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# Random walk models and probabilistic techniques for inhomogeneous polymer chains

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# Introduction

Modeling of polymer chains, that is long linear molecules made up of a sequence of simpler units called monomers, has, for a lot of time, received a lot of attention in physics, chemistry, biology, ... Mathematics belongs to this list too. For example, probabilistic models that naturally arise in statistical mechanics have been widely studied by mathematicians for the very challenging and novel problems that they pose. This is true to the extent that, in probability, the word *polymer* has become synonymous with *self–avoiding walk*, a basic and extremely difficult mathematical entity. The interaction of a polymer with the environment leads to even more challenging questions: these are often tackled in the framework of *directed walks*. Restricting attention to directed trajectories is a way of enforcing the self–avoiding constraint that leads to much more tractable models. Still, the interaction with the environment may quickly lead to extremely difficult questions.

A particularly interesting situation is that of an inhomogeneous polymer (or *copolymer*) in the proximity of an interface between two selective solvents. The polymer is inhomogeneous in that its monomers may differ in some characteristics and, consequently, the interaction with the solvents and the interface may vary from monomer to monomer. In interesting cases there can be a phase transition between a state in which the polymer sticks very close to the interface (localized regime) and a state in which it wanders away from it (delocalized regime). The typical mechanism underlying such phase transitions is an *energy/entropy competition*.

The main task of this Ph.D. thesis is to introduce and study *random walk models* of polymer chains with the purpose of understanding this competition in a deep and quantitative way. Since a random walk can be regarded as an example of an abstract polymer, the idea of modeling real polymers using random walks is quite natural and it has proved to be very successful.

The models we are going to consider are modifications of a basic model introduced in the late eighties by T. Garel, D. A. Huse, S. Leibler and H. Orland [**34**] that

in turn had translated into the language of theoretical physics ideas that were developing in the applied sciences. Despite the fact that the definition of these models is extremely elementary, their analysis is not simple at all. For a number of interesting issues there is still no agreement in the physical literature. From a mathematical viewpoint it has taken quite a lot of time and effort to rigorously derive their basic properties, and several interesting questions are still open.

In this Ph.D. thesis we present new results that answer some of these questions. The approach taken here is essentially probabilistic, and it is interesting to note how the analysis performed has required the application of a wide range of techniques including Large Deviations and Concentration Inequalities (Ch. 2), Perron–Frobenius Theory (Ch. 3), Renewal Theory (Ch. 4) and Fluctuation Theory for random walks (Ch. 5 and 6). A numerical and statistical study has also been performed (Ch. 2). Reciprocally, the study of the models stimulates the extension of these techniques, see, for instance, the Local Limit Theorem for random walks conditioned to stay positive presented in Chapter 6.

The thesis is organized as follows. The definition of the models we consider is given in detail in Chapter 1a, where we also give some motivation and we collect the known results from the literature. The following five chapters contain original results. A detailed outline of the thesis may be found in Section 5 of Chapter 1a.

# Introduction (en Français)

La modélisation des polymères, c'est-à-dire des longues chaînes de molécules élémentaires (les monomères), a fait l'objet de nombreuses études en physique, chimie, biologie, ... Les mathématiques ne font pas exception. De nombreux modèles probabilistes issus de la mécanique statistique ont ainsi été étudiés par les mathématiciens pour les problèmes très intéressants et complexes qu'ils posent. Cela est vrai au point que dans le domaine des probabilités le mot *polymère* est devenu synonyme de *marche aléatoire auto–évitante*, une entité mathématique élémentaire par sa définition mais stimulante par sa complexité. L'interaction d'un polymère avec le milieu conduit à des problèmes encore plus difficiles, souvent traités dans le cadre des *marches dirigées*. En effet, restreindre l'étude aux trajectoires dirigées permet d'obtenir simplement des polymères auto-évitants, tout en rendant techniquement possible l'analyse du modèle. D'autre part, l'introduction d'une interaction avec le milieu conduit à des situations tout à fait non triviales.

Une situation particulièrement intéressante est celle où un polymère inhomogène (ou *copolymère*) est placé à proximité d'une interface séparant deux solvants sélectifs. L'inhomogénéité du polymère signifie que tous les monomères ne sont pas équivalents, en particulier l'interaction avec les solvants et l'interface peut différer d'un monomère à l'autre. Dans les cas les plus intéressants il peut y avoir une transition de phase entre un état où le polymère reste très proche de l'interface (régime localisé) et un état où il évolue dans un des solvants, loin de l'interface (régime délocalisé). Typiquement, le mécanisme à la base de cette transition de phase est une *compétition entre énergie et entropie*.

L'objectif de cette thèse est de présenter plusieurs modèles de polymères élaborés à partir de *marches aléatoires* et de les étudier dans le but de comprendre cette compétition en profondeur, suivant une approche quantitative. Dans la mesure où la marche aléatoire est un exemple de polymère abstrait, l'idée de représenter les polymères par des marches aléatoires est plutôt naturelle et s'est révélée très efficace.

Les modèles qu'on considérera sont des modifications d'un modèle basique introduit à la fin des années quatre-vingts par T. Garel, D. A. Huse, S. Leibler et H. Orland [**34**], qui ont traduit dans le langage de la physique théorique les idées qui étaient en train de se développer dans les sciences appliquées. En dépit du fait que la définition de ces modèles est élémentaire, leur analyse est loin d'être facile. Ainsi, sur certaines questions fondamentales il n'y a pas encore de consensus parmi la communauté physique. Du côté mathématique un certain temps et beaucoup d'efforts ont été nécessaires pour établir rigoureusement les propriétés de base de ces modèles, et plusieurs questions restent toujours sans réponse.

Dans cette thèse nous présentons des résultats nouveaux qui donnent une réponse à quelques–unes de ces questions. L'approche adoptée est essentiellement probabiliste, et il est intéressant de remarquer comment dans l'analyse de ces modèles on a eu l'occasion d'appliquer une vaste gamme de techniques, y compris les Grandes Déviations et les Inégalités de Concentration (Ch. 2), la Théorie de Perron–Frobenius (Ch. 3), la Théorie du Renouvellement (Ch. 4) et la Théorie des Fluctuations pour les marches aléatoires (Ch. 5 et 6). Une étude numérique et statistique a aussi été réalisée (Ch. 2). Réciproquement, l'analyse de ces modèles stimule le développement de ces techniques : citons par exemple le Théorème Limite Local pour des marches aléatoires conditionnées à rester positives prouvé dans le Chapitre 6.

La thèse est organisée de la manière suivante. Les définitions des modèles que nous considérons sont données en détail dans le Chapitre 1b, où nous présentons de plus un certain nombre de raisons qui nous ont conduit à étudier ces modèles et résumons les résultats déjà établis dans le domaine. Les cinq chapitres suivants (écrits en Anglais) contiennent pour leur part des résultats originaux. Enfin, un résumé détaillé de la thèse est donné dans la Section 5 du Chapitre 1b.

CHAPTER 1a

# Inhomogeneous polymer chains

In this first chapter, we introduce the class of models that are the center of our analysis, providing some motivations for their study and recalling the known results in the literature. The exposition takes inspiration from [**35**].

## 1. Introduction and motivations

**1.1. Polymers and random walks.** The notion of polymer has originated in the field of chemistry to indicate a natural or synthetic compound consisting of large molecules which are made up of a linked series of repeated simple molecules called monomers. However this concept has spread out and nowadays polymers appear in a variety of different fields with the broader meaning of *linear structures* which are built up by *joining together* a large number of *simpler structures* (everything possibly in an abstract sense). A very relevant example of an abstract polymer is given by a *random walk*, where the increments are thought of as monomers.

As a matter of fact both the above meanings of polymers, the concrete one and the abstract one, are of interest to us. In fact our main purpose is to build probabilistic models based on random walks that try to mimic the phenomenology of true chemical polymers in some interesting situations. We stress from now that the models we are going to consider are *very simple* and nevertheless they pose very challenging problems.

**1.2. Copolymers at selective interfaces.** The polymer set–up to which we mainly dedicate our attention is the problem of a *copolymer in the proximity of an interface between two selective solvents*, say oil and water. "Copolymer" is simply a synonym of inhomogeneous polymer, that is a polymer whose monomers are not identical but can be of different types. In our case we suppose that the monomers can differ in only one characteristics: they may be hydrophobic (+) or hydrophilic (−) (see Figure 1a.1) and to be definite we assume that there is a majority of hydrophobic monomers.

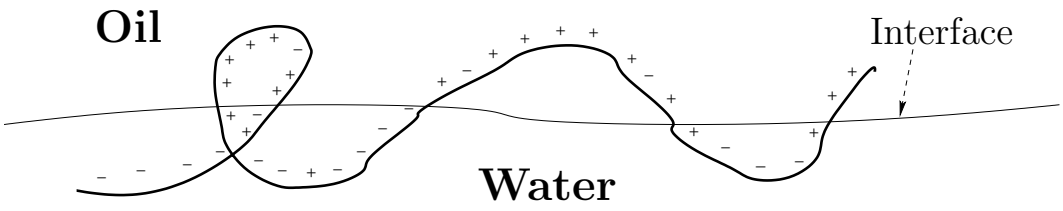


FIGURE 1A.1. A copolymer in the proximity of an interface between two selective solvents.

At first one could be led to think that the copolymer should prefer to live in oil, because of the substantial hydrophobicity of the chain. However a second scenario is also possible: the polymer could decide to stick very close to the interface, in order to place each monomer (or at least a big part of them) in the right solvent, that is + in oil and − in water. Observe that this second strategy produces an *energetic gain*, arising from the fact that a greater fraction of monomers is placed in the preferred solvent, but it also entails an *entropic cost*, because the polymer has access to a much smaller portion of the configuration space (the trajectories that stay close to the interface are much less than those who are free to wander in oil).

It should be clear that we are facing a typical energy/entropy competition: our aim is to build a probabilistic model of this situation following the paradigms of Statistical Mechanics. This will allow us a quantitative study of this competition, in order to decide –in function of the characteristic of the polymer chain and of other physical quantities, such as the temperature– which one is the winning strategy, that is the strategy followed by the polymer.

We stress from now that our interest is in describing the thermodynamic behavior of the copolymer *at a fixed time*: nothing will be said in this thesis about the problem of dynamical evolution.

**1.3. Random walk models.** Let us be a bit more specific about how to build a random walk model for our copolymer (the precise definitions will be given in the next section). We first take a random walk with values in $\mathbb{R}^d$ (or a sublattice of it) and we fix a large integer $N$, the size of the polymer. The idea is to look at the random walk trajectories up to epoch $N$ as describing the configurations of the polymer chain when there are no interactions. Then we modify the law of the walk

in the way prescribed by Statistical Mechanics, that is by giving to each trajectory an exponential weight (Boltzmann factor) which takes into account the interaction of the copolymer with the solvents. This new law is the *copolymer measure*, which describes the statistical behavior of the copolymer in thermodynamic equilibrium.

A basic issue is how to choose the random walk. To avoid trivialities we assume that the space in which the random walk lives is at least two–dimensional, that is $d \geq 2$. Moreover, since real polymers do occupy a physical space, one would rather like to deal with *self–avoiding walks*. In the lattice case, by this we mean a random walk which is conditioned not to visit again the sites it has already visited (defining self–avoiding walks in the continuum case requires some more care, but we don't want to get in details at this point).

However the point is that self–avoiding walks are a very difficult object to deal with. One possibility to bypass the problem is to impose a much simpler excluded volume constraint, by working with *directed walks*. By this we mean walks in which one of the coordinates is forced to be strictly increasing: a typical example is the case of $(1+m)$–dimensional directed walks, that is $\{(n, S_n)\}_n$ where $\{S_n\}_n$ is a random walk in $\mathbb{R}^m$. Although this may appear a too drastic solution, it has been widely used in the literature and it is the one that we will adopt too: more precisely, we will work with an $(1+1)$–dimensional directed walk.

