On the 2d KPZ and Stochastic Heat Equation

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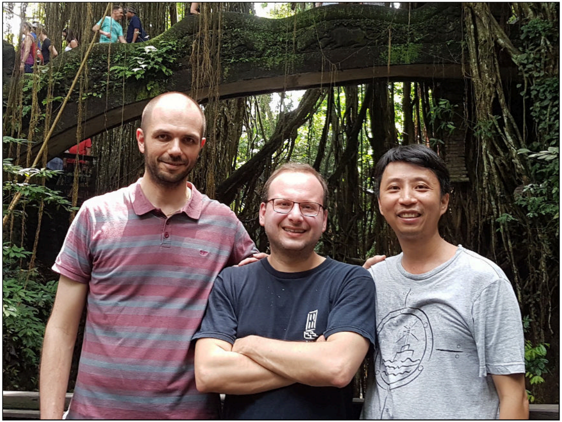
Collaborators



Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

Overview

Two stochastic PDEs on $\mathbb{R}^d$ (mainly $d = 2$)

- Kardar-Parisi-Zhang Equation (KPZ)
- Stochastic Heat Equation (SHE) with multiplicative noise

Very interesting yet ill-defined equations

Plan:

1. Consider a regularized version of these equations
2. Study the limit of the solution, when regularisation is removed

Stochastic Analysis $\leftrightsquigarrow$ Statistical Mechanics

White noise

Space-time white noise $\xi = \xi(t,x)$ on $\mathbb{R}^{1+d}$

Random distribution of negative order (Schwartz) [not a function!]

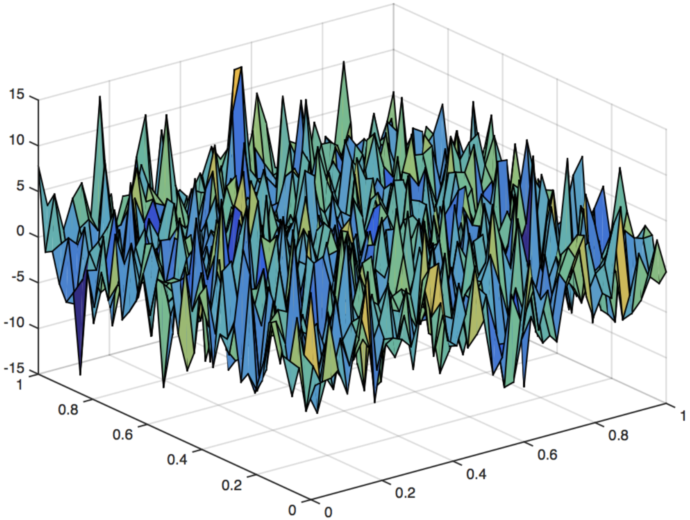
Gaussian: $$\langle \xi, \phi \rangle = \int_{\mathbb{R}^{1+d}} \xi(t,x)\,\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x \sim \mathcal{N}(0, \|\phi\|_{L^2}^2)$$

$$\mathrm{Cov}[\,\xi(t,x), \xi(t',x')\,] = \delta(t-t')\,\delta(x-x')$$

Case $d=0$: $\xi(t) = \frac{\mathrm{d}}{\mathrm{d}t} B(t)$ where (B_t) is Brownian motion

ξ = scaling limit of i.i.d. RVs indexed by $\mathbb{Z}^{1+d}$

White noise



The KPZ equation

KPZ [Kardar Parisi Zhang 86]

$$\partial_t h = \frac{1}{2} \Delta_x h + \frac{1}{2} |\nabla_x h|^2 + \beta \, \xi \qquad \text{(KPZ)}$$

Model for random interface growth

$h = h(t,x) =$ interface height at time $t \geq 0$, space $x \in \mathbb{R}^d$

$\xi = \xi(t,x) =$ space-time white noise $\qquad \beta > 0$ noise strength

$|\nabla_x h|^2$ ill-defined

For smooth ξ

$$u(t,x) := e^{h(t,x)} \qquad \text{(Cole-Hopf)}$$

The multiplicative Stochastic Heat Equation (SHE)

SHE $(t > 0, \quad x \in \mathbb{R}^d)$

$$\partial_t u = \frac{1}{2} \Delta_x u + \beta \, u \, \xi \qquad \text{(SHE)}$$

