On the 2d KPZ and Stochastic Heat Equation via directed polymers

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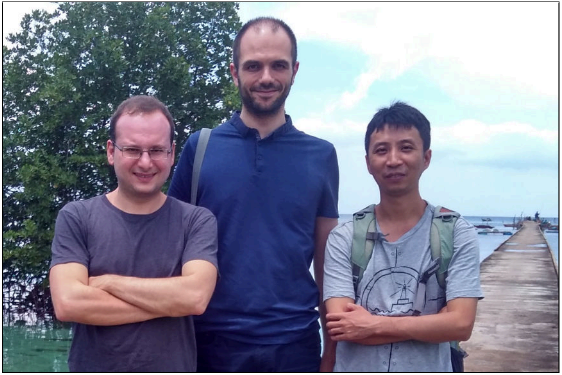
Collaborators



Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

Overview

I will talk about two stochastic PDEs on $\mathbb{R}^d$ (mainly $d = 2$)

- Kardar-Parisi-Zhang Equation (KPZ)
- Multiplicative Stochastic Heat Equation (SHE)

In a nutshell

- KPZ and SHE ill-defined due to singular terms
- Regularized versions (mollified, or discretized)
- Do regularized solutions converge? (as regularization is removed)

Not a minicourse in stochastic analysis! $\rightsquigarrow$ Statistical Mechanics

Overview

Main focus on dimension $d = 2$. Recent progress on "subcritical" regime

... and some results in the "critical" regime (many questions still open!)

Edwards-Wilkinson fluctuations

Regularized solutions converge to explicit Gaussian random field

Plan

- Main results + general picture in dim. $d = 1$, $d = 2$, $d \geq 3$
- Connection and intuition with Directed Polymer
- Sketch of the proof + main tools

References

- [CSZ 17]
  Universality in marginally relevant disordered systems
  AAP 2017

- [CSZ 18a]
  On the moments of the (2+1)-dimensional directed polymer and Stochastic Heat Equation in the critical window
  arXiv, Aug 2018

- [CSZ 18b]
  The two-dimensional KPZ equation in the entire subcritical regime
  arXiv, Dec 2018

$(d = 2)$ [Bertini Cancrini 98] [Chatterjee Dunlap 18]

$(d \geq 3)$ [Magnen Unterberger 18] [Gu Ryzhik Zeitouni 18] [Mukherjee Shamov Zeitouni 16] [Comets Cosco Mukherjee 18]

KPZ Equation

Random interface growth [Kardar-Parisi-Zhang PRL'86]

$$\partial_t h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{1}{2}|\nabla h(t,x)|^2 + \beta\,\xi(t,x) \qquad \text{(KPZ)}$$

$h(t,x)$ = interface height at time $t \geq 0$, space $x \in \mathbb{R}^d$

$\xi(t,x)$ = space-time white noise

(δ-correlated Gaussian field $\rightsquigarrow$ Continuum analogue of i.i.d. random field)

$\beta > 0$ noise strength

Singular term $|\nabla h(t,x)|^2$ undefined (∇h is a distribution)

Take $\xi(t,x)$ smooth. KPZ is linearized by Cole-Hopf transformation

$$u(t,x) := e^{h(t,x)}$$

Stochastic Heat Equation (SHE)

Multiplicative Stochastic Heat Equation (SHE) $t \geq 0,\ x \in \mathbb{R}^d$

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \beta\, u(t,x)\, \xi(t,x) \qquad \text{(SHE)}$$

Linear equation, in principle easier

SHE well-posed in $d = 1$ by Ito theory (stochastic integration)

Initial datum $u(0,x) \equiv 1$ (for simplicity)

Mild formulation: $$u(t,x) = 1 + \beta \int_0^t \int_{\mathbb{R}} g_{t-s}(x-y)\, u(s,y)\, \xi(\mathrm{d}s, \mathrm{d}y)$$

where $g_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ heat kernel on $\mathbb{R}$

One space dimension $d = 1$

- SHE solution $u(t,x)$ well-defined, random continuous function
- Continuous and strictly positive [Mueller 91]

Explicit Wiener chaos representation

$$u(t,x) = 1 + \sum_{k=1}^{\infty} \beta^k \iint_{\substack{0<t_1<\ldots<t_k<t \\ (x_1,\ldots,x_k)\in\mathbb{R}^k}} g_{t_1}(x_1)\, g_{t_2-t_1}(x_2-x_1) \, \ldots \prod_{i=1}^{k} \xi(\mathrm{d}t_i, \mathrm{d}x_i)$$

