THE 2d DIRECTED POLYMER AND THE DICKMAN SUBORDINATOR

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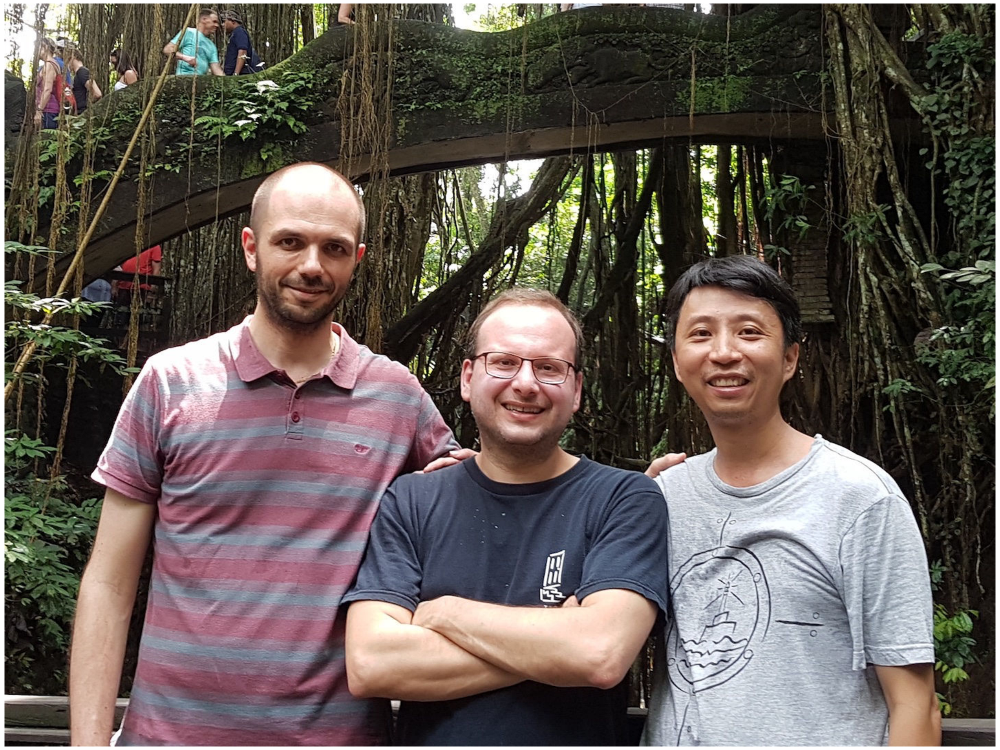
WORKSHOP "RANDOM POLYMERS AND NETWORKS"
PORQUEROLLES, 11 SEPTEMBER 2020

Coauthors



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INTRODUCTION

TWO INDEPENDENT RANDOM INGREDIENTS

- $S = (S_i)_{i=0,1,2,\dots}$ SRW ON $\mathbb{Z}^2$, LAW P
- $\omega = (\omega(i,x))_{(i,x)\in\mathbb{N}\times\mathbb{Z}^2}$ I.I.D. $N(0,1)$, LAW $\mathbb{P}$

PARTITION FUNCTION

($N \in \mathbb{N}$, $\beta \geq 0$, FIXED ω)

$$Z_N = E\left[e^{\sum_{i=1}^{N} \left\{ \beta\,\omega(i,S_i) - \frac{\beta^2}{2} \right\}} \right]$$

$\mathbb{E}[Z_N] = 1$

$\boxed{d \geq 3}$ $\exists\ \beta_c > 0$: [Bolthausen]

$$Z_N \xrightarrow{A.S.} \begin{cases} Z_\infty > 0 \ \text{(RANDOM)} & \text{IF } \beta < \beta_c \\ 0 & \text{IF } \beta > \beta_c \end{cases}$$

$\boxed{d \leq 2}$ $\beta_c = 0$ [Comets, Vargas, Lacoin, ...]

