Scaling Limits and Universality for Random Pinning Models

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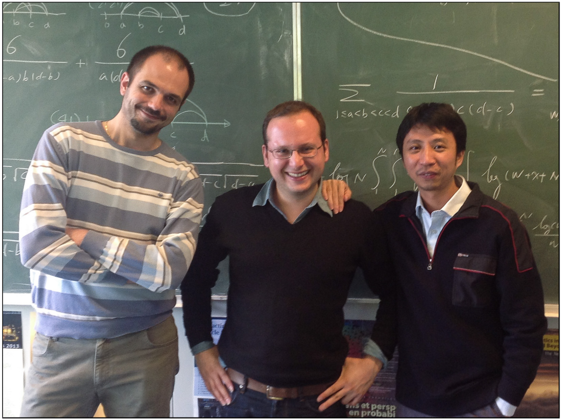
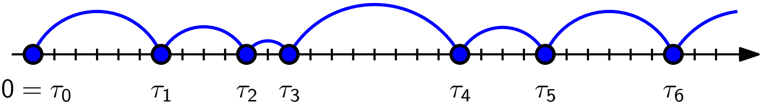
Coworkers



Joint work with Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

Outline

1st ingredient: renewal process



Discrete renewal process $\tau = \{0 = \tau_0 < \tau_1 < \tau_2 < \ldots\} \subseteq \mathbb{N}_0$

Gaps $(\tau_{i+1} - \tau_i)_{i \geq 0}$ are i.i.d. and finite (for simplicity)

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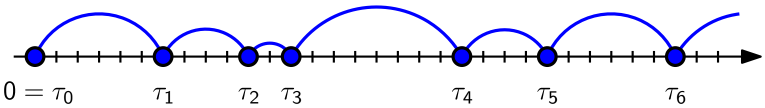

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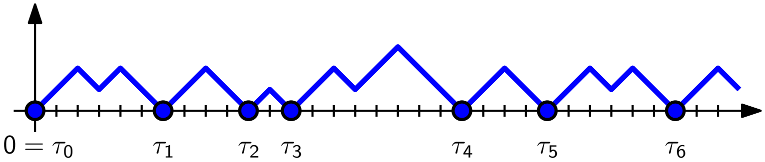

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$\tau = \{n \in \mathbb{N}_0 : \ X_n = 0\}$ zero level set of a Markov chain

(SRW on $\mathbb{Z}^d$, Bessel RWs on $\mathbb{N}_0$)

2nd ingredient: disorder (or charges)

Disorder $\omega = (\omega_i)_{i \in \mathbb{N}}$: i.i.d. real random variables with law $\mathbb{P}$

$$\Lambda(\beta) := \log \mathbb{E}[e^{\beta \omega_1}] < \infty \qquad \mathbb{E}[\omega_1] = 0 \qquad \mathbb{V}\mathrm{ar}[\omega_1] = 1$$

Gaussian case: $\omega_i \sim \mathcal{N}(0,1)$ $\Lambda(\beta) = \frac{1}{2}\beta^2$

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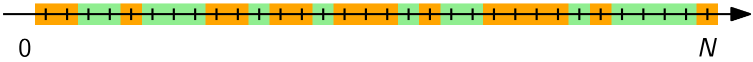
Quenched disorder: (τ, P) and $(\omega, \mathbb{P})$ are independent

Sample a typical ω and use it as a "random environment" for τ

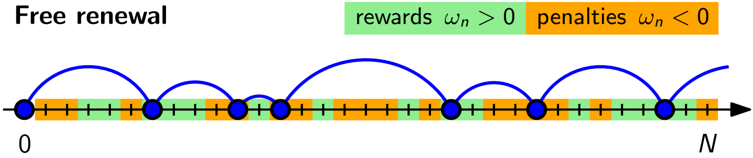
The discrete pinning model

rewards $\omega_n > 0$ penalties $\omega_n < 0$



The discrete pinning model

Free renewal



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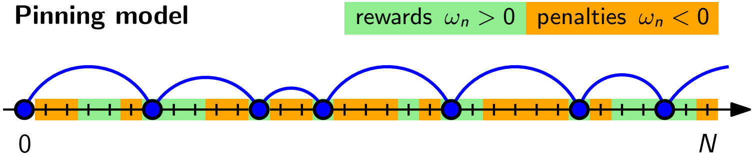
Pinning model



