Hairer's Reconstruction Theorem without Regularity Structures

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0. Summary

- Stochastic analysis & the Sewing Lemma
- Hairer's Reconstruction Theorem

  (elementary approach, optimal assumptions)

- Examples and applications

1. THE SEWING LEMMA

Fix two continuous but non smooth functions

$$X, Y : [0,1] \to \mathbb{R} \qquad \text{(possibly random)}$$

How to define the product $f(t) =$ " $X(t)\,\dot{Y}(t)$ " ?

Define its integral $I(t) =$ " $\int_0^t X(s)\,\dot{Y}(s)\,ds$ "

Key problem in stochastic analysis (Y = Brownian motion)

Robust definition of $I(\cdot)$?

Local description

$$\underbrace{\int_s^t X(u)\,\dot{Y}(u)\,du}_{} = \underbrace{X(s)\big(Y(t)-Y(s)\big)}_{} + o(|t-s|) \qquad \text{(if smooth)}$$

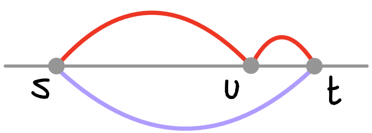
$$I(t) - I(s) = \boxed{A(s,t)} + o(|t-s|) \qquad \textbf{(1)}$$

"GERM"

Sewing Lemma [Gubinelli 04] [Feyel, de La Pradelle 06]

Equation **(1)** has a solution (unique!) if, for some $\gamma > 0$,

$$|A(s,t) - A(s,u) - A(u,t)| \lesssim |t-s|^{1+\gamma} \qquad (s \leq u \leq t)$$



$A(\cdot,\cdot)$ is called coherent germ

- Basic germ $\quad A(s,t) = X(s)\left(Y(t) - Y(s)\right)$

$$\left|A(s,t) - A(s,u) - A(u,t)\right| = \left|\left(X(u) - X(s)\right)\left(Y(t) - Y(u)\right)\right|$$

- Assume $X \in \mathcal{C}^{\alpha}$, $Y \in \mathcal{C}^{\beta}$: $\quad |u-s|^{\alpha}\,|t-u|^{\beta} \lesssim |t-s|^{\alpha+\beta}$

coherent

[Young 1936] $\quad I(t) =$ "$\int_0^t X(u)\,\dot{Y}(u)\,du$" canonical for $\alpha + \beta > 1$

- $Y = B =$ Brownian motion $\quad \beta = \frac{1}{2} - \Rightarrow$ Young requires $\alpha > \frac{1}{2}$

This rules out most interesting cases!

- SDE driven by Brownian motion (σ smooth)

$$\text{“}\ \dot{X}(t) = \sigma(X(t))\,\dot{B}(t)\ \text{”}$$

Expect $X \in C^\alpha$ with $\alpha = \beta = \frac{1}{2}-$ $\Rightarrow$ $\alpha + \beta < 1$

Basic germ $A(s,t) = X(s)\,(B(t)-B(s))$ not coherent

- Enriched germ coherent! (Rough paths...)

$$\tilde{A}(s,t) = X(s)\,(B(t)-B(s)) + \underbrace{X'(s)}_{\sigma'(X(s))\,\sigma(X(s))}\ \underbrace{\boxed{\text{“}\int_s^t (B(u)-B(s))\,\dot{B}(u)\,du\text{”}}}_{\text{Ito, Stratonovich, ...}}$$

2. DISTRIBUTIONS

$d > 1$ $\left\{ \begin{array}{lcl} \text{Functions} & \rightsquigarrow & \text{Distributions} \\ \text{Sewing Lemma} & \rightsquigarrow & \text{Reconstruction Theorem} \end{array} \right.$

$$I(t) - I(s) = A(s,t) + o(|t-s|) \qquad (1)$$

- Local approximation of $I(t) - I(s)$ is the germ $A(s,t)$

- Local approximation of $\boxed{f(t) := \dot{I}(t)}$ is $\boxed{F_s(t) := \frac{\partial}{\partial t} A(s,t)}$

$$\langle f, \mathbb{1}_{[s,t)} \rangle = \langle F_s, \mathbb{1}_{[s,t)} \rangle + o(|t-s|) \qquad (1')$$

distributions

$$\mathcal{D} := \{ \varphi : \mathbb{R}^d \to \mathbb{R} \,,\ \varphi \in C_c^\infty \} = \text{test functions}$$

$$\mathcal{D}' := \{ f : \mathcal{D} \to \mathbb{R} \text{ linear \& continuous} \} = \text{distributions}$$