Product $u \, \xi$ ill-defined

$(d = 1)$ SHE is well-posed by Ito integration [Walsh 80's]

$u(t, x)$ is a function $\rightsquigarrow$ "KPZ solution" $h(t, x) := \log u(t, x)$

$(d = 1)$ SHE and KPZ well-understood in a robust sense ("pathwise")

Regularity Structures (Hairer)
Paracontrolled Distributions (Gubinelli, Imkeller, Perkowski)
Energy Solutions (Goncalves, Jara), Renormalization (Kupiainen)

Higher dimensions $d \geq 2$

In dimensions $d \geq 2$ there is no general theory

We mollify the white noise $\xi(t,x)$ in space on scale $\varepsilon > 0$

$$\xi^\varepsilon(t,\cdot) := \xi(t,\cdot) * \varrho_\varepsilon$$

Solutions $h^\varepsilon(t,x)$, $u^\varepsilon(t,x)$ are well-defined. Convergence as $\varepsilon \downarrow 0$?

Renormalization: we need to tune disorder strength as $\varepsilon \downarrow 0$

$$\beta = \beta_\varepsilon \to 0 \quad \text{as} \quad \begin{cases} \dfrac{\hat\beta}{\sqrt{|\log \varepsilon|}} & (d = 2) \\ \hat\beta\, \varepsilon^{\frac{d-2}{2}} & (d \geq 3) \end{cases} \qquad \hat\beta \in (0,\infty)$$

Mollified and renormalized equations

Mollified and renormalized SHE

$$\begin{cases} \partial_t u^\varepsilon = \frac{1}{2}\Delta u^\varepsilon + \beta_\varepsilon \, u^\varepsilon \, \xi^\varepsilon \\ u^\varepsilon(0,\cdot) \equiv 1 \end{cases} \qquad (\varepsilon\text{-SHE})$$

$u^\varepsilon(t,x) > 0$ Cole-Hopf $h^\varepsilon(t,x) := \log u^\varepsilon(t,x)$ $\rightsquigarrow$ Ito formula

Mollified and renormalized KPZ

$$\begin{cases} \partial_t h^\varepsilon = \frac{1}{2}\Delta h^\varepsilon + \frac{1}{2}|\nabla h^\varepsilon|^2 + \beta_\varepsilon \, \xi^\varepsilon - c\, \beta_\varepsilon^2 \, \varepsilon^{-d} \\ h^\varepsilon(0,\cdot) \equiv 0 \end{cases} \qquad (\varepsilon\text{-KPZ})$$

Main results

Space dimension $d = 2$ $\qquad \beta_\varepsilon = \dfrac{\hat{\beta}}{\sqrt{|\log \varepsilon|}} \qquad \hat{\beta} \in (0, \infty)$

I. Phase transition for SHE and KPZ [CSZ 17]

Solutions $u^\varepsilon(t,x)$ and $h^\varepsilon(t,x)$ undergo phase transition at $\hat{\beta}_c = \sqrt{2\pi}$

II. Sub-critical regime of SHE and KPZ [CSZ 17] [CSZ 20+]

$(\hat{\beta} < \hat{\beta}_c)$ LLN + fluctuations of solutions $u^\varepsilon(t,x)$ and $h^\varepsilon(t,x)$

III. Critical regime of SHE [CSZ 19]

$(\hat{\beta} = \hat{\beta}_c)$ Non-trivial limit(s) of SHE $u^\varepsilon(t,x)$ via moment bounds

References

With Rongfeng Sun and Nikos Zygouras:

- [CSZ 17] *Universality in marginally relevant disordered systems*
  Ann. Appl. Probab. 2017
- [CSZ 19] *On the moments of the (2+1)-dimensional directed polymer and Stochastic Heat Equation in the critical window*
  Commun. Math. Phys. 2019
- [CSZ 20+] *The two-dimensional KPZ equation in the entire subcritical regime*
  Ann. Probab. (to appear)

$(d = 2)$ [Bertini Cancrini 98] [Dell'Antonio Figari Teta 94] [Chatterjee Dunlap 18] [Gu 18] [Gu Quastel Tsai 19]

$(d \geq 3)$ [Magnen Unterberger 18] [Gu Ryzhik Zeitouni 18] [Dunlap Gu Ryzhik Zeitouni 19] [Comets Cosco Mukherjee 18 19a 19b]