Of course another possibility could be to give up any excluded volume constraint and to work with genuine random walks. However we stress that, for the model we consider, working with a $d$–dimensional random walk $\{S_n\}_n$ is equivalent to work with a suitable $(1+1)$–dimensional directed walk (this point will be clarified in Chapter 5). This consideration gives somehow more value to the directed walk approach.

## 2. Copolymers at selective interfaces

We are going to define a random walk model for the copolymer near a selective interface, that will be the main object of this work.

**2.1. Definition of the model.** Let $S = \{S_n\}_{n=0,1,\dots}$ be a simple symmetric random walk on $\mathbb{Z}$, that is

$$S_0 = 0 \qquad\qquad S_n = \sum_{j=1}^{n} X_j \,,$$

where $\{X_j\}_j$ is a sequence of IID random variables with $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = 1/2$. We take the directed walk point of view, looking at the trajectories of $\{(n, S_n)\}_n$ as the configurations of our polymer chain.

For $\lambda \geq 0$, $h \geq 0$, $N \in 2\mathbb{N}$ and $\omega = \{\omega_j\}_{j=1,2,\ldots} \in \mathbb{R}^{\mathbb{N}}$ we introduce the *copolymer measure* $\mathbf{P}_{N,\omega}^{\lambda,h}$ by giving the density w.r.t. $\mathbf{P}$:

$$\begin{aligned} \frac{\mathrm{d}\mathbf{P}_{N,\omega}^{\lambda,h}}{\mathrm{d}\mathbf{P}}(S) &= \frac{1}{\widetilde{Z}_{N,\omega}^{\lambda,h}} \exp\left(\mathcal{H}_{N,\omega}^{\lambda,h}(S)\right) \\ &:= \frac{1}{\widetilde{Z}_{N,\omega}^{\lambda,h}} \exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h) \operatorname{sign}(S_n)\right), \end{aligned} \tag{1a.1}$$

where $\operatorname{sign}(S_{2n})$ is set to be equal to $\operatorname{sign}(S_{2n-1})$ for any $n$ such that $S_{2n} = 0$ (this is a natural choice, as it is explained in the caption of Fig. 1a.2). The term $\widetilde{Z}_{N,\omega}^{\lambda,h}$ is simply a normalization constant to make $\mathbf{P}_{N,\omega}^{\lambda,h}$ a probability measure, that is

$$\widetilde{Z}_{N,\omega}^{\lambda,h} = \mathbf{E}\left[\exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h) \operatorname{sign}(S_n)\right)\right],$$

and it is called the *partition function* of the model.

We refer to the caption of Fig. 1a.2 for a visual interpretation of the copolymer measure (1a.1). The expression in the exponential, in the r.h.s. of (1a.1) is called the *Hamiltonian* (we stress that there is a minus sign of difference with respect to the physicists' conventions). Let us discuss the meaning of the parameters appearing in the definition of $\mathbf{P}_{N,\omega}^{\lambda,h}$:

- $N$ is of course the size of the copolymer;
- the parameter $\lambda$ tunes the overall strength of the interaction, and physically it corresponds to the inverse of the *temperature*;
- $\operatorname{sign}(\omega_n + h)$ tells whether the $n$–th monomer is hydrophobic $(+)$ or hydrophilic $(-)$, and $|\omega_n + h|$ gives the intensity of the hydrophobicity (or hydrophily) of the monomer. The reason for writing $(\omega_n + h)$ is to isolate in the parameter $h$ the overall asymmetry of the hydrophobicity/hydrophily, as it will be clear in a moment.

It remains to specify how to choose the sequence $\omega$, that will be referred to as the *charges* or the *environment* of our copolymer. Two possibilities will be considered in this thesis:

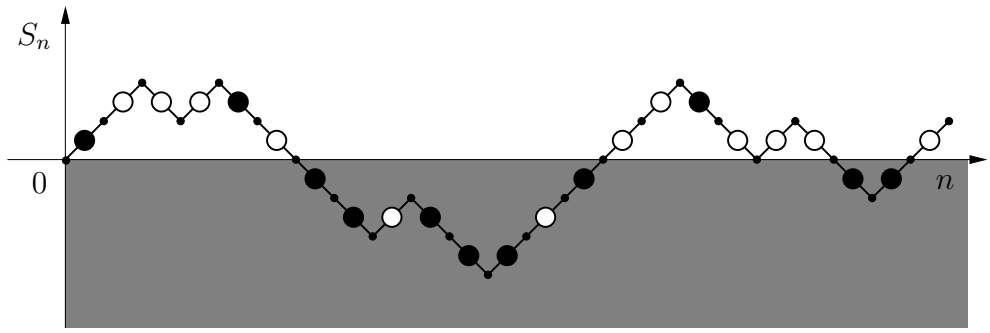


FIGURE 1A.2. The process we have introduced is a model for a non–homogeneous polymer, or *copolymer*, near an interface, the horizontal axis, between two selective solvents, say oil (white) and water (grey). In the drawing the monomer *junctions* are the small black rounds and the monomers are the bonds of the random walk. The big round in the middle of each monomer gives the sign of the charge (white = positive charge = hydrophobic monomer, black = negative charge = hydrophilic monomer). When $h > 0$ water is the unfavorable solvent and the question is whether the polymer is *delocalized* in oil or if it is still more profitable to place a large number of monomers in the preferred solvent, leading in such a way to the *localization at the interface* phenomenon. The conventional choice of sign(0) we have made reflects the fact that the charge is assigned to bonds rather than points.

- *periodic set–up:* $\omega$ is a fixed periodic sequence, that is for some $T \in \mathbb{N}$ we have $\omega_{2T+n} = \omega_n$ for all $n \in \mathbb{N}$: the least such $T$ will be denoted by $T_\omega$ and will be called the half–period of the sequence $\omega$ (the choice of an even period is due to the periodicity of the simple random walk). Up to a redefinition of the parameter $h$, we can (and will) assume that the sequence is centered, namely $\sum_{n=1}^{2T} \omega_n = 0$. Moreover to avoid trivialities we suppose that $\omega_{2n-1} + \omega_{2n} \neq 0$ for some $n$ (remember the periodicity of the walk);

- *random set–up*: $\omega$ is a typical realization of an IID sequence of random variables, whose law is denoted by $\mathbb{P}$. We suppose that

$$\mathsf{M}(\alpha) := \mathbb{E}\left[\exp\left(\alpha\omega_1\right)\right] < \infty \qquad \forall \alpha \in \mathbb{R}\,, \tag{1a.2}$$

that $\mathbb{E}[\omega_1] = 0$ (which simply amounts to redefine $h$) and we also fix $\mathbb{E}[{\omega_1}^2] = 1$. We stress that we are dealing with *quenched randomness*, that is the sequence $\omega$ is chosen at the beginning, according to $\mathbb{P}$, and then is kept fixed to define the copolymer measure $\mathbf{P}_{N,\omega}^{\lambda,h}$.

The differences of the two set-up will be discussed in detail in the sequel.

**2.2. The free energy approach.** We suppose that the sequence of charges $\omega$, periodic or random, has been fixed and we turn to the study of the copolymer measure $\mathbf{P}_{N,\omega}^{\lambda,h}$ when the size $N$ of the copolymer is *very large* (that is we are interested in asymptotic results as $N \to \infty$, the so–called *thermodynamic limit*). More precisely, we would like to understand, in function of the parameters $\lambda \geq 0$ and $h \geq 0$, whether the typical trajectories of the copolymer stay close to the interface (*localized* regime) or if they rather prefer to wander away in the solvents (*delocalized* regime).

In order to have a quantitative criterion to decide between the two situations, it is convenient to introduce the specific *free energy* of the system, defined by

$$f_\omega(\lambda, h) = \lim_{\substack{N\to\infty \\ N\in 2\mathbb{N}}} \frac{1}{N} \log \widetilde{Z}_{N,\omega}^{\lambda,h}. \tag{1a.3}$$

Let us be more precise:

- when the sequence $\omega$ is periodic, the existence of such a limit follows by standard superadditive arguments, see e.g. [**35**];
- in the random setting, the existence of the above limit in the $\mathbb{P}(\mathrm{d}\omega)$–almost sure sense and in $\mathbb{L}_1(\mathbb{P})$ follows by Kingman's Superadditive Ergodic Theorem, see [**35**]. Moreover we stress that in this case the limit does not depend on $\omega$, a phenomenon called *self–averaging*. Therefore in the sequel when treating the random case the $\omega$–dependence of the free energy will be omitted.

In both the periodic and the random setting, by convexity arguments one easily sees that the free energy is a continuous function of $\lambda$ and $h$.

The basic observation is that

$$f_\omega(\lambda, h) \geq \lambda h. \tag{1a.4}$$

In fact if we set $\Omega_N^+ = \{S :\ S_n > 0 \text{ for } n = 1, 2, \dots, N\}$, by restricting the integration over $\Omega_N^+$ (for even values of $N$) we get

$$\begin{aligned} \frac{1}{N} \log \widetilde{Z}_{N,\omega}^{\lambda,h} \ &\geq\ \frac{1}{N} \log \mathbf{E} \left[ \exp \left( \lambda \sum_{n=1}^{N} (\omega_n + h) \operatorname{sign}(S_n) \right) ; \Omega_N^+ \right] \\ &=\ \frac{\lambda}{N} \sum_{n=1}^{N} (\omega_n + h) \ +\ \frac{1}{N} \log \mathbf{P}\left(\Omega_N^+\right) \ \stackrel{N\to\infty}{\longrightarrow}\ \lambda h, \end{aligned} \tag{1a.5}$$

where in the random case the limit has to be understood in the $\mathbb{P}(\mathrm{d}\omega)$–almost sure sense, having used the law of large numbers. We have also applied the well known fact that $\mathbf{P}\left(\Omega_N^+\right)$ behaves like $N^{-1/2}$ for $N$ large [**28**, Ch. III].

The steps in (1a.5) show that $\lambda h$ is the contribution to the free energy coming from paths delocalized in oil. This consideration leads to the following partition of the phase diagram:

- the localized region: $\mathcal{L} = \{(\lambda, h) :\ f_\omega(\lambda, h) > \lambda h\}$;
- the delocalized region: $\mathcal{D} = \{(\lambda, h) :\ f_\omega(\lambda, h) = \lambda h\}$.

This definition of (de)localization in terms of the free energy may seem a bit indirect and it is not a priori obvious whether it corresponds to a really (de)localized behavior of the typical paths of the polymer measure: we will come back in § 2.6 to this important issue.

Now the program is to study in detail the phase diagram, both in the periodic and in the random setting. Notice that a priori it is not even obvious that $\mathcal{L} \neq \emptyset$, while of course $\mathcal{D} \supseteq \{(\lambda, h) :\ \lambda = 0\}$. We start with a basic result, valid in both settings, which says that indeed $\mathcal{L} \neq \emptyset$ and gives the existence of a *critical line*, which will be a central object of our analysis.

Proposition 1a.1. *Both in the periodic and in the random setting, there exists a continuous increasing function $h_c : [0, \infty) \to [0, \infty)$ with $h_c(0) = 0$ such that*

$$\mathcal{D} \,=\, \big\{(\lambda, h) :\ h \geq h_c(\lambda)\big\} \qquad \mathcal{L} \,=\, \big\{(\lambda, h) :\ h < h_c(\lambda)\big\}\,.$$

In particular we have the interesting result that for $h = 0$ and $\lambda > 0$ the copolymer is localized (a fact that was first proven by Sinai in [**62**]).

About the proof of Proposition 1a.1, we point out that just by simple convexity arguments one can prove the existence of the critical line $h_c(\cdot)$ and the fact that for $\lambda > 0$ it can be written as $h_c(\lambda) = U(\lambda)/\lambda$, with $U(\cdot)$ a convex function such

that $U(0) = 0$, cf. [**9**, § 1.2]. From this representation some elementary properties of the critical line follow easily, like for instance the fact that there exists $\ell \in (0, \infty]$ such that $h_c(\cdot)$ is continuous and nondecreasing in $(0, \ell)$ while $h_c(\lambda) = \infty$ for $\lambda > \ell$. It remains to prove that $\ell = \infty$ and that $\lambda \mapsto h_c(\lambda)$ is actually increasing and continuous also at $\lambda = 0$: the easiest way to get these results is to combine convexity arguments with the bounds on $h_c(\cdot)$ described in § 2.4 and § 2.5.