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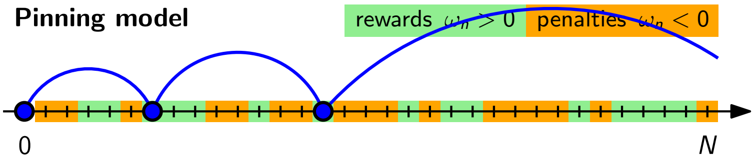

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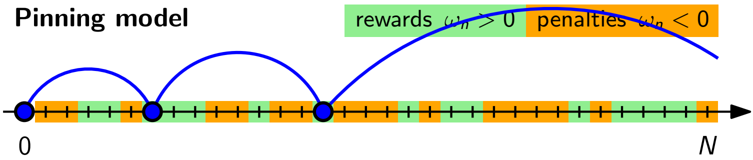

$N \in \mathbb{N}$ (system size) $\quad \beta \geq 0,\ h \in \mathbb{R}$ (disorder strength, bias)

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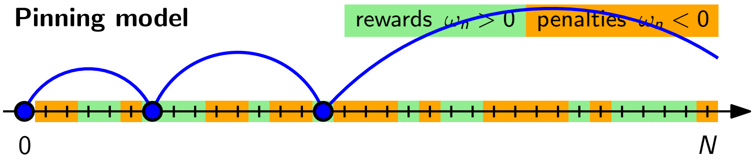

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Gibbs change of measure $\mathrm{P}^{\omega}_{N,\beta,h}$ of the renewal distribution P

$$\frac{\mathrm{dP}^{\omega}_{N,\beta,h}}{\mathrm{dP}}(\tau) := \frac{1}{Z^{\omega}_{N,\beta,h}} \exp\left(H^{\omega}_{N,\beta,h}(\tau) \right)$$

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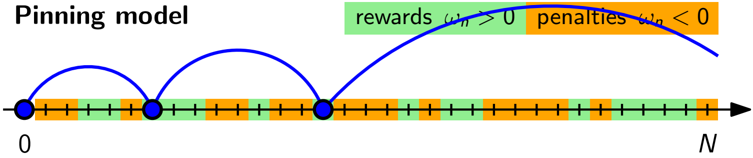

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— Hamiltonian: sum of rewards/penalties visited by τ

normalization constant (partition function)

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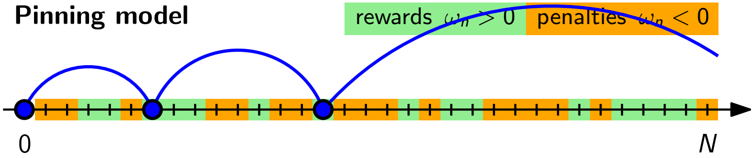

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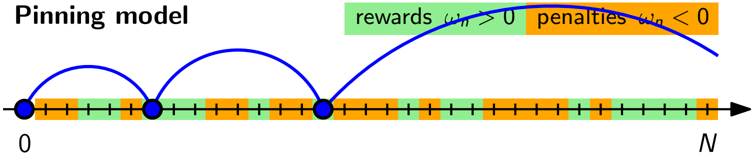

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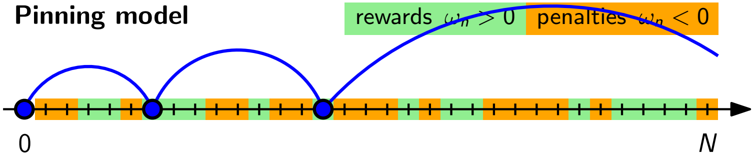