Main result I. Phase transition

Space dimension $d = 2$ $\qquad \beta_\varepsilon = \frac{\hat\beta}{\sqrt{|\log \varepsilon|}} \qquad \hat\beta \in (0, \infty)$

Theorem (Phase transition for SHE) [CSZ 17]

- $(\hat\beta < \sqrt{2\pi})$ $\qquad u^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} \exp\left(\sigma\, Z - \frac{1}{2}\,\sigma^2\right)$

$$Z \sim N(0,1) \qquad \sigma^2 := \log \frac{2\pi}{2\pi - \hat\beta^2}$$

$u^\varepsilon(t,x_i) \xrightarrow[\varepsilon \downarrow 0]{d}$ asympt. independent (for distinct points x_i's)

- $(\hat\beta \geq \sqrt{2\pi})$ $\qquad u^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} 0$

Main result I. Phase transition

Space dimension $d = 2$ $\qquad \beta_\varepsilon = \frac{\hat{\beta}}{\sqrt{|\log \varepsilon|}} \qquad \hat{\beta} \in (0, \infty)$

Theorem (Phase transition for KPZ) [CSZ 17]

- $(\hat{\beta} < \sqrt{2\pi})$ $\qquad h^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} \sigma Z - \frac{1}{2}\sigma^2$

  $$Z \sim N(0,1) \qquad \sigma^2 := \log \frac{2\pi}{2\pi - \hat{\beta}^2}$$

  $h^\varepsilon(t,x_i) \xrightarrow[\varepsilon \downarrow 0]{d}$ asympt. independent (for distinct points x_i's)

- $(\hat{\beta} \geq \sqrt{2\pi})$ $\qquad h^\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} -\infty$

Law of large numbers

Sub-critical regime $\hat{\beta} < \sqrt{2\pi}$ (as $\varepsilon \downarrow 0$)

- $\mathbb{E}[u^\varepsilon(t,x)] \equiv 1$
- $u^\varepsilon(t,x)$ asymptotically independent for distinct x's

- $\mathbb{E}[h^\varepsilon(t,x)] \equiv -\frac{1}{2}\sigma^2 + o(1)$
- $h^\varepsilon(t,x)$ asymptotically independent for distinct x's

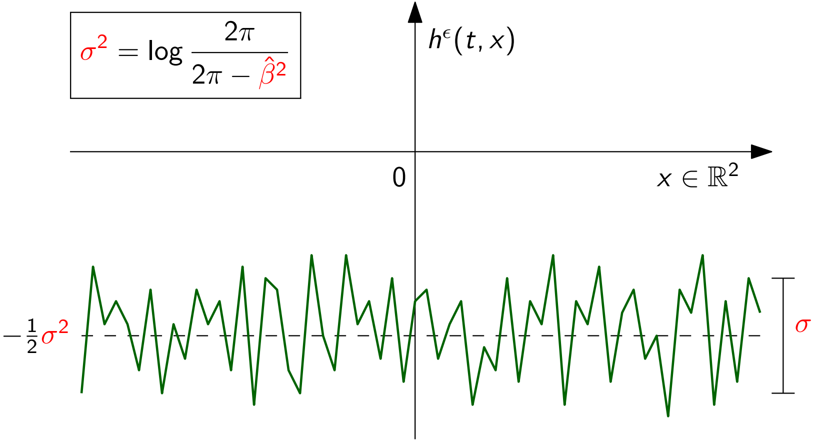
Corollary: LLN as $\varepsilon \downarrow 0$ ($\hat{\beta} < \sqrt{2\pi}$)

as a distribution on $\mathbb{R}^2$ $\quad u^\varepsilon(t,\cdot) \xrightarrow{d} 1 \qquad h^\varepsilon(t,\cdot) \xrightarrow{d} -\frac{1}{2}\sigma^2$

$$\int_{\mathbb{R}^2} h^\varepsilon(t,x)\,\phi(x)\,\mathrm{d}x \xrightarrow{d} -\frac{1}{2}\sigma^2 \int_{\mathbb{R}^2} \phi(x)\,\mathrm{d}x$$

