On the 2d KPZ Equation

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Half day in Stochastic Analysis and applications

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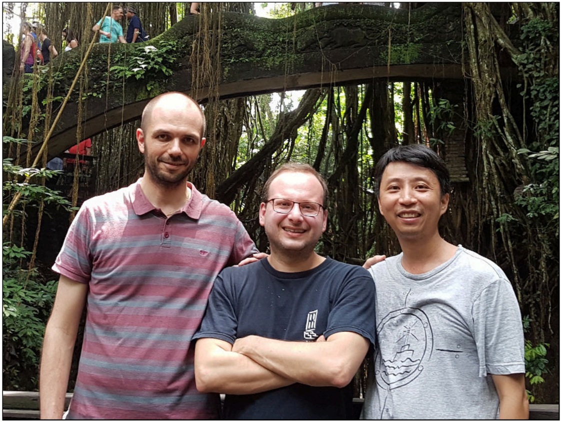
Collaborators



Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

Overview

This talk is about a stochastic PDE on $\mathbb{R}^d$: (mainly $d = 2$)

- the Kardar-Parisi-Zhang Equation (KPZ)

Very interesting, yet ill-defined object

Plan:

1. Consider a regularized version of the equation
2. Study the limit of the solution, when regularisation is removed

Stochastic Analysis $\leftrightsquigarrow$ Statistical Mechanics

White noise

Space-time white noise $\xi = \xi(t,x)$ on $\mathbb{R}^{1+d}$

Random distribution of negative order (Schwartz) [not a function!]

Gaussian: $$\langle \phi, \xi \rangle = \int_{\mathbb{R}^{1+d}} \phi(t,x)\, \xi(t,x)\, \mathrm{d}t\, \mathrm{d}x \sim \mathcal{N}(0, \|\phi\|_{L^2}^2)$$

$$\mathrm{Cov}[\,\xi(t,x), \xi(t',x')\,] = \delta(t-t')\,\delta(x-x')$$

Case $d=0$: $\xi(t) = \frac{\mathrm{d}}{\mathrm{d}t} B(t)$ where (B_t) is Brownian motion

The KPZ equation

KPZ [Kardar Parisi Zhang 86]

$$\partial_t h = \frac{1}{2}\Delta_x h + \frac{1}{2}|\nabla_x h|^2 + \beta\,\xi \qquad \text{(KPZ)}$$

Model for random interface growth

$h = h(t,x) =$ interface height at time $t \geq 0$, space $x \in \mathbb{R}^d$

$\xi = \xi(t,x) =$ space-time white noise $\qquad \beta > 0$ noise strength

$|\nabla_x h|^2$ ill-defined

For smooth ξ

$$u(t,x) := e^{h(t,x)} \qquad \text{(Cole-Hopf)}$$

The multiplicative Stochastic Heat Equation (SHE)

SHE $(t > 0, \ x \in \mathbb{R}^d)$

$$\partial_t u = \tfrac{1}{2} \Delta_x u + \beta \, u \, \xi \qquad \text{(SHE)}$$

Product $u \, \xi$ ill-defined

$(d = 1)$ SHE is well-posed by Ito integration [Walsh 80's]

$u(t, x)$ is a function $\rightsquigarrow$ "KPZ solution" $h(t, x) := \log u(t, x)$

$(d = 1)$ SHE and KPZ well-understood in a robust sense ("pathwise")

Regularity Structures (Hairer)
Paracontrolled Distributions (Gubinelli, Imkeller, Perkowski)
Energy Solutions (Goncalves, Jara), Renormalization (Kupiainen)

Higher dimensions $d \geq 2$

In dimensions $d \geq 2$ there is no general theory

We mollify the white noise $\xi(t,x)$ in space on scale $\varepsilon > 0$

$$\xi^{\varepsilon}(t,\cdot) := \xi(t,\cdot) * \varrho_{\varepsilon}$$

Solutions $h^{\varepsilon}(t,x)$, $u^{\varepsilon}(t,x)$ are well-defined. Convergence as $\varepsilon \downarrow 0$?