In the following sections we are going to study the properties of the critical line $h_c(\cdot)$, and we will see that a closer look shows important differences between the periodic setting and the random one. However before proceeding it is convenient to make some preliminary transformations on our model.

**2.3. A new partition function.** The content of this section is valid both for the periodic and for the random setting. From (1a.4) it is natural to introduce the *excess free energy* $\mathrm{F}_\omega$ defined by

$$\mathrm{F}_\omega(\lambda, h) \;:=\; f_\omega(\lambda, h) - \lambda h \,,$$

so that the condition for localization (resp. delocalization) becomes $\mathrm{F}_\omega(\lambda, h) > 0$ (resp. $\mathrm{F}_\omega(\lambda, h) = 0$). It is clear that we can obtain $\mathrm{F}_\omega$ as the free energy of our copolymer, once we redefine the Hamiltonian $\mathcal{H}_{N,\omega}^{\lambda,h} \to \mathcal{H}_{N,\omega}^{\lambda,h} - \lambda h N$ (observe that adding to the Hamiltonian a term that does not depend on $S$ has no influence on the copolymer measure). However it is more convenient to redefine the Hamiltonian in a slightly different way, by subtracting the term $\lambda \sum_{n=1}^N (\omega_n + h)$ instead of just $\lambda h N$. As this term does not depend on $S$ too, we can write

$$\begin{aligned} \frac{\mathrm{d}\mathbf{P}_{N,\omega}^{\lambda,h}}{\mathrm{d}\mathbf{P}}(S) \;&=\; \frac{1}{Z_{N,\omega}^{\lambda,h}} \exp\left( \lambda \sum_{n=1}^N (\omega_n + h) \big( \mathrm{sign}(S_n) - 1 \big) \right) \\ &=\; \frac{1}{Z_{N,\omega}^{\lambda,h}} \exp\left( -2\lambda \sum_{n=1}^N (\omega_n + h)\, \Delta_n \right) \end{aligned} \tag{1a.6}$$

with $\Delta_n = (1 - \mathrm{sign}(S_n))/2 = \mathbf{1}_{\{\mathrm{sign}(S_n) = -1\}}$ and with a new partition function $Z_{N,\omega}^{\lambda,h}$ given by

$$\begin{aligned} Z_{N,\omega}^{\lambda,h} \;&:=\; \mathbf{E}\left[ \exp\left( -2\lambda \sum_{n=1}^N (\omega_n + h)\, \Delta_n \right) \right] \\ &=\; \widetilde{Z}_{N,\omega}^{\lambda,h} \exp\left( -\lambda \sum_{n=1}^N (\omega_n + h) \right). \end{aligned} \tag{1a.7}$$

Hence from (1a.3) we get

$$\lim_{\substack{N\to\infty \\ N\in 2\mathbb{N}}} \frac{1}{N} \log Z^{\lambda,h}_{N,\omega} = f(\lambda,h) - \lambda h = \mathrm{F}_\omega(\lambda,h)\,, \tag{1a.8}$$

where in the random case this limit has to be interpreted in the $\mathbb{P}(\mathrm{d}\omega)$–a.s. or in the $\mathbb{L}_1(\mathbb{P})$ sense.

We will see that the new partition function $Z_{N,\omega}$ turns out to be substantially more useful than $\widetilde{Z}_{N,\omega}$ (this fact had been already realized in [**12**]). For this reason, in the following with partition function we will always mean $Z_{N,\omega}$, and in the same way $\mathrm{F}_\omega(\lambda,h)$ will be for us the free energy *tout court*.

We will use repeatedly also the partition function associated to the model *pinned* at the right endpoint:

$$Z^{\lambda,h}_{N,\omega}(x) \;:=\; \mathbf{E}\left[\exp\left(-2\lambda\sum_{n=1}^{N}(\omega_n+h)\,\Delta_n\right)\,;\,S_N=x\right]. \tag{1a.9}$$

It is worth recalling that one can substitute $Z^{\lambda,h}_{N,\omega}$ with $Z^{\lambda,h}_{N,\omega}(x)$, any fixed even $x$ (with the same parity of $N$), in (1a.8) and the limit is unchanged, see e.g. [**12**] or [**35**].

**2.4. The phase diagram in the periodic case.** As a matter of fact, the periodic case is essentially simpler than the random case. The reason is that by expressing the partition function in terms of the random walk excursions, the problem can be reduced to a finite–dimensional setting, as it has been first point out in [**11**] (this approach will be exploited in detail in Chapter 4). The net result is that the free energy of the model is expressible as the solution of a finite–dimensional Perron–Frobenius problem, from which sharp estimates for the critical line can be obtained.

To express the results, we introduce the Abelian group $\mathbb{S} := \mathbb{Z}/(T_\omega\mathbb{Z})$ (we recall that $2T_\omega$ is the period of the sequence $\omega$), and we define the following $\mathbb{S}\times\mathbb{S}$ matrix

$$\Xi_{\alpha,\beta} \;:=\; \sum_{i=2a+1}^{2b}\omega_i\,,$$

which is well defined by choosing representatives $a\in\alpha$ and $b\in\beta$ with $a<b$. We also introduce for $x\in\mathbb{N}$, $\alpha,\beta\in\mathbb{S}$ and $\lambda,h\ge 0$

$$\Phi^{\lambda,h}_{\alpha,\beta}(x) \;:=\; \log\left(\frac{1+2\exp\big(-2(\lambda\Xi_{\alpha,\beta}+\lambda h x)\big)}{2}\right).$$

Then, denoting by $K(x) := \mathbf{P}(\tau_1 = 2x)$ where $\tau_1 := \inf\{n > 0 : S_n = 0\}$ is the first return time to zero of the random walk, we define for $b \geq 0$ the following $\mathbb{S} \times \mathbb{S}$ matrix with nonnegative entries:

$$A_{\alpha,\beta}(b;\lambda,h) \;:=\; \sum_{x\in\mathbb{N}} \exp\big(\Phi^{\lambda,h}_{\alpha,\beta}(2x) - b(2x)\big)\, K(2x)\, \mathbf{1}_{(x\in\beta-\alpha)}\,,$$

and we denote by $\mathtt{Z}(b;\lambda,h)$ its Perron–Frobenius eigenvalue, cf. [**5**]. Observe that $\mathtt{Z}$ is a decreasing function of $b$ and $h$, since $A_{\alpha,\beta}$ are so for all $\alpha,\beta$. Then the free energy of the model is given by the following theorem (cf. [**11**, Th. 1.2]):

THEOREM 1a.2. *For $\lambda, h \geq 0$ we denote by $b = \widetilde{b}(\lambda,h)$ the unique solution of the implicit equation $\mathtt{Z}(b;\lambda,h) = 1$, if such a solution exists, and we set $\widetilde{b}(\lambda,h) = 0$ otherwise. Then $\widetilde{b}(\lambda,h)$ is exactly the free energy of the model:*

$$\mathrm{F}_\omega(\lambda,h) \;=\; \widetilde{b}(\lambda,h)\,.$$

It follows in particular that the critical line $h = h_c(\lambda)$ is determined by the implicit equation $\mathtt{Z}(0,\lambda,h_c(\lambda)) = 1$, and from this relation one can extract the asymptotic behavior of $h_c(\lambda)$ both for $\lambda \to 0$ and for $\lambda \to \infty$ (cf. [**11**, Th. 1.3]):

THEOREM 1a.3. *There exist two positive constants $m_\omega > 0$, $M_\omega > 0$ such that:*

$$\begin{aligned} &\text{as } \lambda \to 0 \qquad && h_c(\lambda) \;=\; m_\omega \lambda^3 \big(1 + o(1)\big) \\ &\text{as } \lambda \to \infty \qquad && h_c(\lambda) \;=\; \max_{n=1,\dots,T} \Big( -\frac{\omega_{2n-1} + \omega_{2n}}{2} \Big) - \frac{\big(M_\omega + o(1)\big)}{\lambda}\,. \end{aligned}$$

These results give a satisfactory characterization of the phase diagram of the copolymer in the periodic setting. We point out that the original proof of Theorem 1a.2, cf. [**16**], is based on Large Deviations techniques. We do not report it here because in Chapter 4 we present an approach based on Renewal Theory that allows a much more detailed analysis for a wide class of periodic inhomogeneous polymer models, including the periodic copolymer near a selective interface, and Theorem 1a.2 will come as a byproduct of our main results (cf. Theorem 4.5 in Chapter 4).

**2.5. The phase diagram in the random case.** From now on, when speaking of the random case we will omit the $\omega$–dependency on the free energy, that will be simply denoted by $\mathrm{F}(\lambda,h)$. We sum up in the following theorem what is known about the critical line of the random model (see Fig. 1a.3 for a graphical representation).

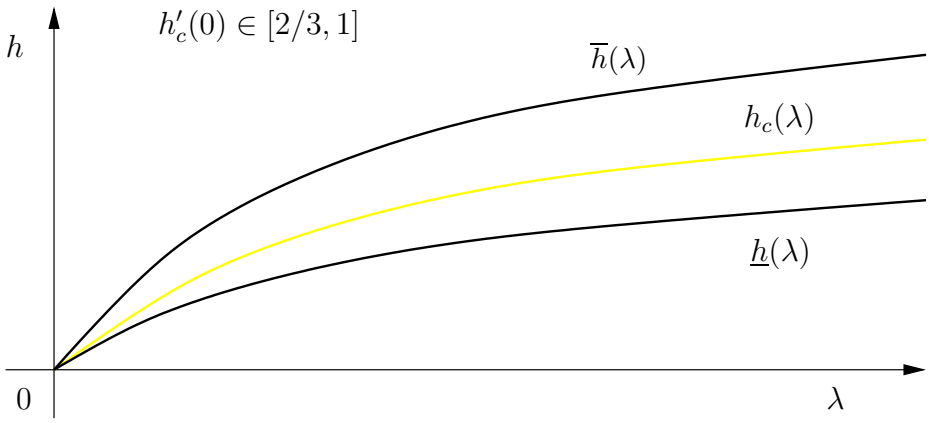


FIGURE 1A.3. The phase diagram in the random case.

THEOREM 1a.4. *For every $\lambda > 0$ the following bounds hold true:*

$$\underline{h}(\lambda) := \frac{1}{4\lambda/3} \log \mathsf{M}(-4\lambda/3) \le h_c(\lambda) \le \frac{1}{2\lambda} \log \mathsf{M}(-2\lambda) =: \overline{h}(\lambda). \qquad (1a.10)$$

*In particular the slope at the origin of $h_c(\cdot)$ belongs to $[2/3, 1]$, in the sense that the inferior limit of $h_c(\lambda)/\lambda$ as $\lambda \searrow 0$ is not smaller than $2/3$ and the superior limit is not larger than $1$.*

We recall that $\mathsf{M}(\cdot)$ it the moment generating function of $\omega_1$, see (1a.2), and we observe that the last statement in the theorem follows easily from (1a.10) applying the asymptotic expansion $\mathsf{M}(\alpha) = 1 + \alpha^2/2 + O(\alpha^3)$ as $\alpha \to 0$ (remember that we have fixed $\mathbb{E}[\omega_1^2] = 1$).

Notice that the main difference with the periodic case, cf. Theorem 1a.3, is given by the behavior of $h_c(\lambda)$ as $\lambda \to 0$: we could say that when $\lambda$ is small for the copolymer it is easier to localize in the random case than in the periodic one. This is easily understood by considering that for $\lambda$ small a major role is played by the *long excursions* of the walk, and observing that the energetic contribution to an excursion of length $L$ is $O(\sqrt{L})$ in the random case by the CLT, while in the periodic case it is of course $O(1)$.