$$Z_N \xrightarrow{A.S.} 0 \qquad \forall\, \beta > 0$$

INTERMEDIATE DISORDER TUNE $\beta = \beta_N \downarrow 0$ SUCH THAT

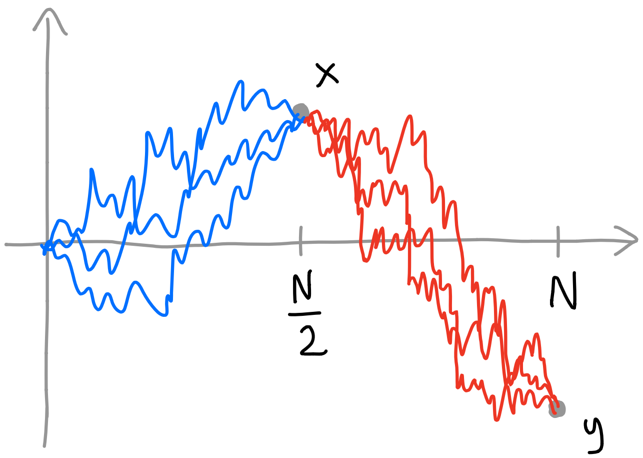
$$Z_N^{\beta_N} \xrightarrow{d} Z_\infty > 0 \quad \text{(RANDOM)}$$

WE FOCUS ON $d=2$ (CASE $d=1$ SOLVED BY [Alberts, Khanin, Quastel])

MOTIVATION: PATH PROPERTIES

POLYMER MEASURE

$$\frac{dP_N}{dP}(S) = \frac{e^{\sum_{i=1}^{N}\left\{\beta\omega(i,S_i) - \frac{\beta^2}{2}\right\}}}{Z_N}$$



$$Z_{A,B}(x,y) = E\left[e^{\sum_{A+1}^{B}\{\dots\}}\mathbb{1}_{\{S_B=y\}}\,\middle|\,S_A=x\right]$$

$$P_N\left(S_{\frac{N}{2}}=x,\ S_N=y\right) = \frac{Z_{0,\frac{N}{2}}(0,x)\ Z_{\frac{N}{2},N}(x,y)}{Z_N}$$

AVERAGED PARTITION FUNCTIONS

$\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ TEST FUNCTIONS

$$Z_N(\varphi, \psi) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi(x)\, \psi(y)\, Z_N(\lfloor x\sqrt{N} \rfloor, \lfloor y\sqrt{N} \rfloor)\, dx\, dy$$

FOR SIMPLICITY, WE FOCUS ON THE LEFT B.C. (STARTING POINT)

- $$Z_N(x) = E\left[e^{\sum_1^N \{\cdots\}} \,\middle|\, S_0 = x \right] = \sum_y Z_N(x,y)$$

- $$Z_N(\varphi) = \int_{\mathbb{R}^2} \varphi(x)\, Z_N(\lfloor x\sqrt{N} \rfloor)\, dx = Z_N(\varphi, \psi \equiv 1)$$

Other motivations

- Stochastic heat equation & KPZ equation

$$Z_N(x) \rightsquigarrow \partial_t u = \Delta u + \xi \cdot u \qquad \text{(SHE)}$$

$$\log Z_N(x) \rightsquigarrow \partial_t h = \Delta h + |\nabla h|^2 + \xi \qquad \text{(KPZ)}$$

- Random fields in $\mathbb{R}^2$

MAIN RESULTS

THEOREM 1 (PHASE TRANSITION) [CSZ 17]

- FOR $\beta = \dfrac{\hat{\beta}}{\sqrt{\log N}}$

EXPLICIT LOG-NORMAL!

$$Z_N \xrightarrow{d} \begin{cases} Z_\infty > 0 \quad \text{(RANDOM)} & \text{IF } \hat{\beta} < \sqrt{\pi} \\ 0 & \text{IF } \hat{\beta} \geq \sqrt{\pi} \end{cases}$$

- $Z_N(x)$ AND $Z_N(y)$ ASYMPTOTICALLY INDEPENDENT FOR $|x-y| \approx \sqrt{N}$

COROLLARY: LLN FOR AVERAGED PARTITION FUNCTION

$\forall \hat{\beta} < \sqrt{\pi}$ $\quad Z_N(\varphi) \xrightarrow{d} \int_{\mathbb{R}^2} \varphi(x)\, dx \quad [Z_N(\lfloor \cdot \sqrt{N} \rfloor) \to \text{LEB}]$