We need to tune disorder strength $\beta = \beta_{\varepsilon}$

$$\beta_{\varepsilon} = \begin{cases} \hat{\beta}\, \dfrac{1}{\sqrt{|\log \varepsilon|}} & (d = 2) \\ \hat{\beta}\, \varepsilon^{\frac{d-2}{2}} & (d \geq 3) \end{cases} \qquad \hat{\beta} \in (0, \infty)$$

Back to KPZ

We now plug $\xi \rightsquigarrow \xi^\varepsilon$ and $\beta \rightsquigarrow \beta_\varepsilon$ into KPZ

We also subtract a diverging constant ("Renormalization")

Renormalized and Mollified KPZ

$$\begin{cases} \partial_t h^\varepsilon = \frac{1}{2}\Delta h^\varepsilon + \frac{1}{2}|\nabla h^\varepsilon|^2 + \beta_\varepsilon \, \xi^\varepsilon - c \, \beta_\varepsilon^2 \, \varepsilon^{-d} \\ h^\varepsilon(0,\cdot) \equiv 0 \end{cases} \qquad (\varepsilon\text{-KPZ})$$

We present some convergence results for $h^\varepsilon(t,x)$ as $\varepsilon \downarrow 0$

Without renormalization, the solution $h^\varepsilon(t,x)$ does not converge!

Main results

Space dimension $d = 2$ $\beta_\varepsilon = \frac{\hat{\beta}}{\sqrt{|\log \varepsilon|}}$ $\hat{\beta} \in (0, \infty)$

I. Phase transition [CSZ 17]

KPZ solution $h^\varepsilon(t, x)$ undergoes a phase transition at $\hat{\beta}_c = \sqrt{2\pi}$

II. Sub-critical regime [CSZ 17] [CSZ 18b]

For all $\hat{\beta} < \hat{\beta}_c$: convergence of $h^\varepsilon(t, x)$ as $\varepsilon \downarrow 0$ (LLN + CLT)

Analogous results for the SHE solution $u^\varepsilon(t, x)$

Critical regime $\hat{\beta} = \hat{\beta}_c$? Recent progress for SHE (nothing for KPZ)

Main result I. Phase transition

Space dimension $d = 2$ $\qquad \beta_\varepsilon = \frac{\hat{\beta}}{\sqrt{|\log \varepsilon|}}$ $\qquad \hat{\beta} \in (0, \infty)$

Theorem (Phase transition for 2d KPZ) [CSZ 17]

- For $\hat{\beta} < \sqrt{2\pi}$ $\qquad h^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} \sigma Z - \frac{1}{2}\sigma^2$

  $$Z \sim N(0,1) \qquad \sigma^2 := \log \frac{2\pi}{2\pi - \hat{\beta}^2}$$

  $h^\varepsilon(t, x_i) \xrightarrow[\varepsilon \downarrow 0]{d}$ asympt. independent (for distinct points x_i's)

- For $\hat{\beta} \geq \sqrt{2\pi}$ $\qquad h^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} -\infty$

Law of large numbers

Consider the sub-critical regime $\hat{\beta} < \sqrt{2\pi}$

- $\mathbb{E}[h^\varepsilon(t,x)] = -\frac{1}{2}\sigma^2 + o(1)$
- $h^\varepsilon(t,x)$ asymptotically independent for distinct x's

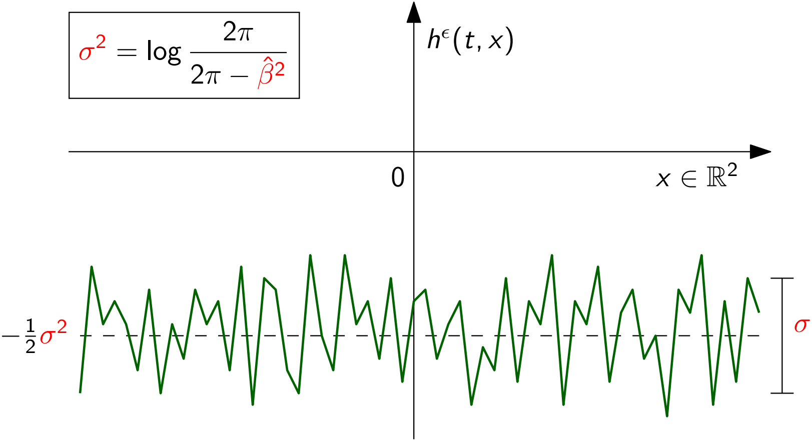
Corollary: LLN $(\hat{\beta} < \sqrt{2\pi})$

$$h^\varepsilon(t,\cdot) \xrightarrow[\varepsilon\downarrow 0]{d} -\frac{1}{2}\,\sigma^2 \qquad \text{as a distribution on } \mathbb{R}^2$$

$$\int_{\mathbb{R}^2} h^\varepsilon(t,x)\,\phi(x)\,\mathrm{d}x \xrightarrow[\varepsilon\downarrow 0]{d} -\frac{1}{2}\,\sigma^2 \int_{\mathbb{R}^2} \phi(x)\,\mathrm{d}x$$