Forget the definition of KPZ equation, focus on its solution

Cole-Hopf "solution" of KPZ

$$h(t,x) := \log u(t,x)$$

This is indeed the "right" solution

One space dimension $d = 1$

In support of KPZ Cole-Hopf solution

- Arises as a limit of interacting particle systems [Bertini Giacomin '97]
- Fluctuations of 1d exactly solvable models of interface growth

  $\rightsquigarrow$ KPZ universality class Surveys: [Corwin '12] [Quastel Spohn '15]

Robust justification by solution theories for singular stochastic PDEs

- Regularity Structures [Hairer '13] [Hairer '14]
- Paracontrolled Distributions [Gubinelli Imkeller Perkowski '15]
- Energy Solutions [Goncalves Jara '14]
- Renormalization Approach [Kupiainen '16]

All these approaches only work for KPZ in $d = 1$ (sub-critical)

The general setting

General dimensions d: how to find a "solution" of KPZ and SHE ?

Mollify (regularize) the white noise $\xi(t,x)$ in space on scale $\varepsilon > 0$

$$\xi^\varepsilon(t,x) := \big(\xi(t,\cdot) * j_\varepsilon\big)(x)$$

- $j_\varepsilon(x) := \varepsilon^{-d} j(\varepsilon^{-1}x)$ $\quad j \in C_c^\infty(\mathbb{R}^d)$ probability density
- $t \mapsto W^\varepsilon(t,x) := \int_0^t \xi^\varepsilon(\mathrm{d}s, x)$ $\quad$ Brownian motions

  (correlated in x, variance $\sigma_\varepsilon^2 := \varepsilon^{-d} \, \|j\|_{L^2}^2$)

Replace ξ by ξ^ε $\rightsquigarrow$ (KPZ) and (SHE) well-posed by Ito theory

Do mollified solutions $h^\varepsilon(t,x)$ and $u^\varepsilon(t,x)$ have a limit as $\varepsilon \downarrow 0$?

Disorder strength $\beta = \beta_\varepsilon$ needs to be renormalized!

Mollified equations

Mollified SHE

$$\begin{cases} \partial_t u^\varepsilon = \frac{1}{2}\Delta u^\varepsilon + \beta_\varepsilon\, u^\varepsilon\, \xi^\varepsilon \\ u^\varepsilon(0,\cdot) \equiv 1 \end{cases} \qquad (\varepsilon\text{-SHE})$$

By Ito's formula $h^\varepsilon(t,x) := \log u^\varepsilon(t,x)$ satisfies

Mollified KPZ

$$\begin{cases} \partial_t h^\varepsilon = \frac{1}{2}\Delta h^\varepsilon + \frac{1}{2}|\nabla h^\varepsilon|^2 + \beta_\varepsilon\, \xi^\varepsilon - C_\varepsilon \\ h^\varepsilon(0,\cdot) \equiv 0 \end{cases} \qquad (\varepsilon\text{-KPZ})$$

$$C_\varepsilon := \beta_\varepsilon^2\, \sigma_\varepsilon^2 = \beta_\varepsilon^2\, \varepsilon^{-d}\, \|j\|_{L^2}^2$$

Key problem

Can we choose $\beta_\varepsilon \in (0, \infty)$ so that

$u^\varepsilon(t, x)$ and $h^\varepsilon(t, x)$ admit non-trivial limits as $\varepsilon \downarrow 0$?

YES! $(\ldots)$

$$\beta_\varepsilon = \begin{cases} \hat\beta \text{ (fixed)} & d = 1 \\ \dfrac{\sqrt{2\pi}\,\hat\beta}{\sqrt{\log \varepsilon^{-1}}} & d = 2 \\ \hat\beta\,\varepsilon^{\frac{d-2}{2}} & d \geq 3 \end{cases} \qquad \hat\beta \in (0, \infty)$$

Note that $\beta_\varepsilon \to 0$ for $d = 2$ and $d \geq 3$

Choice of β_ε will be clear later ($\rightsquigarrow$ directed polymers)