$\forall K \subseteq \mathbb{R}^d$ compact $\quad \exists\, r = r_K \in \mathbb{N} \quad \forall \varphi \in \mathcal{D} : \operatorname{supp}(\varphi) \subseteq K$

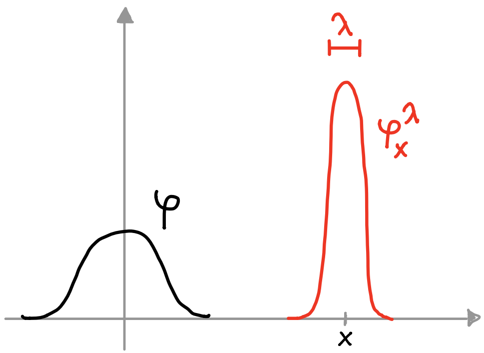
$$|\langle f, \varphi \rangle| \lesssim \|\varphi\|_{C^r} := \max_{|k| \leq r} \|\partial^k \varphi\|_\infty$$

Definition. A GERM $(F_x)_{x \in \mathbb{R}^d}$ is a family of distributions.

Problem. Given a germ $(F_x)_{x \in \mathbb{R}^d}$, can we find a distribution $f \in \mathcal{D}'$ "locally approximated by F_x" ?

What do we mean by "locally well approximated by F_x" ?

Fix a test function $\varphi \in \mathcal{D}$ with $\int \varphi \neq 0$



Rescaled test function

$$\varphi_x^\lambda(z) := \frac{1}{\lambda^d} \varphi\left(\frac{z-x}{\lambda}\right)$$

We can require the same as (1'): $\quad (\varphi = \mathbb{1}_{[0,1)})$

$$\langle f, \varphi_x^\lambda \rangle = \langle F_x, \varphi_x^\lambda \rangle + o(1) \qquad \text{as } \lambda \downarrow 0$$

(unif. for x in compact sets)

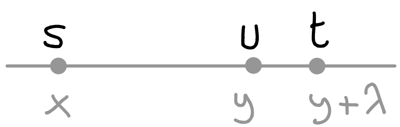
We require slightly more : $o(1) \rightsquigarrow O(\lambda^\gamma)$

Problem. Given a germ $(F_x)_{x\in\mathbb{R}^d}$, find $f \in \mathcal{D}'$ such that

$$\left| \langle f - F_x , \varphi_x^\lambda \rangle \right| \lesssim \lambda^{\gamma} \qquad (2)$$

for some $\varphi \in \mathcal{D}$ and $\gamma > 0$ (unif. $\lambda \in (0,1]$, x in compact)

Uniqueness is for free. Existence requires coherence!



$$\left| A(s,t) - A(s,u) - A(u,t) \right| \lesssim (t-s)^{1+\gamma}$$

$$\Downarrow$$

$$\left| \langle F_x - F_y , \varphi_x^\lambda \rangle \right| \lesssim \lambda^{-1} (|y-x| + \lambda)^{1+\gamma}$$

3. Coherence

Definition. Fix $\gamma \in \mathbb{R}$. A germ $(F_x)_{x\in\mathbb{R}^d}$ is γ-coherent if there is one test function $\varphi \in \mathcal{D}$ with $\int \varphi \neq 0$ s.t.

$$\forall K \subseteq \mathbb{R}^d \text{ compact: } \begin{cases} |\langle F_x - F_y, \varphi_y^\lambda \rangle| \lesssim \lambda^\alpha (|x-y|+\lambda)^{\gamma-\alpha} \\ \text{unif. for } x,y \in K \text{ and } \lambda \in (0,1] \end{cases}$$

for some $\alpha = \alpha_K \leq 0$ such that $\alpha \leq \gamma$.