We point out that in [**36**], using Concentration Inequalities techniques, it has been proven that the limit of $h_c(\lambda)/\lambda$ as $\lambda \to 0$ actually exists and it is *independent of the distribution of* $\omega_1$, at least when $\omega_1$ is a bounded symmetric random variable or when $\omega_1$ is a standard Gaussian variable. Moreover we will see that the slope at the origin is also closely related to the phase diagram of a Brownian copolymer

model which emerges as a scaling limit of our copolymer model as $\lambda, h \to 0$, see § 3.2 below. This *universal character* of the slope at the origin makes this quantity very interesting.

Theorem 1a.4 is a mild generalization of the results proven in [**12**] and [**9**]: the extension lies in the fact that $\omega_1$ is not necessarily symmetric and it requires minimal changes. Despite of the fact that the *lower bound* $\underline{h}(\cdot)$ and the *upper bound* $\overline{h}(\cdot)$ differ only by a scale factor, their origin is actually quite different, as we are going to see. We also point out that in the physical literature both the conjectures that $h_c(\cdot) = \underline{h}(\cdot)$ [**50, 64**] and that $h_c(\cdot) = \overline{h}(\cdot)$ [**34, 69**] have been set forth.

2.5.1. *The upper bound.* For completeness we report the proof given in [**12**] of the upper bound $h_c(\cdot) \le \overline{h}(\cdot)$. As a matter of fact, it is completely elementary: using the fact that the limit in (1a.8) holds also in $\mathbb{L}_1(\mathbb{P})$ and applying Jensen's inequality we can write

$$\mathrm{F}(\lambda, h) \;=\; \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z^{\lambda,h}_{N,\omega} \;\le\; \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z^{\lambda,h}_{N,\omega}\,,$$

and from (1a.7) we have

$$\mathbb{E} Z^{\lambda,h}_{N,\omega} = \mathbf{E}\left[ \exp\left( \sum_{n=1}^{N} \big( \log \mathsf{M}(-2\lambda) - 2\lambda h \big)\, \Delta_n \right) \right]. \tag{1a.11}$$

Then the upper bound follows immediately, because for $h \ge \overline{h}(\lambda)$ the argument of the exponential is nonpositive and hence $\mathbb{E} Z^{\lambda,h}_{N,\omega} \le 1$. Moreover if $h < \overline{h}(\lambda)$ it is easy to check that $\lim_{N\to\infty} \big( \log \mathbb{E} Z^{\lambda,h}_{N,\omega} \big)/N > 0$, hence $\overline{h}(\lambda)$ is indeed the best upper bound one can derive from (1a.11).

This approach to get an upper bound by performing the integration $\mathbb{E}$ over the disorder *before* taking the logarithm is a standard tool in the Statistical Mechanics of disordered systems and it is known as *annealed bound*. We stress however that in our case this approach is not as trivial as it may appear: for instance it is easy to check that by making the same steps with the old partition function $\widetilde{Z}^{\lambda,h}_{N,\omega}$ one would end up with an useless bound. The reason is that $Z^{\lambda,h}_{N,\omega}$ has been obtained by adding to the Hamiltonian the term $-\lambda \sum_{n=1}^{N} (\omega_n + h)$, that does not depend on $S$ (and therefore it leaves the copolymer measure invariant) but that has a strong dependence on $\omega$, which is able to change in a drastic way the annealed bound.

At this point it is clear that one could go further, searching for other $\omega$–depending terms to add to the Hamiltonian in order to improve the upper bound. Unfortunately the standard application of this technique, known as *constrained annealing*, to our

copolymer model cannot improve the basic annealed bound $\overline{h}(\cdot)$ on the critical line: this point is the object of Chapter 3, where this technique is explained in more detail.

Of course the difficulties in improving the upper bound $\overline{h}(\cdot)$ could be due to the fact that $\overline{h}(\cdot)$ is indeed the true critical line. However, the numerical analysis performed in Chapter 2 suggests that this is not the case.

2.5.2. *The lower bound.* The proof given in [**9**] of the lower bound $h_c(\cdot) \geq \underline{h}(\cdot)$ is obtained by computing explicitly the energy–entropy contribution to the partition function given by a suitable *strategy* of the copolymer. Roughly speaking, the strategy chosen is to force the copolymer to spend most of his time in the upper half plane, making it descend in the lower half plane only in correspondence of *long stretches* of the sequence of charges $\omega = \{\omega_n\}_n$ that have an *atypically negative sample mean*. The statistics of such stretches is governed by the so–called Large Deviations functional [**21**] for sums of IID random variables distributed like $\omega_1$, which is nothing but the Legendre transform of $\log \mathsf{M}(\cdot)$: this is the reason why also the lower bound $\underline{h}(\cdot)$ is of this form.

We do not report here the details of the proof because we will give an alternative proof of the lower bound in Chapter 2: see Section 3 for an outline and § 6.2 for the details. We stress that the idea behind the above strategy (and also behind our proof) takes inspiration from a (non rigorous) *renormalization scheme* for one–dimensional disordered systems applied to the copolymer model by C. Monthus [**50**]. This approach was first proposed by D. S. Fisher in the context of quantum Ising model with transverse random magnetic field [**30**] and then applied to random walk in random environment [**45**] with remarkable success.

We point out that the lower bound $\underline{h}(\cdot)$ on the critical line appears to be a very robust one: several attempts have been performed to enrich the above strategy (that is to keep many more random walk trajectories) in order to get a better lower bound, but all of them have failed. There could be of course the possibility that $h_c(\cdot) = \underline{h}(\cdot)$, but we anticipate that in Chapter 2 we present several numerical observations and a rigorous statistical test which strongly suggest that indeed $h_c(\cdot) > \underline{h}(\cdot)$.

**2.6. The path behavior.** The question of whether splitting the phase diagram into the regions $\mathcal{L}$ and $\mathcal{D}$, which are defined in terms of the free energy, does correspond to really different behaviors of the typical paths of the copolymer measure has a positive answer, at least if we do not consider the critical case, that is if we

consider the path behavior for $(\lambda, h) \in \mathcal{L}$ and for $(\lambda, h)$ in the interior of $\mathcal{D}$ (that will be called *strictly delocalized region*). However, while the localized regime is rather well understood, the delocalized one remains somewhat elusive. We first consider the periodic setting.

2.6.1. *The periodic case.* Strong path localization statement can be obtained applying the technique used in [**62**] by Sinai to study the random case. More precisely, if $(\lambda, h) \in \mathcal{L}$ then for every $\varepsilon > 0$ there exist positive constants $N_0 > 0$, $L_0 > 0$ such that for all $N \geq N_0$

$$\sup_{n=1,\dots,N} \mathbf{P}^{\lambda,h}_{N,\omega}\big(|S_n| > L\big) \;\leq\; \exp\big(-(\mathrm{F}_\omega(\lambda,h) - \varepsilon)\, L\big) \qquad \forall L \geq L_0\,. \tag{1a.12}$$

Furthermore, the Large Deviations approach taken in [**11**] gives detailed information on the returns to zero under the copolymer measure, that form a set with positive density, see [**11**, § 1.7].

On the other hand, for the delocalized phase the available results are less precise: the only result known in complete generality is that in the strictly delocalized regime the polymer spends almost all the time above any prefixed level, that is for any $L > 0$

$$\lim_{N\to\infty} \mathbf{E}^{\lambda,h}_{N,\omega}\left[\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{(S_n \geq L)}\right] = 1\,. \tag{1a.13}$$

Much stronger results are known to hold in more specific instances: for example in [**51**] the case of $\omega_n = (-1)^n$ is considered, for a copolymer model which differs from the ours in the definition of sign(0) (we refer to [**11**, § 1.5] for more details on the implications of this change). The authors compute the law of the returns to zero under the polymer measure, from which using the ideas in [**40**] or the general and more robust approach we take in Chapter 4 one can extract the Brownian scaling limits of the model. More precisely, one can prove that for $(\lambda, h)$ in the interior of $\mathcal{D}$ the law of the process $\{S_{\lfloor tN\rfloor}/\sqrt{N}\}_{t\in[0,1]}$ under the polymer measure $\mathbf{P}^{\lambda,h}_{N,\omega}$ converges weakly to the law of the Brownian meander process (that is the law of a standard Brownian motion conditioned not to enter the lower half plane, cf. [**60**]). The analysis can be performed in the critical case too, that is when $h = h_c(\lambda)$, showing that this time the scaling limit process is the absolute value of a Brownian motion.

We point out that the proof of these results has been obtained essentially by explicit computations, because by taking the marginals of a period–2 copolymer over the even sites one gets to an homogeneous pinning model, which is known to

be exactly solvable [**40**] (see also § 3.1 below). However it is widely believed that these results should hold for any periodic $\omega$.

In Chapter 4 we are going to show that this is indeed the case, proving that both the strictly delocalized and the critical Brownian scaling limit holds in complete generality for a wide class of periodic inhomogeneous polymer models, including the copolymer near a selective interface (and also pinning/wetting models, that will be described in § 3.1 below). Furthermore we will also give a precise description of the *local path properties* of the copolymer measure (thermodynamic limit) in all regimes, including the localized case.

2.6.2. *The random case.* Also in the random case it is known that for $(\lambda, h) \in \mathcal{L}$ the copolymer paths are localized in a strong sense. The random analogue of (1a.12) has been proved by Sinai in [**62**] for the case $\lambda > 0, h = 0$ (but the method can be extended to the whole localized region, cf. [**35**]), and it requires some care to state it properly. It is convenient to work with two–sided sequence of charges, that is for the sake of this section we assume that $\omega = \{\omega_n\}_{n\in\mathbb{Z}}$ is an element of the space $\Omega := \mathbb{R}^{\mathbb{Z}}$ and $\mathbb{P}$ is of course the product probability measure on $\Omega$. We also define for $n \in \mathbb{N}$ the translation $\theta^n$ on $\Omega$ by $(\theta^n\omega)_k := \omega_{n+k}$. Then Sinai's result reads as follows: for every $\varepsilon > 0$ there exist positive random variables $N_0(\omega)$, $L_0(\omega) : \Omega \to \mathbb{N}$ such that for $\mathbb{P}$–almost every $\omega$ and for all $N \geq N_0(\omega)$ the following relation holds:

$$\begin{aligned} \forall n \in \big\{ \log^\gamma N, \dots, N - \log^\gamma N \big\} \qquad \forall L \geq L_0\big(\theta^n(\omega)\big) \\ \mathbf{P}^{\lambda,h}_{N,\omega}\big(|S_n| > L\big) \;\leq\; \exp\big( - \big(\mathrm{F}(\lambda,h) - \varepsilon\big)\, L \big)\,, \end{aligned} \tag{1a.14}$$

where $\gamma > 0$ is an absolute constant (depending neither on $\varepsilon$ nor on $\omega$).

Some observations are in order. The restriction on the values of $n$ is made only for convenience: an analogous statement holding for all $n \leq N$ is possible but the notations become more involved. The key point is rather the condition $L \geq L_0\big(\theta^n(\omega)\big)$, which is saying that the "radius of localization" depends on $n$, and can actually be arbitrarily large because the the random variable $L_0$ is essentially unbounded. This fact may make the estimate (1a.14) appear unsatisfactory, but in fact it is unavoidable, and the reason for this is to be sought in the presence of *arbitrary long atypical stretches* in the sequence $\omega$: in fact if a site $n$ is surrounded by a stretch $\{\omega_{n-k}, \dots, \omega_{n+k}\}$ with an atypically positive sample mean, this will have a repulsive effect pushing $S_n$ to height of order $\approx \sqrt{k}$ (in Chapter 2 we will see that one can also take advantage of atypical stretches).

