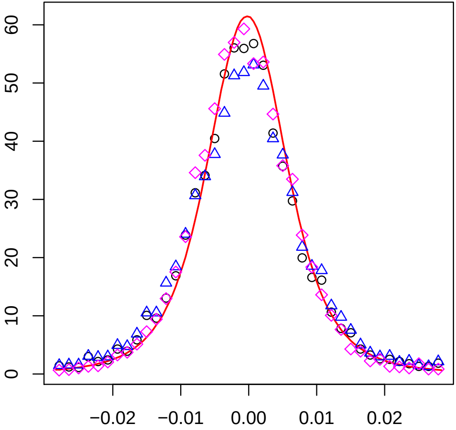
Simulated Data (75 years)

Simulated volatility autocorrelation of our model over sub-periods of 30 years



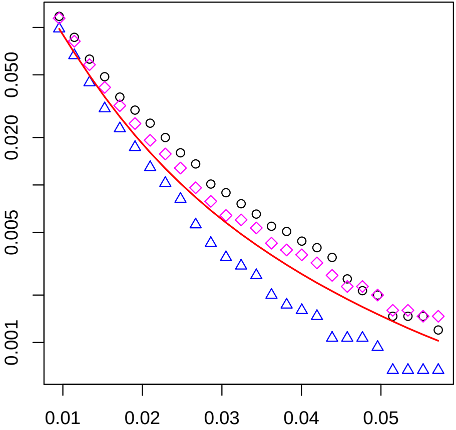
Simulated Data (75 years)

Simulated distribution of our model over sub-periods of 30 years



Simulated Data (75 years)

Simulated tails of our model over sub-periods of 30 years



Baldovin & Stella's Model

Empirically: $$\hat{p}_h(\mathrm{d}r) \simeq \frac{1}{\sqrt{h}}\, g\left(\frac{r}{\sqrt{h}}\right) \mathrm{d}r\,.$$

Assume g is symmetric and let g^* be its Fourier transform.

Baldovin & Stella's Model

Empirically: $$\hat{p}_h(\mathrm{d}r) \simeq \frac{1}{\sqrt{h}}\, g\left(\frac{r}{\sqrt{h}}\right) \mathrm{d}r \,.$$

Assume g is symmetric and let g^* be its Fourier transform.

Let $(Y_t)_{t\geq 0}$ be the process with finite dimensional densities

$$p(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = h\left(\frac{x_1}{\sqrt{t_1}}, \frac{x_2 - x_1}{\sqrt{t_2 - t_1}}, \ldots, \frac{x_n - x_{n-1}}{\sqrt{t_n - t_{n-1}}}\right),$$

Baldovin & Stella's Model

Empirically: $$\hat{p}_h(\mathrm{d}r) \simeq \frac{1}{\sqrt{h}}\, g\left(\frac{r}{\sqrt{h}}\right) \mathrm{d}r\,.$$

Assume g is symmetric and let g^* be its Fourier transform.

Let $(Y_t)_{t\geq 0}$ be the process with finite dimensional densities

$$p(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = h\left(\frac{x_1}{\sqrt{t_1}}, \frac{x_2 - x_1}{\sqrt{t_2 - t_1}}, \ldots, \frac{x_n - x_{n-1}}{\sqrt{t_n - t_{n-1}}}\right),$$

where $h : \mathbb{R}^n \to \mathbb{R}$ has Fourier transform h^* given by

$$h^*(u_1, u_2, \ldots, u_n) := g^*\left(\sqrt{u_1^2 + \ldots + u_n^2}\right).$$

Baldovin & Stella's Model

Empirically: $$\hat{p}_h(\mathrm{d}r) \simeq \frac{1}{\sqrt{h}}\, g\left(\frac{r}{\sqrt{h}}\right) \mathrm{d}r\,.$$

Assume g is symmetric and let g^* be its Fourier transform.

Let $(Y_t)_{t\geq 0}$ be the process with finite dimensional densities

$$p(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = h\left(\frac{x_1}{\sqrt{t_1}}, \frac{x_2 - x_1}{\sqrt{t_2 - t_1}}, \ldots, \frac{x_n - x_{n-1}}{\sqrt{t_n - t_{n-1}}}\right),$$

where $h : \mathbb{R}^n \to \mathbb{R}$ has Fourier transform h^* given by

$$h^*(u_1, u_2, \ldots, u_n) := g^*\left(\sqrt{u_1^2 + \ldots + u_n^2}\right).$$

- If g is standard Gaussian $\rightarrow$ $(Y_t)_{t\geq 0}$ is Brownian motion.

Baldovin & Stella's Model

Empirically: $$\hat{p}_h(\mathrm{d}r) \simeq \frac{1}{\sqrt{h}}\, g\left(\frac{r}{\sqrt{h}}\right) \mathrm{d}r \,.$$

Assume g is symmetric and let g^* be its Fourier transform.