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reward/penalty to visit site n

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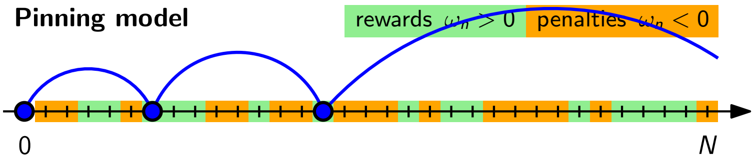

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$H^{\omega}_{N,\beta,h}(\tau)$ = sum of the rewards/penalties visited by τ

The phase transition

How are the typical paths τ of the pinning model $\mathrm{P}^{\omega}_{N,\beta,h}$?

Contact number $\mathcal{C}_N := \big|\tau \cap (0, N]\big| = \sum_{n=1}^{N} \mathbb{1}_{\{n \in \tau\}}$

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$\exists$ continuous, non decreasing, deterministic critical curve $h_c(\beta)$:

- *Localized regime: for $h > h_c(\beta)$ one has $\mathcal{C}_N \approx N$*
- *Deocalized regime: for $h < h_c(\beta)$ one has $\mathcal{C}_N = O(\log N)$*

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$$\exists \mu_{\beta,h} > 0 : \quad \mathrm{P}^{\omega}_{N,\beta,h}\left(\left| \frac{\mathcal{C}_N}{N} - \mu_{\beta,h} \right| > \epsilon \right) \xrightarrow[N\to\infty]{} 0 \,, \quad \omega\text{–a.s.}$$

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1. Discrete pinning model
2. Weak disorder regime
3. Continuum partition function
4. Sketch of the proof
5. Continuum pinning model

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correlation length diverges

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YES, but the road is not straight...

The partition function

As a first step, we look at the partition function

$$Z_{N,\beta,h}^{\omega} := \mathrm{E}\left[e^{H_N(\tau)}\right] = \mathrm{E}\left[e^{\sum_{n=1}^{N}(h+\beta\omega_n-\Lambda(\beta))\mathbb{1}_{\{n\in\tau\}}}\right]$$

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- Free energy $\quad \mathrm{F}(\beta,h) := \lim_{N\to\infty} \frac{1}{N}\log Z^{\omega}_{N,\beta,h} \geq 0 \qquad \mathbb{P}(\mathrm{d}\omega)\text{-a.s.}$

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- Critical curve $h_c(\beta) = \sup\{h\in\mathbb{R}:\ \mathrm{F}(\beta,h)=0\}$

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- Free energy $\quad \mathrm{F}(\beta,h) := \lim_{N\to\infty} \frac{1}{N}\log Z^{\omega}_{N,\beta,h} \geq 0 \qquad \mathbb{P}(\mathrm{d}\omega)\text{-a.s.}$

$$Z^{\omega}_{N,\beta,h} \geq \mathrm{E}\Big[e^{H_N(\tau)}\,\mathbb{1}_{\{\tau\cap(0,N]=\emptyset\}}\Big] = \mathrm{P}(\tau\cap(0,N]=\emptyset) \sim \frac{(const.)}{N^{\alpha}}$$

- Critical curve $\quad h_c(\beta) = \sup\{h\in\mathbb{R} : \ \mathrm{F}(\beta,h)=0\}$

Our first result

Taking $N\to\infty$ and $\beta_N, h_N \to 0$ with appropriate rates, $Z^{\omega}_{N,\beta_N,h_N}$ converges in distribution to a universal limit

Outline

Continuum partition function: preparation

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$$\text{if } \alpha \in \left(\tfrac{1}{2}, 1\right) : \quad \begin{cases} \beta_N := \dfrac{\hat{\beta}}{N^{\alpha - 1/2}} \\ h_N := \dfrac{\hat{h}}{N^{\alpha}} \end{cases}$$

$$\left(h_N \approx \beta_N^{\frac{2\alpha}{2\alpha - 1}} \right)$$

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if $\alpha > 1$: $\begin{cases} \beta_N := \dfrac{\hat\beta}{\sqrt{N}} \\ h_N := \dfrac{\hat h}{N} \end{cases}$