Main result I. Phase transition for SHE

Space dimension $d = 2$ $\qquad \beta_\varepsilon = \dfrac{\sqrt{2\pi}\,\hat{\beta}}{\sqrt{\log \varepsilon^{-1}}} \qquad \hat{\beta} \in (0\infty)$

Theorem (SHE one-point distribution) [CSZ 17]

Phase transition ("weak to strong disorder") with critical value $\hat{\beta}_c = 1$

Fix $t > 0, x \in \mathbb{R}^2$:

$$u^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} \begin{cases} \exp\left(\sigma_{\hat{\beta}}\, Z - \frac{1}{2}\,\sigma_{\hat{\beta}}^2\right) & \text{if } \hat{\beta} < 1 \\ 0 & \text{if } \hat{\beta} \geq 1 \end{cases}$$

$Z \sim N(0,1) \qquad \sigma_{\hat{\beta}}^2 := \log \frac{1}{1-\hat{\beta}^2}$

Subcritical regime $\hat{\beta} < 1$. For distinct $x_1, \ldots, x_n \in \mathbb{R}^2$

$u^\varepsilon(t, x_i)$ become asymptotically independent (!) as $\varepsilon \downarrow 0$

Main result I. Phase transition for KPZ

Space dimension $d = 2$ $\qquad \beta_\varepsilon = \frac{\sqrt{2\pi}\,\hat{\beta}}{\sqrt{\log \varepsilon^{-1}}}$ $\qquad \hat{\beta} \in (0\infty)$

Theorem (KPZ one point distribution) [CSZ 17]

Phase transition ("weak to strong disorder") with critical value $\hat{\beta}_c = 1$

Fix $t > 0, x \in \mathbb{R}^2$:

$$h^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} \begin{cases} \sigma_{\hat{\beta}}\, Z - \frac{1}{2}\,\sigma_{\hat{\beta}}^2 & \text{if } \hat{\beta} < 1 \\ -\infty & \text{if } \hat{\beta} \ge 1 \end{cases}$$

$Z \sim N(0,1) \qquad \sigma_{\hat{\beta}}^2 := \log \frac{1}{1-\hat{\beta}^2}$

Subcritical regime $\hat{\beta} < 1$. For distinct $x_1, \ldots, x_n \in \mathbb{R}^2$

$h^\varepsilon(t, x_i)$ become asymptotically independent (!) as $\varepsilon \downarrow 0$

Sub-critical regime $\hat\beta < 1$

For $\hat\beta < 1$ $u^\varepsilon(t,x)$ and $h^\varepsilon(t,x)$ are very irregular functions of x

Look at $u^\varepsilon(t,\cdot)$ and $h^\varepsilon(t,\cdot)$ as random distributions on $\mathbb{R}^2$

$$\mathbb{E}[u^\varepsilon(t,x)] \equiv 1 \qquad \mathbb{E}[h^\varepsilon(t,x)] = -\tfrac{1}{2}\,\sigma^2_{\hat\beta} + o(1) \quad \text{as } \varepsilon \downarrow 0$$

Law of large numbers

$$u^\varepsilon(t,\cdot) \xrightarrow[\varepsilon\downarrow 0]{d} 1 \qquad h^\varepsilon(t,\cdot) \xrightarrow[\varepsilon\downarrow 0]{d} -\tfrac{1}{2}\,\sigma^2_{\hat\beta} \qquad \text{as distributions}$$

$$\forall \phi \in C_c(\mathbb{R}^2): \qquad \int_{\mathbb{R}^2} u^\varepsilon(t,x)\,\phi(x)\,\mathrm{d}x \xrightarrow[\varepsilon\downarrow 0]{d} \int_{\mathbb{R}^2} \phi(x)\,\mathrm{d}x$$

$$\int_{\mathbb{R}^2} h^\varepsilon(t,x)\,\phi(x)\,\mathrm{d}x \xrightarrow[\varepsilon\downarrow 0]{d} \big(-\tfrac{1}{2}\,\sigma^2_{\hat\beta}\big) \int_{\mathbb{R}^2} \phi(x)\,\mathrm{d}x$$

Fluctuations?