A picture

$$\sigma^2 = \log \frac{2\pi}{2\pi - \hat{\beta}^2}$$



Main result II. Fluctuations for SHE

Rescaled SHE solution $\mathcal{U}^\varepsilon(t,x) := \big(u^\varepsilon(t,x) - 1\big)/\beta_\varepsilon$

$$\partial_t \mathcal{U}^\varepsilon = \tfrac{1}{2}\Delta_x \mathcal{U}^\varepsilon + \xi^\varepsilon + \beta_\varepsilon \mathcal{U}^\varepsilon \xi^\varepsilon$$

Theorem (Fluctuations for SHE) [CSZ 17]

for $\hat\beta < \sqrt{2\pi}$ $\quad \mathcal{U}^\varepsilon(t,\cdot) \xrightarrow[\varepsilon\downarrow 0]{d} v(t,\cdot)$ as a distrib.

v = Gaussian = solution of additive SHE (Edwards-Wilkinson)

$$\partial_t v = \tfrac{1}{2}\Delta_x v + \gamma\,\xi \qquad \text{where} \qquad \gamma = \sqrt{\frac{2\pi}{2\pi - \hat\beta^2}} > 1$$

Remarkably $\beta_\varepsilon \mathcal{U}^\varepsilon \xi^\varepsilon$ does not vanish as $\varepsilon \downarrow 0$! $\quad (\beta_\varepsilon \to 0)$

Converges to $\sqrt{\gamma^2 - 1}\,\xi'$ independent white noise $\quad$ ("resonances")

Main result II. Fluctuations for KPZ

Rescaled KPZ solution $\mathcal{H}^\varepsilon(t,x) := \big(h^\varepsilon(t,x) - \mathbb{E}[h^\varepsilon]\big)/\beta_\varepsilon$

Theorem (Fluctuations for KPZ) [CSZ 20+]

for $\hat\beta < \sqrt{2\pi}$ $\qquad \mathcal{H}^\varepsilon(t,\cdot) \xrightarrow[\varepsilon\downarrow 0]{d} v(t,\cdot)$ as a distrib.

$v =$ Gaussian $=$ solution of additive SHE (Edwards-Wilkinson)

$$\partial_t v = \tfrac{1}{2}\Delta_x v + \gamma\,\xi \qquad \text{where} \qquad \gamma = \sqrt{\frac{2\pi}{2\pi - \hat\beta^2}} > 1$$

$$\partial_t \mathcal{H}^\varepsilon = \tfrac{1}{2}\Delta_x \mathcal{H}^\varepsilon + \xi^\varepsilon + \underbrace{\beta_\varepsilon\left(|\nabla \mathcal{H}^\varepsilon|^2 - c\,\varepsilon^{-2}\right)}_{\text{converges to indep. white noise}}$$

Fluctuations: from SHE to KPZ

Proof based on Wiener Chaos expansions, not available for KPZ

$$h^\varepsilon(t,x) = \log u^\varepsilon(t,x) \qquad \text{(Cole-Hopf)}$$

We might hope that

$$h^\varepsilon(t,\cdot) = \log\big(1 + (u^\varepsilon(t,\cdot) - 1)\big) \approx \big(u^\varepsilon(t,\cdot) - 1\big) \ ?$$

NO, because $u^\varepsilon(t,x)$ is not close to 1 pointwise

Correct comparison (non trivial!)

$$h^\varepsilon(t,\cdot) - \mathbb{E}[h^\varepsilon] \approx \big(u^\varepsilon(t,\cdot) - 1\big)$$

Sketch of the proof

We approximate $u^\varepsilon(t,x)$ by "local version" $\tilde{u}^\varepsilon(t,x)$ which samples noise ξ in a tiny region around (t,x)

Then we approximate KPZ solution $h^\varepsilon(t,x)$ by Taylor expansion

$$h^\varepsilon \;=\; \log u^\varepsilon \;=\; \log \tilde{u}^\varepsilon + \log\left(1 + \frac{u^\varepsilon - \tilde{u}^\varepsilon}{\tilde{u}^\varepsilon}\right) \approx \; \log \tilde{u}^\varepsilon + \frac{u^\varepsilon - \tilde{u}^\varepsilon}{\tilde{u}^\varepsilon} + R^\varepsilon$$