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Continuum disorder (replacing $(\beta\omega_i + h)_i$)

Take a standard Brownian motion $(W_t)_{t\ge0}$ and set

$$W_t^{\hat\beta,\hat h} := \hat\beta\, W_t + \hat h\, t \qquad \text{(BM with drift)}$$

Continuum partition function: main result

Theorem [C., Sun, Zygouras] – inspired by [Alberts, Khanin, Quastel (2012)]

- (Functional) convergence of discrete partition function

$$Z^{\omega}_{\hat{t}N,\beta_N,h_N} \xrightarrow[N\to\infty]{d} \mathscr{Z}^{W}_{\hat{t},\hat{\beta},\hat{h}} \quad =: \text{continuum partition function}$$

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with explicit function $\varphi^{(k)}$ (scaling of renewal correlations)

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- Allows to define the continuum pinning model (cf. later)

Outline

A direct approach?

Discrete partition function: $Z^{\omega}_{N,\beta,h} := \mathrm{E}\Big[e^{\sum_{n=1}^{N}(h+\beta\omega_n-\Lambda(\beta))\mathbb{1}_{\{n\in\tau\}}} \Big]$

Replace $\sum_{n=1}^{N} \rightsquigarrow \int_0^t \mathrm{d}s$ $\quad \Lambda(\beta) \rightsquigarrow \frac{1}{2}\beta^2$ $\quad \omega_n \rightsquigarrow ?$ $\quad \tau \rightsquigarrow ?$

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  [Zero level set of Brownian Motion ($\alpha = \frac{1}{2}$) or Bessel process]

A naive guess

Continuum partition function?

$$\mathscr{Z}^{W}_{\hat{t},\hat{\beta},\hat{h}} := \mathrm{E}\left[\exp\left(\int_0^{\hat{t}} (\hat{h} + \hat{\beta}\,\dot{W}_s - \tfrac{1}{2}\hat{\beta}^2)\,\mathbb{1}_{\{s\in\hat{\tau}\}}\,\mathrm{d}s \right) \right]$$

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- For $\alpha \in (0,1)$ the random set $\hat{\tau}$ has zero Lebesgue measure

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  Guess: $$\mathscr{Z}^{W}_{\hat{t},\hat{\beta},\hat{h}} \overset{d}{=} \exp\left[\hat{h} \frac{\hat{t}}{\mathrm{E}(\tau_1)} + \hat{\beta} W_{\frac{\hat{t}}{\mathrm{E}(\tau_1)}} - \tfrac{1}{2}\hat{\beta}^2 \frac{\hat{t}}{\mathrm{E}(\tau_1)} \right]$$

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  No! Some care is required.

Sketch of the proof (1/3)

High temperature (cluster) expansion:

$$Z^{\omega}_{N,\beta,h} = \mathrm{E}\left[\prod_{n=1}^{N} e^{(h+\beta\omega_n-\Lambda(\beta))\mathbb{1}_{\{n\in\tau\}}} \right]$$

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$$= 1 + \sum_{k=1}^{N} \sum_{1\le n_1<\ldots<n_k\le N} g(\omega_{n_1})\cdots g(\omega_{n_k})\,\mathrm{P}(\{n_1,\ldots,n_k\}\subseteq\tau)$$

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$$\mathbb{E}[g(\omega_n)] = h + o(h) \qquad \mathbb{V}\mathrm{ar}[g(\omega_n)] = \beta^2 + o(\beta^2)$$

Sketch of the proof (2/3)

Pretend that $g(\omega_n) \sim \mathcal{N}(h, \beta)$ and look at the first term $(k = 1)$

$$\sum_{n=1}^{N} g(\omega_n) \, \mathrm{P}(n \in \tau)$$

Sketch of the proof (2/3)