Main result II. Fluctuations for SHE

Recall that $\beta_\varepsilon = \frac{\sqrt{2\pi}\,\hat{\beta}}{\sqrt{\log \varepsilon^{-1}}}$ sub-critical $\hat{\beta} \in (0,1)$

Rescaled SHE solution $\mathcal{U}^\varepsilon(t,x) := \frac{1}{\beta_\varepsilon}\big(u^\varepsilon(t,x) - \mathbb{E}[u^\varepsilon]\big)$

Theorem (EW fluctuations for SHE) [CSZ 17]

$$\forall \phi \in C_c(\mathbb{R}^2) \qquad \int_{\mathbb{R}^2} \mathcal{U}^\varepsilon(t,x)\,\phi(x)\,\mathrm{d}x \xrightarrow[\varepsilon\downarrow 0]{d} \int_{\mathbb{R}^2} v^{(c_{\hat{\beta}})}(t,x)\,\phi(x)\,\mathrm{d}x$$

$c_{\hat{\beta}} = \frac{1}{\sqrt{1-\hat{\beta}^2}}$ $\qquad v^{(c)}(t,x)$ solution of Additive SHE

$$\partial_t v^{(c)}(t,x) = \frac{1}{2}\Delta v^{(c)}(t,x) + c\,\xi(t,x) \qquad \text{(EW)}$$

known as Edwards-Wilkinson equation

Explicit reformulation

EW solution well-defined (in any dimension)

$$v^{(c)}(t,x) = \int_0^t \int_{\mathbb{R}^2} g_{t-s}(x-y)\, \xi(\mathrm{d}s, \mathrm{d}y) \qquad g_t(x) = \frac{1}{2\pi t} e^{-\frac{|x|^2}{2t}}$$

It is a (distribution valued) Gaussian process

$$\int_{\mathbb{R}^2} v^{(c)}(t,x)\, \phi(x)\, \mathrm{d}x \sim N(0, c^2\, \sigma_\phi^2)$$

- $\sigma_\phi^2 = \int_{(\mathbb{R}^2)^2} \phi(x)\, K_t(x,y)\, \phi(y)\, \mathrm{d}x\, \mathrm{d}y$
- $K_t(x,y) := \int_0^t g_{2u}(x-y)\, \mathrm{d}u \sim \frac{1}{4\pi} \log \frac{4t}{|x-y|^2}$

We will understand better how EW emerges from SHE

Fluctuations: from SHE to KPZ?

Mollified SHE solution $u^\varepsilon(t,x)$ admits explicit Wiener-Chaos expansion

Key tool to prove EW fluctuations, not available for KPZ sol. $h^\varepsilon(t,x)$

How to prove EW fluctuations for KPZ?

Naive idea

$$h^\varepsilon(t,x) = \log u^\varepsilon(t,x) \qquad u^\varepsilon(t,x) \to 1 \text{ (as a distribution)}$$

Taylor expansion $h^\varepsilon(t,x) \approx \big(u^\varepsilon(t,x) - 1\big)$?

NO, because $u^\varepsilon(t,x)$ is not close to 1 pointwise

However, with careful analysis, we can correct and control the expansion

$\leadsto$ The same EW fluctuations hold for KPZ

Main result II. Sub-critical fluctuations for KPZ

Recall that $\beta_\varepsilon = \frac{\sqrt{2\pi}\,\hat{\beta}}{\sqrt{\log \varepsilon^{-1}}}$ sub-critical $\hat{\beta} \in (0,1)$

Rescaled KPZ solution $\mathcal{H}^\varepsilon(t,x) := \frac{1}{\beta_\varepsilon}\big(h^\varepsilon(t,x) - \mathbb{E}[h^\varepsilon]\big)$

Theorem (EW fluctuations for KPZ) [CSZ 18b]

$$\forall \phi \in C_c(\mathbb{R}^2) \qquad \int_{\mathbb{R}^2} \mathcal{H}^\varepsilon(t,x)\,\phi(x)\,\mathrm{d}x \xrightarrow[\varepsilon\downarrow 0]{d} \int_{\mathbb{R}^2} v^{(c_{\hat{\beta}})}(t,x)\,\phi(x)\,\mathrm{d}x$$

- $c_{\hat{\beta}} = \frac{1}{\sqrt{1-\hat{\beta}^2}}$ (same constant as before)
- $v^{(c)}(t,x)$ solution of Additive SHE