A picture

$$\sigma^2 = \log \frac{2\pi}{2\pi - \hat{\beta}^2}$$



Main result II. Fluctuations

Rescale $\mathcal{H}^\varepsilon(t,x) := \big(h^\varepsilon(t,x) - \mathbb{E}[h^\varepsilon]\big)/\beta_\varepsilon$

Theorem (Sub-critical Fluctuations for 2d KPZ) [CSZ 18b]

for $\hat\beta < \sqrt{2\pi}$ $\qquad \mathcal{H}^\varepsilon(t,\cdot) \xrightarrow[\varepsilon\downarrow 0]{d} v(t,\cdot)$ $\qquad$ as a distrib.

v = Gaussian = solution of Edwards-Wilkinson equation

$$\partial_t v = \tfrac{1}{2}\Delta_x v + \gamma\,\xi \qquad \text{where} \qquad \gamma = \sqrt{\tfrac{2\pi}{2\pi - \hat\beta^2}} > 1$$

$$\partial_t \mathcal{H}^\varepsilon = \tfrac{1}{2}\,\Delta_x\,\mathcal{H}^\varepsilon + \xi^\varepsilon + \big\{\beta_\varepsilon\,|\nabla_x \mathcal{H}^\varepsilon|^2 - c\,\beta_\varepsilon\,\varepsilon^{-2}\big\}$$

Last term $\{\ldots\}$ produces "extra" white noise! (Independent of ξ)

Other works

Alternative proof by [Gu 18] via Malliavin calculus (only for small $\hat{\beta}$)

[Chatterjee and Dunlap 18] first considered fluctuations for 2d KPZ

They proved tightness of $\mathcal{H}^{\varepsilon}$ (only for small $\hat{\beta}$)

We identify the limit (EW) in the entire sub-critical regime $\hat{\beta} < \sqrt{2\pi}$

Results in dimensions $d \geq 3$ by many authors

References

With Rongfeng Sun and Nikos Zygouras:

- [CSZ 17] *Universality in marginally relevant disordered systems* Ann. Appl. Probab. 2017
- [CSZ 18a] *On the moments of the (2+1)-dimensional directed polymer and Stochastic Heat Equation in the critical window* Commun. Math. Phys. (to appear)
- [CSZ 18b] *The two-dimensional KPZ equation in the entire subcritical regime* Ann. Probab. (to appear)

$(d = 2)$ [Bertini Cancrini 98]

[Chatterjee Dunlap 18] [Gu 18] [Gu Quastel Tsai 19]

$(d \geq 3)$ [Magnen Unterberger 18] [Gu Ryzhik Zeitouni 18]

[Dunlap Gu Ryzhik Zeitouni 19] [Comets Cosco Mukherjee 18 19a 19b]

Renormalized KPZ vs. SHE

Renormalized and Mollified KPZ

$$\begin{cases} \partial_t h^\varepsilon = \frac{1}{2}\Delta h^\varepsilon + \frac{1}{2}|\nabla h^\varepsilon|^2 + \beta_\varepsilon \, \xi^\varepsilon - c\, \beta_\varepsilon^2 \, \varepsilon^{-d} \\ h^\varepsilon(0,\cdot) \equiv 0 \end{cases} \qquad (\varepsilon\text{-KPZ})$$

We can write $h^\varepsilon(t,x) =: \log u^\varepsilon(t,x)$ and apply Ito's formula

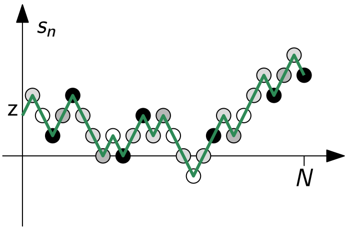
Mollified SHE

$$\begin{cases} \partial_t u^\varepsilon = \frac{1}{2}\Delta u^\varepsilon + \beta_\varepsilon \, u^\varepsilon \, \xi^\varepsilon \\ u^\varepsilon(0,\cdot) \equiv 1 \end{cases} \qquad (\varepsilon\text{-SHE})$$

Facts: $u^\varepsilon(t,x) > 0$ and $\mathbb{E}[u^\varepsilon(t,x)] \equiv 1 \quad \rightsquigarrow \quad \exists$ subseq. limits