However the situation is not so bad. On the one hand, the random variable $L_0$ can be chosen such that $\mathbb{E}\big[\exp(\alpha L_0)\big] < \infty$ for some $\alpha > 0$ and therefore for $\mathbb{P}$–a.e. $\omega$ we have that eventually $L_0(\theta^n\omega) \le \alpha^{-1}\log n$: thus the radius of localization is in any case much smaller than the polymer size. On the other hand, by Birkhoff's Ergodic Theorem we have that, for every $K > 0$, $\mathbb{P}(\mathrm{d}\omega)$–a.s.

$$\lim_{N\to\infty} \sum_{n=1}^{N} \mathbf{1}_{\{L_0(\theta^n\omega)>K\}} \;=\; \mathbb{P}\big(L_0 > K\big)\,,$$

hence by choosing $K$ large we have that the localization radius is smaller than $K$ *most of the time*.

We observe that strong localization results are available also for the thermodynamic limit of the copolymer measure: we do not report them here and we refer to [**8**] and [**1**] for details.

Turning to the delocalized regime, we point out that almost no result is at present available for the critical case. In the strictly delocalized case the situation is somewhat better: for instance it is known that (1a.13) holds for $\mathbb{P}$–almost every $\omega$. However this is quite a weak information on the paths, above all if compared to what is available for the periodic case (and more generally for related non disordered models, see e.g. [**22**] and references therein), namely Brownian scaling.

The standard way to prove this scaling limit for the strictly delocalized regime is to show that under the copolymer measure $\mathbf{P}_{N,\omega}^{\lambda,h}$ the epoch of the last visit to the lower half plane is $o(N)$. For non disordered models actually much more is true: in fact in the limit $N\to\infty$ the polymer becomes transient and it visits the lower half plane, or any point below a fixed level, only a *finite number* of times. The situation appears to be different for the random copolymer: in fact in [**36**] it has been shown that for $h < \overline{h}(\lambda)$ the number of visit to the lower half plane for the *quenched averaged measure* $\mathbb{E}\mathbf{E}_{N,\omega}^{\lambda,h}[\,\cdot\,]$ is $O(\log N)$. This fact alone does not suffice to yield the scaling limit, because besides showing that there are $o(N)$ visit to the lower half plane, one should prove that they all happen close to the origin: we refer to [**36**] for more details and for a discussion on what is still missing.

We stress that in answering this kind of questions an important role is played by the asymptotic behavior as $N\to\infty$ of the partition function $Z_{N,\omega}$ in the interior of the delocalized phase. In the non–disordered case it is known that $Z_{N,\omega} \approx N^{-1/2}$, see for instance Theorem 4.5 of Chapter 4. On the other hand, this asymptotic behavior is known not to hold anymore in the random case: more precisely, for every $(\lambda, h)$

in the interior of $\mathcal{D}$ there exists $\varepsilon > 0$ and a subsequence $\{\tau_N(\omega)\}_N$ such that $N^{1/2-\varepsilon} Z_{\tau_N(\omega),\omega} \to \infty$ as $N \to \infty$, see Proposition 4.1 in [**36**].

The issue of the delocalized path behavior in the random case is taken up again in § 4.1 of Chapter 2.

## 3. Other related polymer models

**3.1. Pinning at an interface and wetting models.** Another problem that has received much attention is the situation in which a polymer chain is attracted (or repelled) by an interface, which may be penetrable or impenetrable. We can model this situation by giving a reward (or a penalization) to each monomer lying on the interface, and this reward/penalization may vary from one monomer to another if the polymer chain is heterogeneous. As in the copolymer model analyzed so far, this modification may alter the paths of the walk inducing a localization/delocalization transition.

Let us define a probabilistic model for these situations when the interface is flat (for us it will be the $x$–axis). We start with the case when the interface is penetrable (*pinning models*): as in the preceding section, we take a simple random walk $\{S_n\}_n$ with law $\mathbf{P}$, and for $N \in 2\mathbb{N}$, $\beta \in \mathbb{R}$ and $\omega = \{\omega_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ we define the new law $\mathbf{P}^{\beta}_{N,\omega}$ by

$$\frac{\mathrm{d}\mathbf{P}^{\beta}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) \;\propto\; \exp\left(\beta \sum_{n=1}^{N} \omega_n \mathbf{1}_{(S_n=0)}\right). \tag{1a.15}$$

The case of an impenetrable interface is obtained by restricting to paths that stay nonnegative up to epoch $N$, that is multiplying the r.h.s. above by $\mathbf{1}_{(S_1\geq 0,\dots,S_N\geq 0)}$. This second case will be called a *wetting model*, as it can be also interpreted as the model of an interface interacting with an impenetrable wall.

Again we will stick to the case when the sequence charges $\omega$ is either (deterministic and) periodic or (quenched) random. Of course the main interest is in understanding the behavior of the above measure when $N$ is large. Localization/delocalization can be defined in terms of the corresponding free energy, exactly as in the previous section.

A particularly simple case is the homogeneous one, that is when the sequence $\omega$ is constant: $\omega_n = \omega_1$ for all $n \in \mathbb{N}$, and up to a redefinition of the parameter $\lambda$ we may assume that $\omega_1 = 1$. We point out that in this case both the pinning and the wetting models are completely solvable, not only for the purpose of finding

the phase diagram (see e.g. [**35**] for an elementary derivation) but also for a very detailed analysis of the polymer path behavior [**40, 22**]. We do not spend much time here on this issue, because in Chapter 4 we will treat in full detail the case of periodic $\omega$. Nevertheless we make some observations: using convexity arguments it is easy to check that the phase diagram in the homogeneous case is encoded in a single number $\beta_c$ such that for $\beta > \beta_c$ (resp. $\beta \le \beta_c$) the polymer is localized (resp. delocalized). Moreover:

- in the pinning case $\beta_c = 0$, that is an arbitrarily small reward is sufficient to localize the polymer;
- in the wetting case on the contrary $\beta_c > 0$.

The reason why the wetting model is more difficult to localize than the pinning model is that conditioning the walk to stay nonnegative up to step $N$ induces a repulsion effect of order $\sqrt{N}$ on the paths, a phenomenon which goes under the name of *entropic repulsion*. In our one dimensional setting, a more precise version of this statement is provided by the following invariance principle [**10**]: the process $\{S_{\lfloor Nt \rfloor}/\sqrt{N}\}_{t\in[0,1]}$ conditionally on the event $\{S_1 \ge 0, \dots, S_N \ge 0\}$ converges weakly as $N \to \infty$ to the Brownian meander process, that is to a Brownian motion conditioned to stay nonnegative [**60**]. In Chapter 6 we will prove a local version of this weak convergence.

As already anticipated, the periodic version of these models will be analyzed in Chapter 4. On the other hand, nothing will be said in this thesis about the random case. We only mention that, as in the copolymer case, in the physical literature there is no agreement on the phase diagram of the model, especially for small values of the coupling constants: for more details on this issue and for the available rigorous results see [**2, 57**] and references therein.

To conclude we would like to point out the relevance that random walk models have for the modeling of DNA molecules. DNA is normally in a double–stranded state, however it may happen that the two strands get detached, for example when the temperature is sufficiently high (*denaturation transition*) or due to the effect of an external force (*pulling induced unzipping*). Since the interaction between the two strands may be described (at least at a first level) by an Hamiltonian of the form (1a.15), the energy/entropy competition that gives origin to such phase transitions may be understood in terms of suitable modifications of the pinning/wetting models just described.

**3.2. A Brownian motion model: the coarse graining issue.** One of the main results in the paper of Bolthausen and Den Hollander [**12**] is that in the limit of weak coupling the copolymer model described in Section 2 can be approximated by a continuous model built with Brownian motions instead of random walks. This continuous model is defined in complete analogy with the discrete one: we take two Brownian motions $B = \{B_t\}_{t\geq 0}$ (the polymer) and $\beta = \{\beta_t\}_{t\geq 0}$ (the charges), with respective laws $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbb{P}}$, and for $t > 0$, $\lambda, h \geq 0$ and a $\widetilde{\mathbb{P}}$–typical path $\{\beta_s\}_s$ we introduce the polymer measure $\widetilde{\mathbf{P}}_{t,\beta}^{\lambda,h}$ on paths of length $t$ defined by

$$\frac{\mathrm{d}\widetilde{\mathbf{P}}_{t,\beta}^{\lambda,h}}{\mathrm{d}\widetilde{\mathbf{P}}}(B) \;:=\; \frac{1}{Z_{t,\beta}^{\lambda,h}} \;\exp\left(\lambda \int_0^t \mathrm{sign}(B_s)\left(\mathrm{d}\beta_s + h\mathrm{d}s\right)\right),$$

where the integral with respect to $\beta_s$ is an Ito integral. The partition function of the model is of course

$$Z_{t,\beta}^{\lambda,h} \;=\; \widetilde{\mathbf{E}} \exp\left(\lambda \int_0^t \mathrm{sign}(\beta_s)\left(\mathrm{d}\beta_s + h\mathrm{d}s\right)\right),$$

and the free energy $\widetilde{f}(\lambda, h)$ is defined as

$$\widetilde{f}(\lambda, h) \;:=\; \lim_{t\to\infty} \frac{1}{t} \log Z_{t,\beta}^{\lambda,h},$$

where the limit holds both $\widetilde{\mathbb{P}}$–a.s. and in $\mathbb{L}_1(\widetilde{\mathbb{P}})$ and $\widetilde{f}(\lambda, h)$ is nonrandom (see [**35**] for a detailed proof of the existence of such a limit).

As in the discrete case, we have

$$\widetilde{f}(\lambda, h) \;\geq\; \lambda h,$$

and consequently we distinguish between a delocalized regime ($\widetilde{f}(\lambda, h) = \lambda h$) and a localized regime ($\widetilde{f}(\lambda, h) > \lambda h$). Notice however that the scaling properties of Brownian motions entail that for all $a > 0$

$$\widetilde{f}(\lambda, h) \;=\; \frac{1}{a^2}\widetilde{f}(a\lambda, ah),$$

from which it follows immediately that the critical curve of this model is a *straight line*, that is

$$\exists K_c > 0 \;:\quad \widetilde{f}(\lambda, h) \begin{cases} = \lambda h & \text{if } h \geq K_c\lambda \\ > \lambda h & \text{if } h < K_c\lambda \end{cases}.$$

Despite the apparent simplicity of the phase diagram, we could say that this continuous model retains the full complexity of the discrete model, which is hidden in the constant $K_c$. This statement is made precise by the following fundamental theorem

(cf. [**12**, Th. 5 and 6]), which also clarifies in which sense the continuous model is an approximation of the discrete one.

THEOREM 1a.5. *Let us consider the free energy $f(\lambda, h)$ of the discrete model (see eq.* (1a.3)*) in the case when $\mathbb{P}(\omega_1 = +1) = \mathbb{P}(\omega_1 = -1) = 1/2$, and the corresponding critical line $h = h_c(\lambda)$ (see Prop.* (1a.1)*). Then the following relations hold:*

$$\lim_{a \to 0} \frac{1}{a^2} f(a\lambda, ah) = \widetilde{f}(\lambda, h) \quad \forall \lambda, h \geq 0 \tag{1a.16}$$

$$h_c'(0) := \lim_{\lambda \to 0} \frac{h_c(\lambda)}{\lambda} = K_c \,. \tag{1a.17}$$

In particular by (1a.10) it follows that $2/3 \leq K_c \leq 1$.

We point out that (1a.17) does not follow directly from (1a.16): in fact the scaling limit of the free energy expressed by (1a.16) yields only the lower bound $h_c'(0) \geq K_c$. The proof of (1a.17) is achieved through sharp comparison inequalities between $f$ and $\widetilde{f}$ and requires very delicate estimates.

We stress that Theorem 1a.5 has been proven for the case when the charges have a symmetric Bernoulli law, but its validity should be very general. The intuitive idea is that as $\lambda, h \to 0$ what really matters are the *long excursions* of the walk, and consequently the microscopic details of the model should become irrelevant.