Summary so far

- Transition at scale $\beta_\varepsilon = \frac{\sqrt{2\pi}\,\hat{\beta}}{\sqrt{\log \varepsilon^{-1}}}$ with explicit critical point $\hat{\beta}_c = 1$
- Edwards-Wilkinson fluctuations

$$\frac{h^\varepsilon(t,x) - \mathbb{E}[h^\varepsilon]}{\beta_\varepsilon} \quad \text{and} \quad \frac{u^\varepsilon(t,x) - \mathbb{E}[u^\varepsilon]}{\beta_\varepsilon} \xrightarrow[\varepsilon\downarrow 0]{d} v^{(c_{\hat{\beta}})}(t,x)$$

- Explicit $c_{\hat{\beta}} = \frac{1}{\sqrt{1-\hat{\beta}^2}}$
- Fluctuations in the entire subcritical regime $0 < \hat{\beta} < 1$

We now discuss related results in the literature

A variation on KPZ

Recently Chatterjee and Dunlap [CD 18] considered a variation

$$\partial_t \tilde{h}^\varepsilon = \frac{1}{2}\Delta \tilde{h}^\varepsilon + \frac{1}{2}\,\beta_\varepsilon\,|\nabla \tilde{h}^\varepsilon|^2 + \xi^\varepsilon$$

The same $\beta_\varepsilon = \dfrac{\sqrt{2\pi}\,\hat{\beta}}{\sqrt{\log \varepsilon^{-1}}}$ now multiplies the non-linearity instead of ξ^ε

Theorem [Chatterjee Dunlap 18]

For $\hat{\beta}$ sufficiently small, the centered solution $\tilde{h}^\varepsilon(t,\cdot) - \mathbb{E}[\tilde{h}^\varepsilon]$ admits subsequential limits in law as $\varepsilon \downarrow 0$ (as a random distribution on $\mathbb{R}^2$)

Any limit is not the solution of Additive SHE (EW) with $c = 1$
(what one would get simply removing the non-linearity)

Relation with our results

Recall "our" KPZ : $\partial_t h^\varepsilon = \frac{1}{2}\Delta h^\varepsilon + \frac{1}{2}|\nabla h^\varepsilon|^2 + \beta_\varepsilon\,\xi^\varepsilon - C_\varepsilon$

Scaling relation

$$\tilde{h}^\varepsilon(t,x) - \mathbb{E}[\tilde{h}^\varepsilon] = \frac{1}{\beta_\varepsilon}\big(h^\varepsilon(t,x) - \mathbb{E}[h^\varepsilon]\big) = \mathcal{H}^\varepsilon(t,x)$$

Theorem [CSZ 18b]

For every sub-critical $\hat{\beta} < 1$, the centered solution $\tilde{h}^\varepsilon(t,\cdot) - \mathbb{E}[\tilde{h}^\varepsilon]$ admits a unique limit in law as $\varepsilon \downarrow 0$ (as a random distribution on $\mathbb{R}^2$)

The limit is the solution of Additive SHE (EW) with $c_{\hat{\beta}} = \frac{1}{\sqrt{1-\hat{\beta}^2}} > 1$

Phase transition for $d \geq 3$

For $d \geq 3$ the right way to scale β_ε is

$$\beta_\varepsilon = \hat{\beta}\, \varepsilon^{\frac{d-2}{2}} \qquad \hat{\beta} \in (0, \infty)$$

Theorem [Mukherjee Shamov Zeitouni 16]

There exists $\hat{\beta}_c \in (0, \infty)$ (unknown) such that

$$u^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} \begin{cases} > 0 & \text{if } \hat{\beta} < \hat{\beta}_c \\ 0 & \text{if } \hat{\beta} > \hat{\beta}_c \end{cases}$$

$$h^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} \begin{cases} \in \mathbb{R} & \text{if } \hat{\beta} < \hat{\beta}_c \\ -\infty & \text{if } \hat{\beta} > \hat{\beta}_c \end{cases}$$

See also [Comets Cosco Mukherjee 18] for related results

Edwards-Wilkinson fluctuations in $d \geq 3$

$$\beta_\varepsilon = \hat{\beta}\, \varepsilon^{\frac{d-2}{2}} \qquad \text{sub-critical } \hat{\beta} \in (0,1)$$

EW fluctuations for KPZ established in [Magnen Unterberger 18]

Theorem [Magnen Unterberger 18]

For $\hat{\beta} < 1$ sufficiently small, one has

$$\frac{h^\varepsilon(t,\cdot) - \mathbb{E}[h^\varepsilon]}{\beta_\varepsilon} \xrightarrow[\varepsilon \downarrow 0]{d} v^{(c_{\hat{\beta}})}(t,\cdot)$$

solution of the Additive SHE (EW) for a suitable noise strength $c_{\hat{\beta}}$.