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$$\sum_{n=1}^{N} g(\omega_n)\, \mathrm{P}(n \in \tau) \sim \mathcal{N}\left(h \sum_{n=1}^{N} \mathrm{P}(n \in \tau),\ \beta^2 \sum_{n=1}^{N} \mathrm{P}(n \in \tau)^2 \right)$$

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- Mean and variance $O(1)$ $\rightsquigarrow$ Choice of $\beta = \beta_N, \quad h = h_N$

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Set now $\beta = \beta_N$, $h = h_N$ and conclude ($\frac{1}{2} < \alpha < 1$):

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$$\approx 1 + \sum_{k=1}^{\infty} \int \cdots \int_{0<t_1<\cdots<t_k<1} \frac{C^k \, \mathrm{d}W_{t_1}^{\hat\beta,\hat h} \cdots \mathrm{d}W_{t_k}^{\hat\beta,\hat h}}{t_1^{1-\alpha}(t_2-t_1)^{1-\alpha} \cdots (t_k - t_{k-1})^{1-\alpha}}$$

Recall: $\mathrm{d}W_t^{\hat\beta,\hat h} = \hat\beta \, \mathrm{d}W_t + \hat h \, \mathrm{d}t$

$$\mathrm{P}(\{n_1, \ldots, n_k\} \subseteq \tau) = \mathrm{P}(n_1 \in \tau)\mathrm{P}(n_2 - n_1 \in \tau) \cdots \mathrm{P}(n_k - n_{k-1} \in \tau)$$
$$\sim \frac{C^k}{n_1^{1-\alpha}(n_2 - n_1)^{1-\alpha} \cdots (n_k - n_{k-1})^{1-\alpha}}$$

How to make this rigorous? Lindeberg's principle

- $\zeta_1, \zeta_2, \ldots$ independent rvs in L^2: $\mathbb{E}(\zeta_i) = \mu_i$, $\mathbb{V}\mathrm{ar}(\zeta_i) = 1$

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- every variable ζ_i has small influence on Ψ:

$$\mathrm{Inf}_i(\Psi) := \mathbb{E}\left[\mathbb{V}\mathrm{ar}\left(\Psi \middle| \zeta_j, j \neq i\right)\right] = \sum_{\{n_1,\ldots,n_k\} \ni i} \psi(n_1, \ldots, n_k)^2 \ll 1$$

Outline

How to define the continuum pinning model ?

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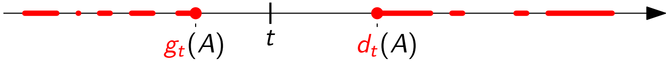
No Hamiltonian $\rightsquigarrow$ No Gibbs measure wrt $\mathscr{P}$

Alternative definition (using continuum partition function)

Finite-dimensional distributions (f.d.d.)

Every closed $A \subseteq \mathbb{R}$ is identified by $t \mapsto g_t(A)$, $t \mapsto d_t(A)$



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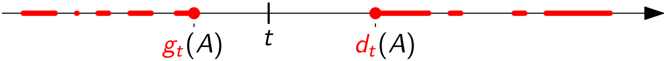

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Random closed subsets of $\mathbb{R}$ (= probabilities on Ω_0) characterized by joint distributions of $(g_{t_1}, d_{t_1}, \ldots, g_{t_k}, d_{t_k})$ for $k \in \mathbb{N}$, $t_i \in \mathbb{R}$

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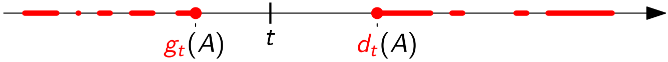

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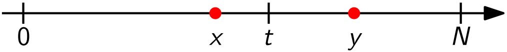
Continuum pinning model $\mathscr{P}^{W}_{T,\hat\beta,\hat h}$ will be defined by its f.d.d.

Discrete pinning model: f.d.d.