THEOREM 2 (CLT ~ EW FLUCT.) $\forall \hat{\beta} < \sqrt{\pi}$ [CSZ 17 & 20]

$$\sqrt{\log N}\,\big(Z_N(\varphi) - 1\big) \xrightarrow{d} \text{GAUSSIAN FIELD}$$

$$\sqrt{\log N}\,\Big(\log Z_N(\varphi) - \mathbb{E}[\log Z_N(\varphi)]\Big) \xrightarrow{d} \text{SAME GAUSSIAN FIELD}$$

SIMILAR RESULTS FOR $d \geq 3$ BY SEVERAL AUTHORS

[Comets, Cosca, Mukerjee] [Magnen, Unterberger] [Gu, Ryzhik, Zeitouni, Dunlap]

CRITICAL POINT $\hat{\beta} = \sqrt{\pi}$, OR EVEN CRITICAL WINDOW

$$\beta = \frac{1}{\sqrt{\log N}}\left(\sqrt{\pi} + \frac{\vartheta}{\log N}\right) \qquad \vartheta \in \mathbb{R}$$

- $\mathbb{E}[Z_N(\varphi)] \equiv \int \varphi(x)\, dx$
- $\mathbb{E}[Z_N(\varphi)^2] \longrightarrow \iint \varphi(x)\varphi(y)\, K_\vartheta(y-x)\, dx\, dy < \infty$ [Bertini-Cancrini]

THEOREM 3 (CRITICAL 3RD MOMENT) [CSZ 19]

$$\mathbb{E}[Z_N(\varphi)^3] \longrightarrow \iiint \varphi(x)\,\varphi(y)\,\varphi(z)\, M_\vartheta(x,y,z)\, dx\, dy\, dz < \infty$$

COROLLARY: $\exists$ SUBSEQUENCE (N_k) SUCH THAT

$$Z_N(\lfloor x\sqrt{N}\rfloor)\,dx \xrightarrow{d} Z_\infty(dx)$$ RANDOM MEASURE ON $\mathbb{R}^2$

AND ANY SUBSEQ. LIMIT Z_∞ HAS THE SAME COVARIANCE KERNEL $K_\vartheta(x,y)$

LATER [Gu, Quastel, Tsai] PROVED THAT (FOR SHE)

$$\forall k \in \mathbb{N}: \quad \mathbb{E}\left[Z_N(\varphi)^k\right] \longrightarrow \int\cdots\int \{\cdots\} < \infty$$

OPEN PROBLEM: UNIQUENESS OF SUBSEQUENTIAL LIMITS?

(KNOWLEDGE OF ALL MOMENTS IS NOT ENOUGH)

POLYNOMIAL CHAOS

DEFINE $\xi(i,x) = \xi^{\beta}(i,x) = \dfrac{e^{\beta\omega(i,x) - \frac{\beta^2}{2}} - 1}{\beta}$

I.I.D. WITH $\mathbb{E}[\xi(i,x)] = 0$ $\quad \mathbb{E}[\xi(i,x)^2] \simeq 1$ AS $\beta \downarrow 0$

$$Z_N = E\left[e^{\sum_{i=1}^{N} \{\beta\omega(i,S_i) - \frac{\beta^2}{2}\}} \right]$$

$$= E\left[\prod_{i=1}^{N} \left(1 + \beta\,\xi(i,S_i)\right)\right]$$

$$= 1 + \sum_{\substack{A \subseteq \{1,\dots,N\} \\ A \neq \phi}} E\left[\prod_{n\in A} \beta\,\xi(n,S_n)\right]$$

WE SPLIT THE SUM ACCORDING TO $k = |A| = 1, \dots, N$

AND WE WRITE $A = \{n_1 < n_2 < \dots < n_k\}$.

$$Z_N = 1 + \sum_{k=1}^{N} \beta^k \sum_{0 < n_1 < \dots < n_k \leq N} E\left[\xi(n_1, S_{n_1}) \cdots \xi(n_k, S_{n_k}) \right]$$