Analogous EW fluctuations for SHE proved in [Gu Ryzhik Zeitouni 18]

(See also [Comets Cosco Mukherjee 18])

The one-dimensional case

The situation for $d = 1$ is rather different

$$\beta_\varepsilon = \hat{\beta} \in (0, \infty) \qquad \text{(fixed)}$$

- No phase transition:

$$u^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} u(t,x) > 0 \quad \forall \hat{\beta} \in (0,\infty)$$

$$h^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} h(t,x) > 0 \quad \forall \hat{\beta} \in (0,\infty)$$

- EW fluctuations easily established as $\hat{\beta} \to 0$

The critical regime

What about $\hat{\beta} = 1$?

More generally, critical window [Bertini Cancrini 98]

$$\beta_\varepsilon = \sqrt{\frac{2\pi}{\log \varepsilon^{-1}} \left(1 + \frac{\vartheta}{\log \varepsilon^{-1}}\right)} \qquad \text{with} \quad \vartheta \in \mathbb{R}$$

Nothing known for KPZ $h^\varepsilon(t,x)$, some progress for SHE $u^\varepsilon(t,x)$

Key conjecture

$u^\varepsilon(t,\cdot)$ has a limit $\mathcal{U}(t,\cdot)$ for $\varepsilon \downarrow 0$, as a random distribution on $\mathbb{R}^2$

$$\langle u^\varepsilon(t,\cdot), \phi\rangle := \int_{\mathbb{R}^2} u^\varepsilon(t,x)\,\phi(x)\,\mathrm{d}x \xrightarrow[\varepsilon\downarrow 0]{d} \int_{\mathbb{R}^2} \mathcal{U}(t,x)\,\phi(x)\,\mathrm{d}x$$

(actually a random measure, since $u^\varepsilon \geq 0$)

Second moment in the critical window

What is known [Bertini Cancrini 98]

Tightness via second moment bounds

$$\mathbb{E}\big[\langle u^\varepsilon(t,\cdot),\phi\rangle\big] \equiv \langle 1,\phi\rangle \qquad \sup_{\varepsilon>0} \mathbb{E}\big[\langle u^\varepsilon(t,\cdot),\phi\rangle^2\big] < \infty$$

More precisely $\mathbb{E}\big[\langle u^\varepsilon(t,\cdot),\phi\rangle^2\big] \xrightarrow[\varepsilon\downarrow 0]{} \langle \phi,\, K\phi\rangle < \infty$

Explicit kernel $K(x,x') \sim C \log\frac{1}{|x-x'|}$ as $|x-x'| \to 0$

Corollary

$$\exists \text{ subsequential limits } \langle u^{\varepsilon_k}(t,\cdot),\phi\rangle \xrightarrow[k\to\infty]{d} \langle \mathcal{U}\,,\,\phi\rangle$$

Can the limit be trivial $\mathcal{U}(t,\cdot) \equiv 1$?

Main result III. Third moment in the critical window

We determine the sharp asymptotics of third moment

Theorem [CSZ 18a]

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\Big[\langle u^{\varepsilon}(t,\cdot), \phi \rangle^3\Big] = C(\phi) < \infty$$

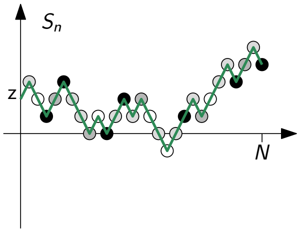
- Explicit expression for $C(\phi)$ (series of multiple integrals)

Corollary

Any subsequential limit $u^{\varepsilon_k}(t,\cdot) \xrightarrow{d} \mathcal{U}(t,\cdot)$ has covariance $K(x,x')$

$$\rightsquigarrow \quad \mathcal{U}(t,\cdot) \not\equiv 1 \quad \text{is non-degenerate!}$$

Directed Polymer in Random Environment [Comets 17]



- $(S_n)_{n\geq 0}$ simple random walk on $\mathbb{Z}^d$
- Disorder: i.i.d. random variables $\omega(n,x)$
  zero mean, unit variance
  $$\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega(n,x)}] < \infty$$
- (-) Hamiltonian $\quad H_{N,\beta}(\omega, S) := \beta \sum_{n=1}^{N} \omega(n, S_n) - \lambda(\beta)\, N$