$Z^{\omega}_{[a,b]} := \mathrm{E}\left[e^{H^{\omega}_{[a,b]}(\tau)}\right]$ (free) $\qquad \widehat{Z}^{\omega}_{[0,x]} := \mathrm{E}\left[e^{H^{\omega}_{[0,x]}(\tau)} \mathbb{1}_{\{x \in \tau\}}\right]$ (constr.)



Discrete pinning model

$$\mathrm{P}^{\omega}_{N,\beta,h}\big(g_t = x,\ d_t = y\big) =$$

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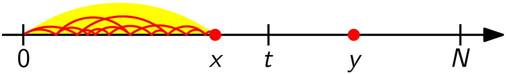

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$$\mathrm{P}^{\omega}_{N,\beta,h}\big(g_t = x,\ d_t = y\big) = \widehat{Z}^{\omega}_{[0,x]}$$

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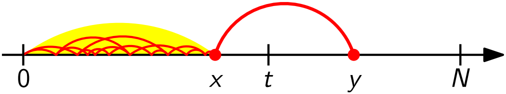

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$$\mathrm{P}^{\omega}_{N,\beta,h}\big(g_t = x,\ d_t = y\big) = \frac{\widehat{Z}^{\omega}_{[0,x]}\, K(y-x)}{}$$

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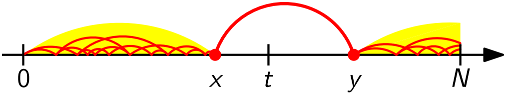

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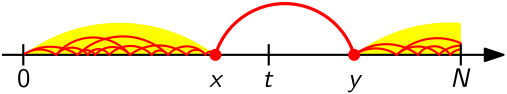

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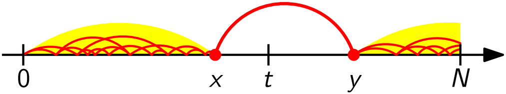

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F.d.d. expressed in terms of partition functions $\widehat{Z}$ and Z

(that have continuum limits $\widehat{\mathscr{Z}}$ and $\mathscr{Z}$ ...)

Continuum pinning model: definition

$\alpha \in (\frac{1}{2}, 1)$, $T > 0$, $W = (W_t)_{t \geq 0}$ BM, $\hat{\beta} \geq 0$, $\hat{h} \in \mathbb{R}$

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Conj.: $\mathscr{P}_T^W \xrightarrow[T \to \infty]{d} \mathscr{P}^W = \mathscr{P}_{\alpha, \frac{\hat{h}}{\hat{\beta}}}^W$ "disordered regenerative set"

Continuum pinning model: main properties

Theorem (scaling limit)

For any discrete pinning model $\mathrm{P}^{\omega}_{TN,\beta_N,h_N}$, the law of the rescaled discrete set $\frac{1}{N}\tau$ converges in distribution to $\mathscr{P}^{W}_{T,\hat{\beta},\hat{h}}$ (as $N \to \infty$)

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Any given a.s. property of the regenerative set $\mathscr{P}$ is an a.s. property of the continuum pinning model $\mathscr{P}^{W}_{T}$, for $\mathbb{P}$-a.e. W

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Theorem (singularity)

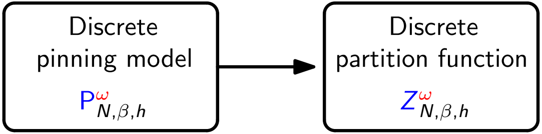
For $\mathbb{P}$-a.e. W, the continuum pinning model $\mathscr{P}^{W}_{T}$ and the regenerative set $\mathscr{P}$ are mutually singular probabilities on Ω_0