Let $(Y_t)_{t \geq 0}$ be the process with finite dimensional densities

$$p(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = h\left(\frac{x_1}{\sqrt{t_1}}, \frac{x_2 - x_1}{\sqrt{t_2 - t_1}}, \ldots, \frac{x_n - x_{n-1}}{\sqrt{t_n - t_{n-1}}}\right),$$

where $h : \mathbb{R}^n \to \mathbb{R}$ has Fourier transform h^* given by

$$h^*(u_1, u_2, \ldots, u_n) := g^*\left(\sqrt{u_1^2 + \ldots + u_n^2}\right).$$

- If g is standard Gaussian $\to$ $(Y_t)_{t \geq 0}$ is Brownian motion.
- Is the definition well-posed? Conditions on g.

Baldovin & Stella's Model

- The increments of Y have diffusive scaling.
  Their (rescaled) marginal density is $g(\cdot)$.

Baldovin & Stella's Model

- The increments of Y have diffusive scaling.
  Their (rescaled) marginal density is $g(\cdot)$.
- The increments of Y are uncorrelated but not independent.

Baldovin & Stella's Model

- The increments of Y have diffusive scaling.
  Their (rescaled) marginal density is $g(\cdot)$.
- The increments of Y are uncorrelated but not independent.
- However, they are exchangeable: no decay of correlations.

Baldovin & Stella's Model

- The increments of Y have diffusive scaling.
  Their (rescaled) marginal density is $g(\cdot)$.
- The increments of Y are uncorrelated but not independent.
- However, they are exchangeable: no decay of correlations.

  By De Finetti's theorem in continuous time [Freedman 1963] the process $(Y_t)_{t \geq 0}$ is a mixture of Brownian motions:

$$Y_t = \sigma \, W_t$$

  where σ is random and independent of the BM $(W_t)_{t \geq 0}$.

Baldovin & Stella's Model

- The increments of Y have diffusive scaling.
  Their (rescaled) marginal density is $g(\cdot)$.
- The increments of Y are uncorrelated but not independent.
- However, they are exchangeable: no decay of correlations.

  By De Finetti's theorem in continuous time [Freedman 1963] the process $(Y_t)_{t\geq 0}$ is a mixture of Brownian motions:

$$Y_t = \sigma \, W_t$$

  where σ is random and independent of the BM $(W_t)_{t\geq 0}$.

  A sample path of $(Y_t)_{t\geq 0}$ cannot be distinguished from a sample path of a BM with constant volatility: no ergodicity.

Baldovin & Stella's Model

- The increments of Y have diffusive scaling.
  Their (rescaled) marginal density is $g(\cdot)$.
- The increments of Y are uncorrelated but not independent.
- However, they are exchangeable: no decay of correlations.

  By De Finetti's theorem in continuous time [Freedman 1963] the process $(Y_t)_{t \geq 0}$ is a mixture of Brownian motions:

$$Y_t = \sigma \, W_t$$

  where σ is random and independent of the BM $(W_t)_{t \geq 0}$.

  A sample path of $(Y_t)_{t \geq 0}$ cannot be distinguished from a sample path of a BM with constant volatility: no ergodicity.

Apart from this issue, there is still no multiscaling of moments.
This is solved introducing a time inhomogeneity in the model.

Baldovin & Stella's Model

Fix a (periodic) sequence of epochs $0 < \tau_1 < \tau_2 < \cdots < \tau_n \uparrow +\infty$ and a parameter $0 < D \leq 1/2$. Define a new process $(X_t)_{t \geq 0}$ by

$$X_t := Y_{t^{2D}} \qquad \text{for } t \in [0, \tau_1)\,,$$

$$X_t := Y_{(t-\tau_n)^{2D} + \sum_{k=1}^{n} (\tau_k - \tau_{k-1})^{2D}} \qquad \text{for } t \in [\tau_n, \tau_{n+1})\,.$$

Baldovin & Stella's Model

Fix a (periodic) sequence of epochs $0 < \tau_1 < \tau_2 < \cdots < \tau_n \uparrow +\infty$ and a parameter $0 < D \leq 1/2$. Define a new process $(X_t)_{t \geq 0}$ by

$$X_t := Y_{t^{2D}} \qquad \text{for } t \in [0, \tau_1)\,,$$

$$X_t := Y_{(t-\tau_n)^{2D} + \sum_{k=1}^{n} (\tau_k - \tau_{k-1})^{2D}} \qquad \text{for } t \in [\tau_n, \tau_{n+1})\,.$$

- For $D = 1/2$ we have the old model $X_t \equiv Y_t$.