Directed Polymers

We can study the SHE solution $u^\varepsilon(t,x)$ via Directed Polymers



- $s = (s_n)_{n\geq 0}$ simple random walk path
- Indep. standard Gaussian RVs $\omega(n,x)$ (Disorder)
- $H_N(\omega, s) := \sum_{n=1}^{N} \omega(n, s_n)$

Directed Polymer Partition Functions $(N \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$\mathscr{Z}_\beta(N,z) := \frac{1}{(2d)^N} \sum_{\substack{s=(s_0,\ldots,s_N) \\ \text{s.r.w. path with } s_0=z}} e^{\beta H_N(\omega,s) - \frac{1}{2}\beta^2 N}$$

Directed Polymers and KPZ

Partition functions $\mathscr{Z}_\beta(N,z)$ are discrete analogues of $u^\varepsilon(t,x)$ (SHE)

- Similar to Feynman-Kac formula for SHE
- They solve a lattice version of the SHE

Theorem

We can approximate (in L^2)

$$u^\varepsilon(t,x) \approx \mathscr{Z}_\beta(N,z) \qquad \text{and} \qquad h^\varepsilon(t,x) \approx \log \mathscr{Z}_\beta(N,z)$$

where we set $$N = \varepsilon^{-2} t\,, \qquad z = \varepsilon^{-1} x\,, \qquad \beta_\varepsilon = \varepsilon^{\frac{d-2}{2}}\,\beta$$

Our results are first proved for partition functions $\mathscr{Z}_\beta(N,z)$

In conclusion

We study KPZ using partition functions of Directed Polymers

We use tools from "discrete stochastic analysis"

- Polynomial chaos (analogous to Wiener chaos)
- 4th Moment Theorems to prove Gaussianity
- Hypercontractivity + Concentration of Measure

together with "classical" probabilistic techniques, esp. Renewal Theory

Robustness + Universality

Next challenges

- Critical regime $\hat{\beta} = \sqrt{2\pi}$
- Robust (pathwise) analysis of sub-critical regime $\hat{\beta} < \sqrt{2\pi}$

Thanks.

Renormalization of KPZ

We have considered the Renormalized Mollified KPZ

$$\partial_t h^\varepsilon = \tfrac{1}{2}\Delta h^\varepsilon + \tfrac{1}{2}|\nabla h^\varepsilon|^2 + \beta_\varepsilon \, \xi^\varepsilon - c\, \beta_\varepsilon^2 \, \varepsilon^{-2}$$

As $\varepsilon \downarrow 0$ we formally obtain (!)

$$\partial_t h = \tfrac{1}{2}\Delta h + \tfrac{1}{2}|\nabla h|^2 + 0\,\xi - \infty$$

Are we entitled to "change the equation"? We started from (ill-posed)

$$\partial_t h = \tfrac{1}{2}\Delta h + \tfrac{1}{2}|\nabla h|^2 + \xi$$

For smooth ξ we can look at a family of equations $(A, B \in \mathbb{R})$

$$\partial_t h = \tfrac{1}{2}\Delta h + \tfrac{1}{2}|\nabla h|^2 + A\,\xi + B$$

Renormalization = appropriate "reference frame" A_ε, B_ε for ξ^ε

Feynman-Kac for SHE

Recall the mollified SHE
$$\begin{cases} \partial_t u^\varepsilon = \frac{1}{2}\Delta u^\varepsilon + \beta_\varepsilon\, u^\varepsilon\, \xi^\varepsilon \\ u^\varepsilon(0,\cdot) \equiv 1 \end{cases}$$

A stochastic Feynman-Kac formula holds

$$u^\varepsilon(t,x) \overset{d}{=} \mathrm{E}_{\varepsilon^{-1}x}\left[\exp\left(\beta_\varepsilon\, \varepsilon^{-\frac{d-2}{2}} \int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^2} \varrho(B_s - y)\,\xi(\mathrm{d}s,\mathrm{d}y) - \text{q.v.}\right)\right]$$

where $\varrho \in C_c^\infty(\mathbb{R}^d)$ is the mollifier and $B = (B_s)_{s\geq 0}$ is Brownian motion

We can identify $u^\varepsilon(t,x) \approx \mathscr{Z}_\beta(N,z)$ with

$$N = \varepsilon^{-2}t \qquad z = \varepsilon^{-1}x \qquad \beta_\varepsilon = \varepsilon^{\frac{d-2}{2}}\,\beta$$