IF WE SET

$$q(n_1, \dots, n_k; x_1, \dots, x_k) := P(S_{n_1} = x_1, \dots, S_{n_k} = x_k)$$

$$= q(n_1, x_1) \cdot \prod_{i=2}^{k} q(n_i - n_{i-1}, x_i - x_{i-1})$$

THEN WE OBTAIN THE POLYNOMIAL CHAOS DECOMPOSITION OF Z_N

$$Z_N = 1 + \sum_{k=1}^{N} Z_N^{(k)}$$

WHERE

$$Z_N^{(k)} = \beta^k \sum_{\substack{0<n_1<\dots<n_k\leq N \\ x_1\dots x_k\in\mathbb{Z}^2}} \underbrace{q(n_1,\dots,n_k;x_1,\dots,x_k)}_{\text{COEFFICIENTS}} \underbrace{\xi(n_1,x_1)\cdots\xi(n_k,x_k)}_{\text{PRODUCT OF K DISTINCT RVs}}$$

IN $d=1$, IF WE FIX $\beta = \frac{1}{N^{1/4}}$, THEN [Alberts, Khanin, Quastel]

$$Z_N^{(k)} \longrightarrow \int_{([0,1]\times\mathbb{R})^k}\cdots\int \underbrace{g(t_1,\dots,t_k;z_1,\dots,z_k)}_{\text{HEAT KERNEL}} \underbrace{\xi(dt_1,dz_1)\cdots\xi(dt_k,dz_k)}_{\text{WHITE NOISE}}$$

SECOND MOMENT

THE RVs $\left(Z_N^{(K)}\right)_{K=1,\dots,N}$ ARE CENTERED AND UNCORRELATED

$$\mathbb{E}\left[Z_N^{(K)}\right]=0 \qquad \mathbb{E}\left[Z_N^{(K)} Z_N^{(K')}\right]=0 \quad \forall K\neq K'$$

(THEY ARE NOT INDEPENDENT). THEN

$$\mathbb{E}\left[Z_N^{2}\right] = 1 + \sum_{K=1}^{N} \mathbb{E}\left[\left(Z_N^{(K)}\right)^2\right]$$

$$= 1 + \sum_{K=1}^{N} \beta^{2K} \sum_{\substack{0<n_1<\dots<n_K\leq N \\ x_1 \cdots x_K \in \mathbb{Z}^2}} q(n_1,\dots,n_K; x_1,\dots,x_K)^2$$

NOTE THAT $\displaystyle \sum_{x \in \mathbb{Z}^2} \underbrace{q(n,x)}_{P(S_n = x)}{}^2 = q(2n, 0) \sim \frac{1}{\pi n}$ HENCE

$$\mathbb{E}[Z_N^2] \sim 1 + \sum_{k=1}^{N} \left(\frac{\beta^2}{\pi}\right)^k \underbrace{\sum_{0 < n_1 < \dots < n_k \leq N} \frac{1}{n_1} \cdot \frac{1}{n_2 - n_1} \cdots \frac{1}{n_k - n_{k-1}}}_{\leq \left(\sum_{n=1}^{N} \frac{1}{n}\right)^k \sim (\log N)^k}$$

THUS $\displaystyle \mathbb{E}[Z_N^2] \leq 1 + \sum_{k=1}^{\infty} \left(\frac{\beta^2}{\pi} \log N\right)^k \leq C < \infty$

IF $\displaystyle \beta \sim \frac{\hat{\beta}}{\sqrt{\log N}}$ WITH $\hat{\beta} < \sqrt{\pi}$

CRITICAL REGIME $\hat{\beta} = \sqrt{\pi}$

WE NEED REFINED ESTIMATES ON

$$\sum_{0 < n_1 < \dots < n_k \leq N} \frac{1}{n_1} \cdot \frac{1}{n_2 - n_1} \cdot \dots \cdot \frac{1}{n_k - n_{k-1}} = P\left(\tau_k^{(N)} \leq N \right) (\log N)^k$$