- Remainder is small $\big(R^\varepsilon(t,\cdot) - \mathbb{E}[R^\varepsilon]\big)/\beta_\varepsilon \xrightarrow{d} 0$
- Local dependence of $\tilde{u}^\varepsilon$ $\big(\log \tilde{u}^\varepsilon(t,\cdot) - \mathbb{E}[\log \tilde{u}^\varepsilon]\big)/\beta_\varepsilon \xrightarrow{d} 0$
- Crucial approximation $\dfrac{u^\varepsilon(t,\cdot) - \tilde{u}^\varepsilon(t,\cdot)}{\tilde{u}^\varepsilon(t,\cdot)} \approx u^\varepsilon(t,\cdot) - 1$

Some Comments

Key tools in our approach are

- Wiener chaos + Renewal Theory $\rightsquigarrow$ sharp L^2 computations
- 4th Moment Theorems to prove Gaussianity
- Hypercontractivity + Concentration of Measure

Alternative proof by [Gu 18] via Malliavin calculus (for small $\hat{\beta}$)

[Chatterjee and Dunlap 18] first considered fluctuations for KPZ and they proved tightness of $\mathcal{H}^\varepsilon$ (for small $\hat{\beta}$)

We identify the limit of $\mathcal{H}^\varepsilon$ (EW) (for every $\hat{\beta} < \sqrt{2\pi}$)

Results in dimensions $d \geq 3$ by many authors (unknown critical point)

A variation on KPZ

Chatterjee and Dunlap [CD 18] looked at a different KPZ

$$\partial_t \tilde{h}^\varepsilon = \tfrac{1}{2}\Delta \tilde{h}^\varepsilon + \tfrac{1}{2}\, \beta_\varepsilon \, |\nabla \tilde{h}^\varepsilon|^2 + \xi^\varepsilon$$

where β_ε tunes the strength of the non-linearity

In our setting, β_ε tunes the strength of the noise

$$\partial_t h^\varepsilon = \tfrac{1}{2}\Delta h^\varepsilon + \tfrac{1}{2}|\nabla h^\varepsilon|^2 + \beta_\varepsilon \, \xi^\varepsilon - c\, \beta_\varepsilon^2 \, \varepsilon^{-d}$$

The two equations have the same fluctuations

$$\tilde{h}^\varepsilon(t,x) - \mathbb{E}[\tilde{h}^\varepsilon] = \frac{1}{\beta_\varepsilon}\left(h^\varepsilon(t,x) - \mathbb{E}[h^\varepsilon]\right) = \mathcal{H}^\varepsilon(t,x)$$

The critical regime

What about the critical point $\hat{\beta} = \sqrt{2\pi}$? [Bertini Cancrini 98]

$$\beta_\varepsilon = \frac{\sqrt{2\pi}}{\sqrt{|\log \varepsilon|}} \left(1 + \frac{\vartheta}{|\log \varepsilon|}\right) \qquad \text{with} \quad \vartheta \in \mathbb{R}$$

So-called critical window

Key conjecture for critical SHE

$$u^\varepsilon(t, \cdot) \xrightarrow[\varepsilon \downarrow 0]{d} \mathcal{U}_\vartheta(t, \cdot) \qquad \text{(random distribution on } \mathbb{R}^2\text{)}$$

Nothing known for KPZ solution $h^\varepsilon(t, \cdot)$

Second moment

Known results [Bertini Cancrini 98]

$$\mathbb{E}\big[\langle u^\varepsilon(t,\cdot),\phi\rangle\big] \equiv \langle 1,\phi\rangle \qquad \sup_{\varepsilon>0} \mathbb{E}\big[\langle u^\varepsilon(t,\cdot),\phi\rangle^2\big] < \infty$$

$$\mathbb{E}\big[\langle u^\varepsilon(t,\cdot),\phi\rangle^2\big] \xrightarrow[\varepsilon\downarrow 0]{} \langle \phi,\, K\phi\rangle \qquad K(x,x') \sim C \log \tfrac{1}{|x-x'|}$$

Corollary: tightness

$\exists$ subseq. limits $\quad u^{\varepsilon_k}(t,\cdot) \xrightarrow[k\to\infty]{d} \mathcal{U}(t,\cdot) \quad$ as random distributions

Could the limit be trivial $\mathcal{U}(t,\cdot) \equiv 1$?