For instance, as we already mentioned, in [**36**] it has been proven with Concentration Inequalities techniques that actually (1a.17) holds whenever $\omega_1$ is bounded and symmetric (and such that $\mathbb{E}[\omega_1] = 0$ and $\mathbb{E}[{\omega_1}^2] = 1$) or if $\omega_1$ is a standard Gaussian. Alternatively, the original proof of Theorem 1a.5 given in [**12**] can be adapted to show that indeed both (1a.16) and (1a.17) hold for any choice of the law of $\omega_1$ satisfying (1a.2) and such that $\mathbb{E}[\omega_1] = 0$, $\mathbb{E}[{\omega_1}^2] = 1$.

In Chapter 5 we introduce another kind of variation on the discrete model, namely we will change the law $\mathbf{P}$ of the underlying walk, taking into account general random walks on $\mathbb{R}$ whose increments are bounded and have a continuous law. This change too is supposed not to have any influence on the conclusions of Theorem 1a.5, however giving a complete proof of this fact appears to be very challenging. Some partial steps have been done in the direction of proving (1a.16) alone: we refer to Chapter 5 for more details on this issue.

## 4. An overview of the literature

The *copolymer in the proximity of an interface* problem has a long literature, mostly in the area of chemistry and physics, but possibly the first article that attracted the attention of mathematicians is [**34**]. The first mathematical study on the subject has been performed by Sinai in [**62**], where he shows that for $h = 0$ and $\lambda > 0$ (we are referring to the parameter of the model introduced in Section 2) the copolymer with random charges is localized in a strong pathwise sense (see § 2.6 above). Further path investigations and a detailed analysis of the free energy (always for the random case and in the symmetric setting $h = 0$) have been performed by Albeverio and Zhou in [**1**].

As already mentioned, our attention on the random copolymer has been mainly focused on the issue of investigating the phase diagram, which entails studying the copolymer for $h > 0$. In this direction the fundamental paper is the one by Bolthausen and den Hollander [**12**], where the existence and some basic properties of the critical line $h_c(\cdot)$ (including the upper bound in (1a.10)) have been proven. However the main result of [**12**] is the coarse graining of the free energy, expressed by Theorem 1a.5 of § 3.2 below. The other fundamental result on the phase diagram in the random case, namely the lower bound in (1a.10), has been proven by Bodineau and Giacomin in [**9**].

The strategy used in [**9**] takes inspiration from the physical paper by Monthus [**50**], where the lower bound curve $\underline{h}(\cdot)$ has been introduced for the first time, as a conjecture for the true critical line. Again from the physical literature, we point out that the conjecture $h_c(\cdot) = \underline{h}(\cdot)$ has been set forth also in [**64**] on the ground of replica computations, while the complementary conjecture that $h_c(\cdot) = \overline{h}(\cdot)$ has been formulated in [**69**] and in [**34**].

Coming back to mathematical papers, a path analysis for the whole localized region $\mathcal{L}$ in the random case has been performed by Biskup and den Hollander in [**8**]: the keywords of their approach are *thermodynamic limit* and *Gibbs measures*. On the other hand, path results for the delocalized region appear to be much more challenging: recent progresses in this direction have been obtained by Giacomin and Toninelli in [**36**].

Turning to the case of periodic charges, the issue of determining the phase diagram has received a complete solution in the paper [**11**] by Bolthausen and Giacomin

(see § 2.4). We refer to this paper also for references to previous works on periodic copolymers.

In the literature one finds also a large number of numerical works on copolymers, see for example [**19, 65**] and references therein: with respect to the numerical approach we take in Chapter 2, the attention is often shifted toward different aspects, notably the issue of critical exponents and the more complex model in which the polymer is not directed but rather self–avoiding.

Finally, about other polymer models dealing with pinning/adsorption phenomena we mention [**40**], [**22**], [**2**], [**71, 72**] and references therein.

## 5. Outline of the thesis

The exposition is organized as follows.

- In Chapter 2 we combine numerical computations with rigorous arguments to study the phase diagram and the path behavior of the copolymer near a selective interface model defined in Section 2. We consider the case of *random charges*. We provide several evidences for the fact that the critical line lies strictly in between the two known bounds given in (1a.10) and for the fact that the scaling limit towards the Brownian meander process holds in the strictly delocalized region. In particular the conjecture that $h_c(\cdot) = \underline{h}(\cdot)$ can be excluded with a high level of confidence, thanks to a rigorous statistical test with explicit error bounds. We also give an alternative self–contained proof of the lower bound $h_c(\cdot) \geq \underline{h}(\cdot)$.

  The article [**17**] has been taken from the content of this chapter.

- In Chapter 3 we address the issue of improving the annealed upper bound for disordered systems (see § 2.5.1) by adding to the Hamiltonian disorder–dependent terms, a technique known as *constrained annealing*. We show that for a number of disordered linear chain models (including the copolymer near selective interfaces and the pinning/wetting model described in the preceding sections) the standard application of this technique using empirical averages of local functions cannot improve the basic annealed bound on the critical curve.

  The article [**16**] has been taken from the content of this chapter.

- In Chapter 4 we consider a general model of a heterogeneous polymer in the proximity of an interface (including as special cases the copolymer near a selective interface and the pinning/wetting model) in the case of *periodic charges*. We propose an approach based on Renewal Theory that yields sharp estimates on the partition function of the model in all the regimes, including the critical one. From these results we obtain a very precise description of both the thermodynamic limit and the scaling limits of the polymer measure.

  The preprint [**18**] has been taken from the content of this chapter.

- In Chapter 5 we consider a modification of the copolymer near a selective interface model in which the reference measure $\mathbf{P}$ is not any more the law of the simple symmetric random walk on $\mathbb{Z}$. More precisely, we allow $\mathbf{P}$ to be the law of a general real random walk whose typical step is centered, bounded and has an absolutely continuous law. We focus on the case of *random charges*. Besides giving a proof of the existence of the free energy, we study the phase diagram of the model, pointing out the close analogies with the simple random walk case. We finally consider the issue of extending to this model the coarse graining of the free energy expressed by Theorem 1a.5 (work in progress), giving some partial result in this direction and discussing what is missing.

- In Chapter 6 we prove a local limit theorem for random walks conditioned to stay positive which is valid in great generality (whenever the walk is attracted to the Gaussian law). This theorem provides a local refinement of the well-known weak convergence of random walks conditioned to stay positive towards the Brownian meander process. Besides being an interesting result in itself, it is an important tool for the purpose of dealing with polymer models built over general random walks, like the one considered in Chapter 5.

  The article [**15**] has been taken from the content of this chapter.

CHAPTER 1b

# Chaînes polymériques inhomogènes

Dans ce chapitre (qui n'est que la traduction en Français du chapitre précédent) nous présentons les différents types de modèles que nous considérerons, en donnant quelques motivations pour leur étude et en rappelant les résultats déjà établis dans la littérature. Notre exposé s'inspire de [**35**].

## 1. Introduction et motivations

**1.1. Polymères et marches aléatoires.** La notion de polymère est à l'origine une notion chimique. Ainsi, en chimie un polymère est une macro-molécule, d'origine naturelle ou synthétique, constituée d'une séquence linéaire de molécules plus petites, appelées monomères. Ce concept s'est généralisé : les polymères apparaissent désormais dans une variété de domaines différents et désignent globalement une *structure linéaire* construite en juxtaposant un grand nombre de *structures simples* (tout cela éventuellement dans un sens abstrait). Un exemple fondamental de polymère abstrait est celui de la *marche aléatoire*, dont les accroissements constituent les monomères.

Dans la suite nous nous intéresserons aux deux aspects ci-dessus, concret et abstrait. Notre but est en effet de construire des modèles basés sur une marche aléatoire qui permettraient de reproduire la phénoménologie des polymères chimiques dans certaines situations. Nous attirons l'attention sur le fait que les modèles que nous allons considérer sont *élémentaires* et qu'ils posent néanmoins des problèmes très intéressants et complexes.

**1.2. Copolymères à proximité d'une interface sélective.** La situation sur laquelle nous focaliserons notre attention est celle d'un *copolymère à proximité d'une interface séparant deux solvants sélectifs*, comme par exemple l'huile et l'eau. "Copolymère" est simplement un synonyme de polymère inhomogène, c'est-à-dire un polymère dont les monomères ne sont pas tous identiques. Dans notre cas on supposera que les monomères peuvent différer par une seule caractéristique : ils

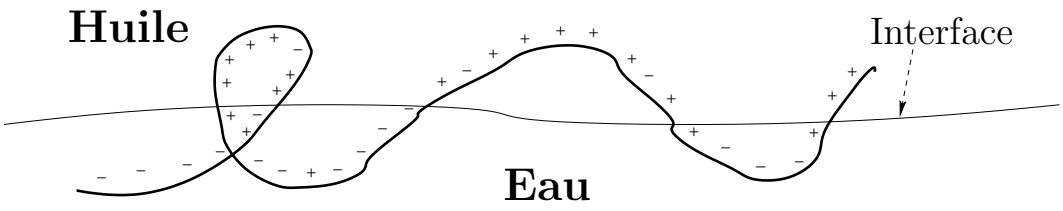


FIGURE 1B.1. Un copolymère à proximité d'une interface sélective.

peuvent être soit hydrophiles (+) soit hydrophobes (−) (voir Figure 1b.1). Dans la suite, nous supposerons que les monomères hydrophobes sont majoritaires.

On pourrait tout d'abord penser que le copolymère devrait préférer vivre dans l'huile, du fait de l'hydrophobie globale de la chaîne. Cependant un autre scénario est envisageable : le polymère pourrait rester très proche de l'interface et placer chaque monomère (du moins, une majorité d'entre eux) dans son solvant préféré, c'est-à-dire + dans l'huile et − dans l'eau. Observons que cette deuxième stratégie permet un *gain d'énergie*, du fait qu'une plus grande fraction des monomères est placée dans le solvant correspondant, mais elle implique aussi une *perte d'entropie*, dans la mesure où le polymère a accès à une partie beaucoup plus petite de l'espace des configurations (les trajectoires qui restent proches de l'interface sont en effet beaucoup moins nombreuses que les trajectoires libres de fluctuer dans l'huile).

Il devrait être clair qu'il y a une compétition entre énergie et entropie : notre but est de construire un modèle probabiliste rendant compte de cette situation, en suivant les principes de la Mécanique Statistique. Cela nous permettra d'étudier de façon quantitative cette compétition, pour décider –en fonction des caractéristiques du polymère et d'autres quantités physiques, comme la température– quelle est la stratégie gagnante, c'est-à-dire celle effectivement suivie par le polymère.

Nous soulignons que nous ne nous intéressons qu'au comportement thermodynamique du copolymère *à un temps fixé* – dans cette thèse nous ne considérerons jamais le problème de l'évolution dynamique.

**1.3. Modèles basés sur une marche aléatoire.** Soyons un peu plus précis quant à la construction d'un modèle de copolymère basé sur une marche aléatoire (les définitions exactes seront données dans la prochaine section). On considère d'abord une marche aléatoire à valeur dans $\mathbb{R}^d$ (ou dans un sous–réseau de $\mathbb{R}^d$) et on fixe un

entier $N$ grand, i.e. la taille du polymère. L'idée est de considérer les trajectoires de la marche aléatoire jusqu'au temps $N$ comme décrivant les configurations de la chaîne polymérique quand il n'y a pas d'interactions. Nous modifions ensuite la loi de la marche suivant les principes de la Mécanique Statistique, c'est-à-dire que nous donnons à chaque trajectoire un poids exponentiel (facteur de Boltzmann) prenant en compte l'interaction du copolymère avec les solvants. Cette nouvelle loi est la *mesure du copolymère*, elle décrit le comportement statistique du copolymère à l'équilibre thermodynamique.

Le choix de la marche aléatoire est une question fondamentale. Pour éviter les cas triviaux, on suppose que l'espace où vit la marche est au moins bidimensionnel, c'est-à-dire $d \geq 2$. D'autre part, comme les polymères réels occupent de l'espace physique, nous aimerions considérer des *marches aléatoires auto–évitantes*. Dans le cas où la marche se déplace dans un réseau, cela nécessite de conditionner la marche à ne pas visiter deux fois un même site (définir une marche auto–évitante dans le cas général est plus délicat ; nous n'entrerons pas dans les détails à ce stade).