Baldovin & Stella's Model

Fix a (periodic) sequence of epochs $0 < \tau_1 < \tau_2 < \cdots < \tau_n \uparrow +\infty$ and a parameter $0 < D \leq 1/2$. Define a new process $(X_t)_{t \geq 0}$ by

$$X_t := Y_{t^{2D}} \qquad \text{for } t \in [0, \tau_1) \, ,$$

$$X_t := Y_{(t-\tau_n)^{2D} + \sum_{k=1}^{n} (\tau_k - \tau_{k-1})^{2D}} \qquad \text{for } t \in [\tau_n, \tau_{n+1}) \, .$$

- For $D = 1/2$ we have the old model $X_t \equiv Y_t$.
- For $D < 1/2$, the process $(X_t)_{t \geq 0}$ is obtained from $(Y_t)_{t \geq 0}$ by a nonlinear time-change, refreshed at each time τ_n.

Baldovin & Stella's Model

Fix a (periodic) sequence of epochs $0 < \tau_1 < \tau_2 < \cdots < \tau_n \uparrow +\infty$ and a parameter $0 < D \leq 1/2$. Define a new process $(X_t)_{t \geq 0}$ by

$$X_t := Y_{t^{2D}} \qquad \text{for } t \in [0, \tau_1)\,,$$

$$X_t := Y_{(t-\tau_n)^{2D} + \sum_{k=1}^{n} (\tau_k - \tau_{k-1})^{2D}} \qquad \text{for } t \in [\tau_n, \tau_{n+1})\,.$$

- For $D = 1/2$ we have the old model $X_t \equiv Y_t$.
- For $D < 1/2$, the process $(X_t)_{t \geq 0}$ is obtained from $(Y_t)_{t \geq 0}$ by a nonlinear time-change, refreshed at each time τ_n.
- Increments are amplified immediately after the times $(\tau_n)_{n \geq 1}$ and then progressively damped out.

Baldovin & Stella's Model

Fix a (periodic) sequence of epochs $0 < \tau_1 < \tau_2 < \cdots < \tau_n \uparrow +\infty$ and a parameter $0 < D \leq 1/2$. Define a new process $(X_t)_{t \geq 0}$ by

$$X_t := Y_{t^{2D}} \qquad \text{for } t \in [0, \tau_1) \,,$$

$$X_t := Y_{(t-\tau_n)^{2D} + \sum_{k=1}^{n} (\tau_k - \tau_{k-1})^{2D}} \qquad \text{for } t \in [\tau_n, \tau_{n+1}) \,.$$

- For $D = 1/2$ we have the old model $X_t \equiv Y_t$.
- For $D < 1/2$, the process $(X_t)_{t \geq 0}$ is obtained from $(Y_t)_{t \geq 0}$ by a nonlinear time-change, refreshed at each time τ_n.
- Increments are amplified immediately after the times $(\tau_n)_{n \geq 1}$ and then progressively damped out.
- Interpretation: $(\tau_n)_{n \geq 1}$ linked to "shocks" in the market.

Baldovin & Stella's Model

Despite the time-change, the process $(X_t)_{t \geq 0}$ remains not ergodic.

Baldovin & Stella's Model

Despite the time-change, the process $(X_t)_{t\geq 0}$ remains not ergodic.

However, Baldovin & Stella show by simulations that this model (with $(\tau_n)_n$ a periodic sequence) fits all mentioned stylized facts.

Baldovin & Stella's Model

Despite the time-change, the process $(X_t)_{t\geq 0}$ remains not ergodic.

However, Baldovin & Stella show by simulations that this model (with $(\tau_n)_n$ a periodic sequence) fits all mentioned stylized facts.

They actually simulate a different model: an autoregressive version of $(X_t)_{t\geq 0}$.

Baldovin & Stella's Model

Despite the time-change, the process $(X_t)_{t\geq 0}$ remains not ergodic.

However, Baldovin & Stella show by simulations that this model (with $(\tau_n)_n$ a periodic sequence) fits all mentioned stylized facts.

They actually simulate a different model: an autoregressive version of $(X_t)_{t\geq 0}$.

Other issue: the density of Y_1 is $g(\cdot)$ by construction. However, the density of X_1 is not $g(\cdot)$ and depends on the choice of $(\tau_n)_n$.

Baldovin & Stella's Model

Despite the time-change, the process $(X_t)_{t \geq 0}$ remains not ergodic.

However, Baldovin & Stella show by simulations that this model (with $(\tau_n)_n$ a periodic sequence) fits all mentioned stylized facts.

They actually simulate a different model: an autoregressive version of $(X_t)_{t \geq 0}$.

Other issue: the density of Y_1 is $g(\cdot)$ by construction. However, the density of X_1 is not $g(\cdot)$ and depends on the choice of $(\tau_n)_n$.