Main result III. Third moment in the critical window

We computed the sharp asymptotics of third moment

Theorem [CSZ 19]

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\left[\langle u^\varepsilon(t,\cdot),\phi\rangle^3\right] = C(\phi) < \infty$$

Corollary

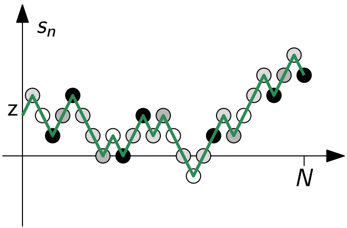
Any subseq. limit $u^{\varepsilon_k}(t,\cdot) \xrightarrow{d} \mathcal{U}(t,\cdot)$ has the same covariance $K(x,x')$

$$\rightsquigarrow \quad \mathcal{U}(t,\cdot) \not\equiv 1 \quad \text{is non-trivial}$$

Recently [Gu Quastel Tsai 19] proved convergence of all moments exploiting link with delta Bose gas [Dell'Antonio Figari Teta 94]

Directed Polymers

We can study the SHE solution $u^\varepsilon(t,x)$ via Directed Polymers



- $s = (s_n)_{n \geq 0}$ simple random walk path
- Indep. standard Gaussian RVs $\omega(n,x)$ (Disorder)
- $H_N(\omega, s) := \sum_{n=1}^{N} \omega(n, s_n)$

Directed Polymer Partition Functions ($N \in \mathbb{N}$, $z \in \mathbb{Z}^d$)

$$\mathscr{Z}_\beta(N,z) := \frac{1}{(2d)^N} \sum_{\substack{s=(s_0,\ldots,s_N) \\ \text{s.r.w. path with } s_0=z}} e^{\beta H_N(\omega,s) - \frac{1}{2}\beta^2 N}$$

Directed Polymers and SHE

Partition functions $\mathscr{Z}_\beta(N,z)$ are discrete analogues of $u^\varepsilon(t,x)$ (SHE)

- They solve a lattice version of the SHE
- They look very close to Feynman-Kac formula for SHE

Theorem

We can approximate (in L^2)

$$u^\varepsilon(t,x) \approx \mathscr{Z}_\beta(N,z) \qquad \text{and} \qquad h^\varepsilon(t,x) \approx \log \mathscr{Z}_\beta(N,z)$$

where $N = \varepsilon^{-2} t\,, \quad z = \varepsilon^{-1} x\,, \quad \beta_\varepsilon = \varepsilon^{\frac{d-2}{2}}\,\beta$

Our results are first proved for partition functions $\mathscr{Z}_\beta(N,z)$

Feynman-Kac for SHE

Recall the mollified SHE
$$\begin{cases} \partial_t u^\varepsilon = \frac{1}{2}\Delta u^\varepsilon + \beta_\varepsilon \, u^\varepsilon \, (\xi * \varrho^\varepsilon) \\ u^\varepsilon(0,\cdot) \equiv 1 \end{cases}$$

A stochastic Feynman-Kac formula holds

$$u^\varepsilon(t,x) \overset{d}{=} \mathrm{E}_{\varepsilon^{-1}x}\left[\exp\left(\beta_\varepsilon \, \varepsilon^{-\frac{d-2}{2}} \int_0^{\varepsilon^{-2}t} \int_{\mathbb{R}^2} \varrho(B_s - y)\, \xi(\mathrm{d}s, \mathrm{d}y) \; - \; \text{q.v.} \right) \right]$$

where $\varrho \in C_c^\infty(\mathbb{R}^d)$ is the mollifier and $B = (B_s)_{s\geq 0}$ is Brownian motion

We can identify $u^\varepsilon(t,x) \approx \mathscr{Z}_\beta(N,z)$ with

$$N = \varepsilon^{-2} t \qquad z = \varepsilon^{-1} x \qquad \beta_\varepsilon = \varepsilon^{\frac{d-2}{2}} \beta$$

In conclusion

Directed Polymers provides a friendly framework for our PDEs

Our results are first proved for Directed Polymer, then for SHE and KPZ

All mentioned tools have "discrete stochastic analysis" analogues:

Polynomial Chaos, 4th Moment Theorems,

Concentration Inequalities, Hypercontractivity

Probabilistic arguments are more transparent in a discrete setting

Robustness + Universality

Next challenges

- Critical regime $\hat{\beta} = \sqrt{2\pi}$
- Robust (pathwise) analysis of sub-critical regime $\hat{\beta} < \sqrt{2\pi}$

Thanks.