Une difficulté importante réside dans le fait que les marches auto–évitantes sont des objets très complexes. Une manière de contourner le problème est de simplifier la contrainte, en considérant des *marches dirigées* – c'est-à-dire, des marches dont une composante est strictement croissante. Un exemple typique est celui de la marche dirigée $(1+m)$–dimensionnelle, c'est-à-dire du processus $\{(n, S_n)\}_n$ où $\{S_n\}_n$ est une marche aléatoire dans $\mathbb{R}^m$. Cette approche peut sembler drastique, mais elle est très répandue dans la littérature et nous l'adopterons dorénavant nous aussi : plus précisément, nous travaillerons avec des marches dirigées $(1+1)$–dimensionnelles.

Bien entendu, une autre possibilité serait de renoncer aux contraintes d'exclusion de volume en travaillant avec des marches aléatoires. Néanmoins, nous attirons l'attention sur le fait que dans notre cas, le modèle construit à partir d'une marche aléatoire $d$–dimensionnelle $\{S_n\}_n$ est équivalent à celui élaboré à partir d'une marche dirigée opportune $(1+1)$–dimensionnelle (cette question est discutée plus en détail dans le Chapitre 5). Cette constatation justifie notre choix d'une approche basée sur les marches dirigées.

## 2. Copolymères à proximité d'une interface sélective

Dans cette section nous définissons un modèle de copolymère à proximité d'une interface sélective basé sur une marche aléatoire. Ce modèle sera le sujet principal de cette thèse.

**2.1. Définition du modèle.** Soit $\{S_n\}_{n=0,1,\dots}$ une marche aléatoire symétrique simple sur $\mathbb{Z}$, c'est-à-dire

$$S_0 = 0 \qquad\qquad S_n = \sum_{j=1}^{n} X_j \,,$$

où les variables $\{X_j\}_j$ sont IID et telles que $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = 1/2$. Nous adoptons le point de vue "dirigé", en considérant les trajectoires de $\{(n, S_n)\}_n$ comme les configurations de la chaîne polymérique.

Pour $\lambda \geq 0$, $h \geq 0$, $N \in 2\mathbb{N}$ et $\omega = \{\omega_j\}_{j=1,2,\dots} \in \mathbb{R}^{\mathbb{N}}$ nous définissons la *mesure du copolymère* $\mathbf{P}_{N,\omega}^{\lambda,h}$ en donnant sa densité par rapport à $\mathbf{P}$ :

$$\begin{aligned} \frac{\mathrm{d}\mathbf{P}_{N,\omega}^{\lambda,h}}{\mathrm{d}\mathbf{P}}(S) \;&=\; \frac{1}{\widetilde{Z}_{N,\omega}^{\lambda,h}} \exp\Big(\mathcal{H}_{N,\omega}^{\lambda,h}(S)\Big) \\ &:=\; \frac{1}{\widetilde{Z}_{N,\omega}^{\lambda,h}} \exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h)\,\mathrm{sign}\,(S_n)\right), \end{aligned} \tag{1b.1}$$

où on définit $\mathrm{sign}\,(S_{2n}) := \mathrm{sign}\,(S_{2n-1})$ pour tous les $n$ tels que $S_{2n} = 0$ (ce qui est un choix naturel, voir la Fig. 1b.2). Le terme $\widetilde{Z}_{N,\omega}^{\lambda,h}$ n'est qu'une constante de normalisation qui fait de $\mathbf{P}_{N,\omega}^{\lambda,h}$ une mesure de probabilité :

$$\widetilde{Z}_{N,\omega}^{\lambda,h} \;=\; \mathbf{E}\left[\exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h)\,\mathrm{sign}\,(S_n)\right)\right].$$

On appelle $\widetilde{Z}_{N,\omega}^{\lambda,h}$ la *fonction de partition* du système.

Nous renvoyons à la Fig. 1b.2 pour une interprétation visuelle de la mesure du copolymère. Dans l'équation (1b.1), l'expression qui apparaît dans l'exponentielle est appelée *Hamiltonien* du système (les conventions des physiciens diffèrent d'un signe moins). Discutons plus en détail la signification des paramètres qui apparaissent dans la définition de $\mathbf{P}_{N,\omega}^{\lambda,h}$:

- $N$ est bien entendu la taille du copolymère ;
- le paramètre $\lambda$ représente la force globale de l'interaction, et correspond d'un point de vue physique à l'inverse de la *température* ;

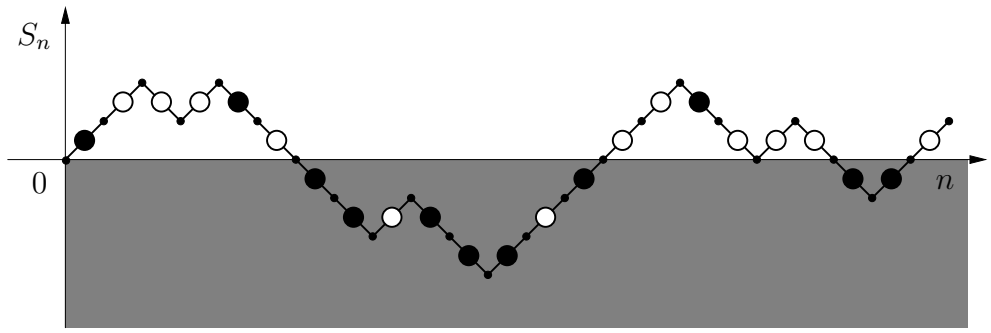


FIGURE 1B.2. Le processus qu'on a introduit est un modèle de polymère inhomogène, ou *copolymère*, à proximité d'une interface (ici l'axe des abscisses) entre deux solvants sélectifs, disons l'huile (blanc) et l'eau (gris). Les liens entre les monomères sont représentés par les points noirs tandis que les accroissements de la marche aléatoire correspondent aux monomères proprement dits. La couleur du disque au centre de chaque monomère indique le signe de la charge (blanc = charge positive = monomère hydrophobe, noir = charge négative = monomère hydrophile). Quand $h > 0$ l'eau est le solvant défavorable et la question est de savoir si le polymère est *délocalisé* dans l'huile ou bien au contraire s'il a plus intérêt à placer un grand nombre de monomères dans le solvant préféré, ce qui conduit au phénomène de *localisation à l'interface*. Enfin, la définition de sign(0) que nous indiquions plus haut est motivée par le fait que la charge est portée par les monomères et non par leurs extrémités.

- sign$(\omega_n + h)$ indique si le $n$–ième monomère est hydrophobe $(+)$ ou hydrophile $(-)$, et $|\omega_n + h|$ donne l'intensité de l'hydrophobicité (ou hydrophilie) du monomère. Le choix d'une telle écriture $(\omega_n + h)$ permet d'isoler dans le paramètre $h$ l'asymétrie globale de l'hydrophobicité/hydrophilie – nous reviendrons sur ce point dans un instant.

Nous devons maintenant indiquer comment choisir la suite $\omega$, c'est-à-dire la suite des *charges*, ou encore le *désordre* du copolymère. Deux possibilités seront considérées dans cette thèse:

- *cas périodique :* $\omega$ est une suite périodique fixée, c'est-à-dire qu'il existe $T \in \mathbb{N}$ tel que $\omega_{2T+n} = \omega_n$ pour tout $n \in \mathbb{N}$ : le plus petit $T$ satisfaisant

cette condition sera noté $T_\omega$ et nommé demi–période de $\omega$ (le choix d'une période paire est dû à la périodicité de la marche aléatoire simple). À une modification du paramètre $h$ près, nous supposerons que la suite est centrée : $\sum_{n=1}^{2T} \omega_n = 0$, et pour éviter les cas triviaux nous supposerons qu'il existe $n$ tel que $\omega_{2n-1} + \omega_{2n} \neq 0$ (rappelons la périodicité de la marche aléatoire) ;

- *cas aléatoire :* $\omega$ est une réalisation typique d'une suite IID de variables aléatoires de loi $\mathbb{P}$. On suppose que

$$\mathsf{M}(\alpha) := \mathbb{E}\left[\exp\left(\alpha\omega_1\right)\right] < \infty \qquad \forall \alpha \in \mathbb{R}\,, \tag{1b.2}$$

que $\mathbb{E}\left[\omega_1\right] = 0$ (il suffit de modifier $h$) et que $\mathbb{E}[{\omega_1}^2] = 1$. Soulignons que nous travaillons avec un aléa de type *quenched*, c'est-à-dire que nous considérons tout d'abord une réalisation de $\omega$ sous la mesure $\mathbb{P}$, avant de définir la mesure du copolymère $\mathbf{P}_{N,\omega}^{\lambda,h}$.

Les différences entre les deux cas seront discutées en détail dans la suite.

**2.2. L'énergie libre.** Supposons que la suite des charges $\omega$, périodique ou aléatoire, est fixée. Nous passons à l'étude de la mesure du copolymère $\mathbf{P}_{N,\omega}^{\lambda,h}$ pour une taille $N$ du copolymère *très grande* (c'est-à-dire que nous sommes intéressés par les résultats asymptotiques dans la limite $N \to \infty$, dite *limite thermodynamique*). Plus précisément, on aimerait comprendre, en fonction des paramètres $\lambda \geq 0$ et $h \geq 0$, si les trajectoires typiques du copolymère restent proche de l'interface (régime *localisé*) ou si elles préfèrent plutôt fluctuer dans les solvants (régime *délocalisé*).

Afin d'avoir un critère quantitatif pour différencier ces deux situations, il est pertinent d'introduire l'*énergie libre* spécifique du système, définie par

$$f_\omega(\lambda, h) \;=\; \lim_{\substack{N\to\infty \\ N\in 2\mathbb{N}}} \frac{1}{N} \log \widetilde{Z}_{N,\omega}^{\lambda,h}\,. \tag{1b.3}$$

Plus précisément :

- quand la suite $\omega$ est périodique, l'existence d'une telle limite peut être prouvée facilement grâce à des arguments standards de super-additivité, voir par exemple [**35**] ;
- dans le cas aléatoire, l'existence de la limite ci–dessus $\mathbb{P}\left(\mathrm{d}\omega\right)$–p.s. et dans $\mathbb{L}_1\left(\mathbb{P}\right)$ découle du Théorème superadditif de Kingman, voir [**35**]. De plus, nous soulignons que dans ce cas la limite ne dépend pas de $\omega$ – ce phénomène

est appelé *self–averaging*. Pour cette raison, lorsque nous traiterons le cas aléatoire nous omettrons la dépendance en $\omega$ de l'énergie libre dans la suite.

Dans les deux cas, périodique et aléatoire, en utilisant des arguments de convexité on peut montrer facilement que l'énergie libre est une fonction continue des paramètres $\lambda$ et $h$.

Une observation simple mais fondamentale est que

$$f_\omega(\lambda, h) \geq \lambda h. \tag{1b.4}$$

En effet si on pose $\Omega_N^+ = \{S : S_n > 0 \text{ for } n = 1, 2, \ldots, N\}$, en restreignant l'intégration sur l'ensemble $\Omega_N^+$ (pour $N$ pair) on obtient

$$\begin{aligned} \frac{1}{N} \log \widetilde{Z}_{N,\omega}^{\lambda,h} &\geq \frac{1}{N} \log \mathbf{E} \left[ \exp \left( \lambda \sum_{n=1}^{N} (\omega_n + h) \operatorname{sign}(S_n) \right) ; \Omega_N^+ \right] \\ &= \frac{\lambda}{N} \sum_{n=1}^{N} (\omega_n + h) + \frac{1}{N} \log \mathbf{P}\left(\Omega_N^+\right) \overset{N\to\infty}{\longrightarrow} \lambda h, \end{aligned} \tag{1b.5}$$

où dans le cas aléatoire la limite s'entend dans le sens $\mathbb{P}(\mathrm{d}\omega)$–presque sûr (par la loi des grands nombres). Cela repose aussi sur le fait bien connu que $\mathbf{P}\left(\Omega_N^+\right)$ est de l'ordre de $N^{-1/2}$ pour $N$ grand [**28**, Ch. III].