If $\alpha_K \equiv \alpha$ $(\forall K)$ then $(F_x)_{x\in\mathbb{R}^d}$ is called (α,γ)-coherent.

Coherence: F_x and F_y are "close" when x and y are close

Example If an "old" germ $(A(s,t))_{s,t\in[0,1]}$ is γ-coherent

$$\left(|A(s,t) - A(s,u) - A(u,t)| \lesssim (t-s)^{1+\gamma} \right)$$

the "new" germ $(F_s := \partial_t A(s,t))_{s\in[0,1]}$ is $(-1,\gamma)$-coherent

Example Fix a function $f \in C^\gamma$ with $\gamma \in (0,\infty)$.

The (degree $\lfloor\gamma\rfloor$) Taylor polynomial of f is $(0,\gamma)$-coherent

$$F_x(\cdot) = \sum_{|k|<\gamma} \partial^k f(x) \frac{(\cdot - x)^k}{k!} \qquad |F_x(z) - F_y(z)| \lesssim (|x-y| + \underbrace{|z-y|}_{\lambda})^\gamma$$

4. The Reconstruction Theorem

Theorem. Let $F = (F_x)_{x \in \mathbb{R}^d}$ be a γ-coherent ($\gamma \in \mathbb{R}$).

Then **(2)** has a solution: there is $f \in \mathcal{D}'$ s.t.

$\forall K \subseteq \mathbb{R}^d$ compact, $\forall \psi \in \mathcal{D}(B(0,1))$:

$$|\langle f - F_x , \psi_x^\lambda \rangle| \leq C \cdot \begin{cases} \lambda^\gamma & \text{if } \gamma \neq 0 \\ (1 + |\log \lambda|) & \text{if } \gamma = 0 \end{cases}$$

unif. for $x \in K$, $\lambda \in (0,1]$

where $C = C(K, F, \psi) \lesssim \|\psi\|_{C^r}$ for a suitable r (*)

If $\gamma > 0$, then $f = \mathcal{R}F$ is unique and linear.

- Coherence is necessary for the Reconstruction Theorem

- For $\gamma \leq 0$ no uniqueness, linearity ok

  (For $\gamma = 0$ the bound $|\log \lambda|$ is optimal)

  Coherent germs form a vector space $(\varphi \rightsquigarrow \text{any } \psi)$

- Local homogeneity of a germ: for suitable $\beta = \beta_K$

  $$|\langle F_x, \varphi_x^\lambda \rangle| \lesssim \lambda^\beta \qquad \text{unif. for } x \in K, \ \lambda \in (0,1]$$

- τ in $(*)$ satisfies $\tau > \max\{-\alpha_{\overline{K}_2}, -\beta_{\overline{K}_2}\}$

5. Regularity Structures

Consider a singular stochastic PDE on $\mathbb{R}^d$ ($\rightsquigarrow$ Lorenzo)

In RS the (distributional) solution $h(t,x)$ of the PDE is represented by a coherent germ $H = (H_{t,x})$

$(\Pi_{t,x})_{t \geq 0, x \in \mathbb{R}^d}$ "model", $\quad h \in \mathcal{D}^\gamma$ "modelled distribution"

$\rightsquigarrow$ γ-coherent germ $H = \left(H_{t,x} := \Pi_{t,x} h(t,x) \right)_{t \geq 0, x \in \mathbb{R}^d}$

If $\gamma > 0$, the solution is $\boxed{h(t,x) = \mathcal{R} H}$ (reconstruction of H)

The germs arising in RS have a graded structure

$$F_x(\cdot) = \sum_{a \in A} c^a(x)\, \Pi_x^a(\cdot)$$

Finite set local "basis" of distributions

This structure allows to define e.g. "products"

Another key ingredient of RS is Schauder estimates

Coherence in RS:

$$|\langle F_x - F_y, \varphi_y^\lambda \rangle| \lesssim \sum_{a \in A,\, a < \gamma} \lambda^a |x-y|^{\gamma - a} \lesssim \lambda^\alpha (|x-y| + \lambda)^{\gamma - \alpha}$$

$\alpha := \min A$

Thanks!