Our aims

- Define a simple model capturing the essential features of Baldovin & Stella's construction.

Baldovin & Stella's Model

Despite the time-change, the process $(X_t)_{t \geq 0}$ remains not ergodic.

However, Baldovin & Stella show by simulations that this model (with $(\tau_n)_n$ a periodic sequence) fits all mentioned stylized facts.

They actually simulate a different model: an autoregressive version of $(X_t)_{t \geq 0}$.

Other issue: the density of Y_1 is $g(\cdot)$ by construction. However, the density of X_1 is not $g(\cdot)$ and depends on the choice of $(\tau_n)_n$.

Our aims

- Define a simple model capturing the essential features of Baldovin & Stella's construction.
- Easy to describe and to simulate.

Baldovin & Stella's Model

Despite the time-change, the process $(X_t)_{t \geq 0}$ remains not ergodic.

However, Baldovin & Stella show by simulations that this model (with $(\tau_n)_n$ a periodic sequence) fits all mentioned stylized facts.

They actually simulate a different model: an autoregressive version of $(X_t)_{t \geq 0}$.

Other issue: the density of Y_1 is $g(\cdot)$ by construction. However, the density of X_1 is not $g(\cdot)$ and depends on the choice of $(\tau_n)_n$.

Our aims

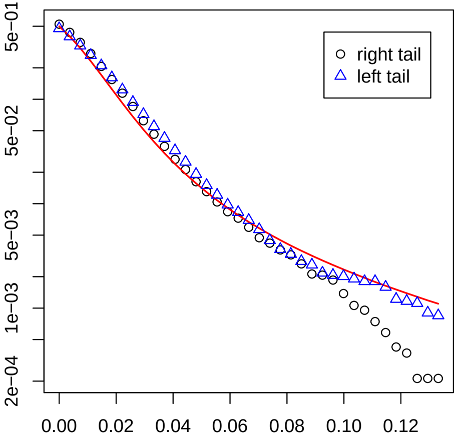
- Define a simple model capturing the essential features of Baldovin & Stella's construction.
- Easy to describe and to simulate.
- Rigorous proofs of the mentioned stylized facts.

Other observables

Is everything going as expected?

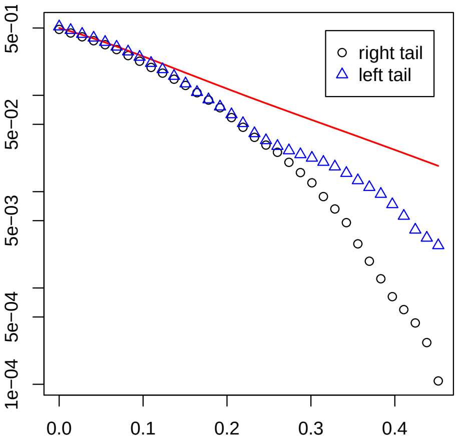
DJIA Time Series (1935-2009)

Empirical and theoretical tails of 5-day log return [log plot]



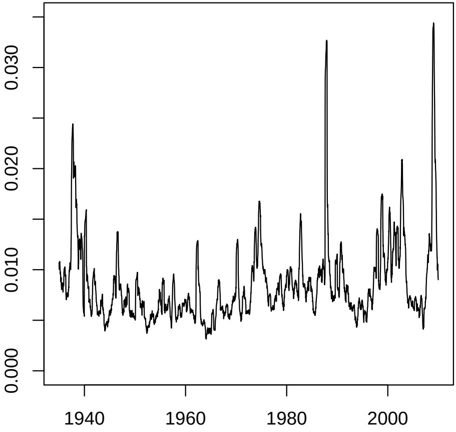
DJIA Time Series (1935-2009)

Empirical and theoretical tails of 400-day log return [log plot]



DJIA Time Series (1935-2009)

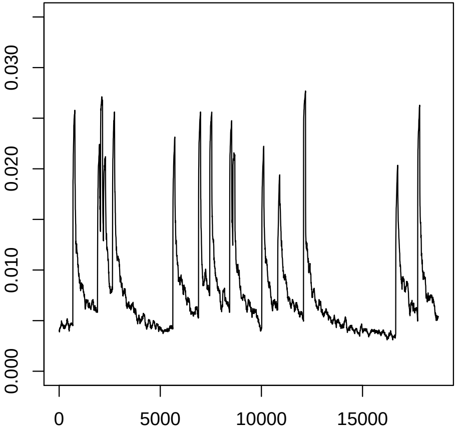
Empirical volatility



Local standard deviation of log-returns in a window of 100 days

Simulated Data (75 years)

Empirical volatility



Local standard deviation of log-returns in a window of 100 days