RENEWAL PROCESS (= POSITIVE RW) $\left(\tau_k^{(N)} \right)_{k = 0, 1, 2, \dots}$

$$\tau_0^{(N)} = 0 \qquad P\left(\tau_i^{(N)} - \tau_{i-1}^{(N)} = n \right) = \frac{1}{n} \cdot \mathbb{1}_{\{1, \dots, N\}}(n) \cdot \frac{1}{\log N}$$

WHAT IS THE ASYMPTOTIC BEHAVIOR OF $P(\tau_K^{(N)} \leq N)$?

$$E[\tau_1^{(N)}] = \frac{N}{\log N} \quad \Rightarrow \quad E[\tau_K^{(N)}] = \frac{K N}{\log N}$$

IT IS NATURAL TO TAKE $K \simeq \log N$

<u>PROPOSITION</u>

[CSZ 19]

$$\left(\frac{\tau^{(N)}_{\lfloor s \log N \rfloor}}{N}\right)_{s \in [0,\infty)} \xrightarrow{d} Y = (Y_s)_{s \in [0,\infty)}$$

WHERE Y IS AN EXPLICIT LÉVY PROCESS (DICKMAN SUBORDINATOR)

THEN AT THE CRITICAL POINT $\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}}$ WE GET

$$\mathbb{E}\left[Z_N^2\right] \sim 1 + \sum_{k=1}^{N} \left(\frac{\beta^2}{\pi}\right)^k \sum_{0<n_1<\dots<n_k\leq N} \frac{1}{n_1} \cdot \frac{1}{n_2-n_1} \cdots \frac{1}{n_k-n_{k-1}}$$

$$\sim 1 + \sum_{k=1}^{N} P\left(\tau_k^{(N)} \leq N\right)$$

$$\sim 1 + \sum_{s \in \frac{1}{\log N}\mathbb{N}} P\left(\tau_{\lfloor s \log N \rfloor}^{(N)} \leq N\right)$$

$$\sim (\log N) \cdot \underbrace{\int_0^\infty P(Y_s \leq 1)\, ds}_{C<\infty}$$

THUS FOR THE PARTITION FUNCTION $Z_N = Z_N(0)$ STARTED AT A FIXED POINT WE HAVE SHOWN THAT

$$\mathbb{E}\left[Z_N^2\right] \sim C \log N$$

FOR THE AVERAGED PARTITION FUNCTION $Z_N(\varphi)$ WE GET

$$\mathbb{E}\left[Z_N(\varphi)^2\right] \longrightarrow \iint \varphi(x)\,\varphi(y)\,K(y-x)\,dx\,dy < \infty$$

WHERE $K(x,y)$ IS ALSO RELATED TO Y.

THE DICKMAN SUBORDINATOR

$Y = (Y_s)_{s \in [0,\infty)}$ IS THE PURE JUMP LÉVY PROCESS WITH

LÉVY MEASURE $\nu(dt) = \frac{1}{t} \mathbb{1}_{(0,1)}(t)$

$$\Longleftrightarrow \quad E\left[e^{\lambda Y_s}\right] = \exp\left\{ s \int_0^1 (e^{\lambda} - 1) \frac{dt}{t} \right\}$$

THEOREM $\quad f_s(t) = \dfrac{P(Y_s \in dt)}{dt} = \dfrac{e^{-\gamma s}\, t^{s-1}}{\Gamma(s)}$ FOR $t \in (0,1]$

γ = EULER-MASCHERONI CST. (RECURSIVE FORMULA FOR $t \in (1,\infty)$)

THE FUNCTION $\rho(t) := e^{\gamma} f_1(t)$ IS CALLED DICKMAN FUNCTION AND APPEARS IN NUMBER THEORY & COMBINATORICS.

BESIDES THE WEAK CONVERGENCE $\frac{\tau^{(N)}_{\lfloor s \log N \rfloor}}{N} \xrightarrow{d} Y_s$ WE HAVE

THEOREM (SHARP RENEWAL THEOREM) [CSZ 19]

$$P(n \in \tau^{(N)}) = \sum_{k=0}^{\infty} P(\tau^{(N)}_k = n) \sim \frac{\log N}{N} G\left(\frac{n}{N}\right)$$

WHERE $G(t) = \int_0^{\infty} f_s(t)\, ds$ IS THE GREEN FUNCTION OF Y

Merci