Conclusion

Discrete
pinning model

$\mathrm{P}^{\omega}_{N,\beta,h}$

Conclusion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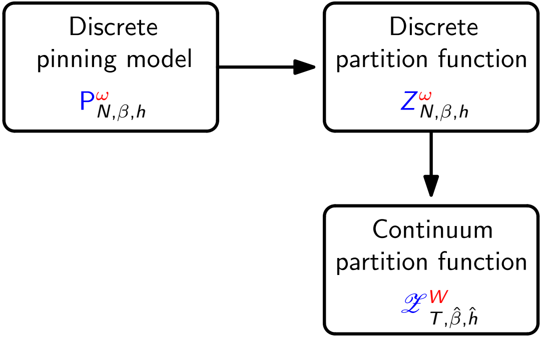
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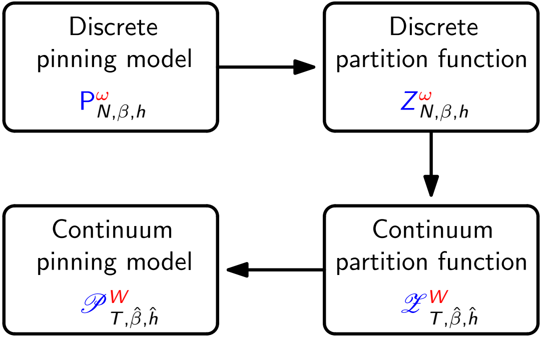
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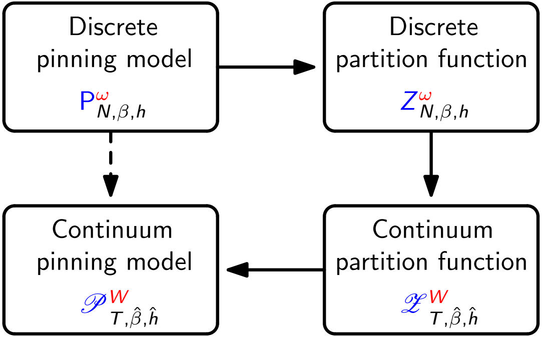
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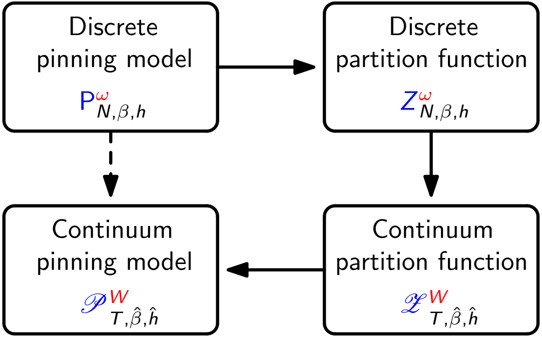

Open problems

- Universality of the critical line $h_c(\beta)$ at weak disorder

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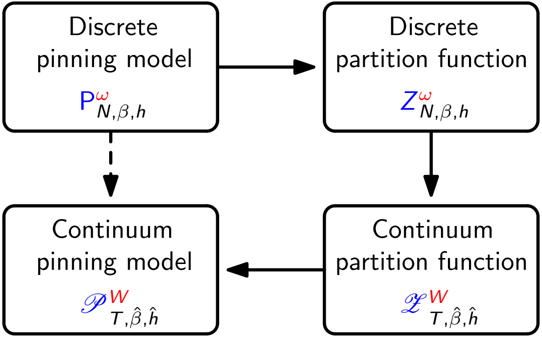

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- Infinite-volume continuum model: $\text{w-lim}_{T\to\infty} \mathscr{P}^{W}_{T,\hat\beta,\hat h}$?

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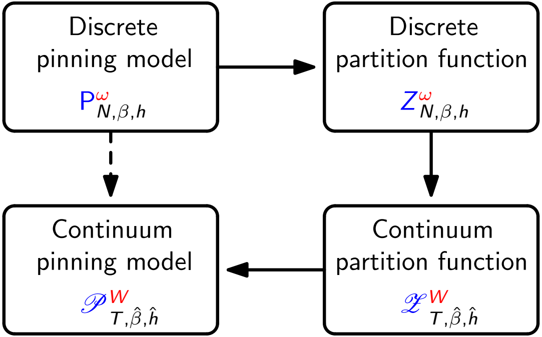

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- Marginal case $\alpha = \frac{1}{2}$ (in progress)

Thanks