Partition Functions $\quad (N \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z_{N,\beta}(z) = \mathbf{E}^{\mathrm{rw}}\Big[e^{H_{N,\beta}(\omega,S)}\Big| S_0 = z\Big] = \frac{1}{(2d)^N} \sum_{(s_0,\ldots,s_N)\ \text{n.n.:}\ s_0=z} e^{H_{N,\beta}(\omega,s)}$$

Directed Polymer and SHE

Partition functions $\boldsymbol{Z}_{N,\beta}(z)$ are discrete analogues of $u^\varepsilon(t,x)$

- They solve a lattice SHE

$$\boldsymbol{Z}_{N+1}(z) - \boldsymbol{Z}_N(z) \;=\; \Delta \boldsymbol{Z}_N(z) \;+\; \beta\, \tilde{\omega}(N+1,z)\, \widetilde{\boldsymbol{Z}}_N(z)$$

$\rightsquigarrow$ Alternative way of regularizing SHE (discretize vs. mollify)

- Quantitative analogy via Feynman-Kac formula for SHE

SHE $\beta_\varepsilon = \varepsilon^{\frac{d-2}{2}}\, \beta$ Directed Polymer

Feynman-Kac formula for SHE

Recall the ε-mollified SHE
$$\begin{cases} \partial_t u^\varepsilon = \frac{1}{2}\Delta h^\varepsilon + \beta_\varepsilon\, u^\varepsilon\, \xi^\varepsilon \\ u^\varepsilon(0,\cdot) \equiv 1 \end{cases}$$

Feynman-Kac

$$u^\varepsilon(t,x) \overset{d}{=} \mathrm{E}_{\varepsilon^{-1}x}\left[\exp\left(\beta_\varepsilon\, \varepsilon^{-\frac{d-2}{2}} \int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^2} j(B_s - y)\, \xi(\mathrm{d}s, \mathrm{d}y) - \ldots\right)\right]$$

$(B_s)_{s\geq 0}$ Brownian motion $\qquad j(\cdot) \in C_c^\infty(\mathbb{R}^d)$ probability density

$u^\varepsilon(t,x)$ corresponds to $\boldsymbol{Z}_{N,\beta}(z)$ with

$$N = \varepsilon^{-2}t \qquad z = \varepsilon^{-1}x \qquad \beta_\varepsilon = \varepsilon^{\frac{d-2}{2}}\,\beta$$

Weak and strong disorder for Directed Polymer

For $d \geq 3$ there is a phase transition: $\exists \beta_c \in (0, \infty)$ such that

$$\text{for } \beta < \beta_c: \qquad \boldsymbol{Z}_N(z) \xrightarrow[N\to\infty]{\text{a.s.}} \mathcal{Z}(z) > 0 \quad \text{(weak disorder)}$$

$$\text{for } \beta > \beta_c: \qquad \boldsymbol{Z}_N(z) \xrightarrow[N\to\infty]{\text{a.s.}} 0 \quad \text{(strong disorder)}$$

For $d = 1$, $d = 2$ we have $\beta_c = 0$, i.e. only strong disorder:

$$\text{for any } \beta > 0: \qquad \boldsymbol{Z}_N(z) \xrightarrow[N\to\infty]{\text{a.s.}} 0$$

[Bolthausen 89] [Comets Shiga Yoshida 03] [Vargas 07] [Lacoin 11] [Comets 17]

Directed Polymer and SHE

To see weak disorder for $d = 1$, $d = 2$ we must take $\beta = \beta_N \to 0$

$$\beta_N \approx \begin{cases} \dfrac{\hat{\beta}}{N^{1/4}} & \text{without transition} \quad d = 1 \quad \text{[Alberts, Khanin, Quastel 14]} \\ \dfrac{\hat{\beta}}{\sqrt{\log N}} & \text{with transition} \quad d = 2 \quad \text{[CSZ 17]} \end{cases}$$

This matches with the scaling for β_ε for SHE and KPZ

- Directed Polymer provides a friendly framework for SHE
- Results first proved for Directed Polymer, then for SHE and KPZ
- We will sketch some of the proofs highlighting key tools:

  Concentration Inequalities Polynomial Chaos Hypercontractivity