L'équation (1b.5) montre que $\lambda h$ est la contribution à l'énergie libre des trajectoires délocalisées dans l'huile. Cette considération conduit au découpage du diagramme de phase suivant :

- la région localisée : $\mathcal{L} = \{(\lambda, h) : f_\omega(\lambda, h) > \lambda h\}$ ;
- la région délocalisée : $\mathcal{D} = \{(\lambda, h) : f_\omega(\lambda, h) = \lambda h\}$.

Cette définition de la (dé)localisation à partir de l'énergie libre peut sembler peu directe, et il n'est pas à priori évident que cela corresponde vraiment à la (dé)localisation des trajectoires typiques sous la mesure du copolymère : cette question est fondamentale, nous la traitons dans § 2.6.

Notre programme est d'étudier en détail le diagramme de phase, à la fois dans le cas périodique et dans le cas aléatoire. Observons qu'à priori il n'est pas évident que $\mathcal{L} \neq \emptyset$, tandis que clairement $\mathcal{D} \supseteq \{(\lambda, h) : \lambda = 0\}$. Commençons par un premier résultat, valide dans les deux cas, qui montre que effectivement $\mathcal{L} \neq \emptyset$ et indique de plus l'existence d'une *courbe critique*, qui sera un des objets essentiels de notre étude.

PROPOSITION 1b.1. *Dans les deux cas, périodique et aléatoire, il existe une fonction continue et croissante* $h_c : [0,\infty) \to [0,\infty)$ *avec* $h_c(0) = 0$ *telle que*

$$\mathcal{D} = \big\{(\lambda,h) :\ h \geq h_c(\lambda)\big\} \qquad \mathcal{L} = \big\{(\lambda,h) :\ h < h_c(\lambda)\big\}\,.$$

En particulier, pour $h = 0$ et $\lambda > 0$ le copolymère est localisé (cela a été prouvé pour la première fois par Sinai dans [**62**]).

En ce qui concerne la preuve de la Proposition 1b.1, observons que de simples arguments de convexité permettent de prouver l'existence d'une courbe critique $h_c(\cdot)$ et le fait que pour $\lambda > 0$ elle est de la forme $h_c(\lambda) = U(\lambda)/\lambda$, avec $U(\cdot)$ une fonction convexe telle que $U(0) = 0$, cf. [**9**, § 1.2]. À partir de cette représentation, plusieurs propriétés élémentaires de cette courbe critique découlent facilement, comme par exemple le fait qu'il existe $\ell \in (0,\infty]$ tel que $h_c(\cdot)$ est continue et non décroissante dans l'intervalle $(0,\ell)$, tandis que $h_c(\lambda) = \infty$ pour $\lambda > \ell$. Il ne reste donc qu'à prouver que $\ell = \infty$ et que $\lambda \mapsto h_c(\lambda)$ est effectivement croissante, et continue en $\lambda = 0$ : la voie la plus simple pour obtenir ces résultats est de combiner les arguments de convexité avec les bornes sur $h_c(\cdot)$ décrites dans § 2.4 et § 2.5.

Dans les sections suivantes nous étudierons les propriétés de la courbe critique $h_c(\cdot)$, qui présente des différences importantes entre le cas périodique et le cas aléatoire. Mais auparavant, nous introduisons quelques modifications sur notre modèle.

**2.3. Une nouvelle fonction de partition.** Le contenu de cette section s'applique à la fois aux cas périodique et aléatoire. D'après (1b.4), il est naturel d'introduire l'*énergie libre résiduelle* $\mathrm{F}_\omega$ définie par

$$\mathrm{F}_\omega(\lambda,h) \ :=\ f_\omega(\lambda,h) - \lambda h\,,$$

de façon à ce que la condition pour la localisation (resp. délocalisation) devienne $\mathrm{F}_\omega(\lambda,h) > 0$ (resp. $\mathrm{F}_\omega(\lambda,h) = 0$). Il est clair qu'on peut obtenir $\mathrm{F}_\omega$ comme l'énergie libre de notre copolymère, en redéfinissant l'Hamiltonien par $\mathcal{H}^{\lambda,h}_{N,\omega} \to \mathcal{H}^{\lambda,h}_{N,\omega} - \lambda h N$ (observons que l'ajout à l'Hamiltonien d'un terme qui ne dépend pas de $S$ n'a pas d'influence sur la mesure du copolymère). Cependant, nous préférons redéfinir l'Hamiltonien d'une manière légèrement différente, en lui soustrayant le terme

$$\lambda \sum_{n=1}^{N} (\omega_n + h)$$

plutôt que $\lambda h N$. Comme ce terme ne dépend pas non plus de $S$, on peut écrire

$$\begin{aligned}\frac{\mathrm{d}\mathbf{P}^{\lambda,h}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) &= \frac{1}{Z^{\lambda,h}_{N,\omega}} \exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h)\left(\operatorname{sign}(S_n) - 1\right)\right) \\ &= \frac{1}{Z^{\lambda,h}_{N,\omega}} \exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h)\,\Delta_n\right) \qquad \text{(1b.6)}\end{aligned}$$

avec $\Delta_n = (1 - \operatorname{sign}(S_n))/2 = \mathbf{1}_{\{\operatorname{sign}(S_n)=-1\}}$ et avec une nouvelle fonction de partition $Z^{\lambda,h}_{N,\omega}$ définie par

$$\begin{aligned}Z^{\lambda,h}_{N,\omega} &:= \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h)\,\Delta_n\right)\right] \qquad \text{(1b.7)} \\ &= \widetilde{Z}^{\lambda,h}_{N,\omega} \exp\left(-\lambda \sum_{n=1}^{N} (\omega_n + h)\right).\end{aligned}$$

Par conséquent, de (1b.3) on obtient

$$\lim_{\substack{N\to\infty \\ N\in 2\mathbb{N}}} \frac{1}{N} \log Z^{\lambda,h}_{N,\omega} = f(\lambda, h) - \lambda h = \mathrm{F}_\omega(\lambda, h)\,, \qquad \text{(1b.8)}$$

où dans le cas aléatoire cette limite a lieu $\mathbb{P}(\mathrm{d}\omega)$–p.s. et dans $\mathbb{L}_1(\mathbb{P})$.

Comme nous le verrons, la nouvelle fonction de partition $Z_{N,\omega}$ se révélera beaucoup plus utile que $\widetilde{Z}_{N,\omega}$ (cela avait été déjà compris dans [**12**]). Pour cette raison, dans la suite nous entendrons toujours $Z_{N,\omega}$ par "fonction de partition", et de la même façon $\mathrm{F}_\omega(\lambda, h)$ sera pour nous l'énergie libre tout court.

Nous utiliserons à plusieurs reprises la fonction de partition associée au modèle *accroché* au point terminal :

$$Z^{\lambda,h}_{N,\omega}(x) := \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h)\,\Delta_n\right) ;\, S_N = x\right]. \qquad \text{(1b.9)}$$

Nous rappelons qu'on peut substituer $Z^{\lambda,h}_{N,\omega}(x)$ à $Z^{\lambda,h}_{N,\omega}$, pour chaque $x$ fixé (avec la même parité que $N$), dans (1b.8) sans que la limite soit modifiée, voir par exemple [**12**] ou [**35**].

**2.4. Le diagramme de phase dans le cas périodique.** Nous verrons que le cas périodique est en essence plus simple que le cas aléatoire. La raison est que, en exprimant la fonction de partition en termes des excursions de la marche aléatoire, on peut se ramener à un problème fini–dimensionnel, comme cela a été montré pour la première fois dans [**11**] (ce point de vue sera exploité en détail dans le Chapitre 4).

Plus précisément, l'énergie libre du modèle peut s'exprimer comme la solution d'un problème de Perron–Frobenius fini–dimensionnel, et à partir de cela on peut obtenir des estimations très précises sur la courbe critique.

Pour présenter ces résultats, nous introduisons le groupe Abelien $\mathbb{S} := \mathbb{Z}/(T_\omega\mathbb{Z})$ (on rappelle que $2T_\omega$ est la période de la suite $\omega$) et définissons la matrice $\mathbb{S} \times \mathbb{S}$ suivante :

$$\Xi_{\alpha,\beta} \;:=\; \sum_{i=2a+1}^{2b} \omega_i \,,$$

qui ne dépend pas des points $a \in \alpha$ et $b \in \beta$ pourvu que $a < b$. Pour $x \in \mathbb{N}$, $\alpha, \beta \in \mathbb{S}$ et $\lambda, h \geq 0$, nous introduisons d'autre part

$$\Phi^{\lambda,h}_{\alpha,\beta}(x) \;:=\; \log\left(\frac{1 + 2\exp\big(-2(\lambda\Xi_{\alpha,\beta} + \lambda h x)\big)}{2}\right).$$

Puis, en notant $K(x) := \mathbf{P}(\tau_1 = 2x)$, où $\tau_1 := \inf\{n > 0 :\, S_n = 0\}$ est le premier temps de retour à zéro de la marche, on définit pour $b \geq 0$ la matrice $\mathbb{S} \times \mathbb{S}$ suivante, dont les entrées sont non négatives :

$$A_{\alpha,\beta}(b; \lambda, h) \;:=\; \sum_{x \in \mathbb{N}} \exp\big(\Phi^{\lambda,h}_{\alpha,\beta}(2x) - b(2x)\big)\, K(2x)\, \mathbf{1}_{(x \in \beta - \alpha)} \,,$$

et on dénote par $\mathtt{Z}(b; \lambda, h)$ sa valeur propre de Perron–Frobenius, cf. [**5**]. Observons que $\mathtt{Z}$ est une fonction décroissante de $b$ et $h$ dans la mesure où les entrées $A_{\alpha,\beta}$ vérifient cette propriété pour tous $\alpha, \beta$. L'énergie libre du modèle est donc donnée par le théorème suivant (cf. [**11**, Th. 1.2]) :

Théorème 1b.1. *Pour tous $\lambda, h \geq 0$ on note $b = \widetilde{b}(\lambda, h)$ l'unique solution de l'équation implicite $\mathtt{Z}(b; \lambda, h) = 1$ si une telle solution existe, et on pose $\widetilde{b}(\lambda, h) = 0$ dans le cas contraire. Alors $\widetilde{b}(\lambda, h)$ est exactement l'énergie libre de notre modèle :*

$$\mathrm{F}_\omega(\lambda, h) \;=\; \widetilde{b}(\lambda, h)\,.$$

En particulier il s'ensuit que la courbe critique $h = h_c(\lambda)$ est déterminée par l'équation implicite $\mathtt{Z}(0, \lambda, h_c(\lambda)) = 1$, et à partir de cette relation on peut extraire le comportement asymptotique de $h_c(\lambda)$ à la fois pour $\lambda \to 0$ et pour $\lambda \to \infty$ (cf. [**11**, Th. 1.3]) :

THÉORÈME 1b.2. *Il existe deux constantes positives $m_\omega > 0$, $M_\omega > 0$ telles que:*

$$\begin{aligned} \textit{quand } \lambda \to 0 \qquad & h_c(\lambda) \;=\; m_\omega \lambda^3 \big(1 + o(1)\big) \\ \textit{quand } \lambda \to \infty \qquad & h_c(\lambda) \;=\; \max_{n=1,\dots,T} \Big( - \frac{\omega_{2n-1} + \omega_{2n}}{2} \Big) - \frac{\big(M_\omega + o(1)\big)}{\lambda} \,. \end{aligned}$$

Ces résultats donnent une caractérisation très satisfaisante du diagramme de phase du copolymère dans le cas périodique. Nous soulignons que la preuve du Théorème 1b.1 donnée dans [**16**] utilise de façon essentielle la théorie des Grandes Déviations. Nous ne la reportons pas ici dans la mesure où dans le Chapitre 4 nous présentons une approche basée sur la Théorie de Renouvellement qui permet une analyse beaucoup plus détaillée pour une vaste classe de modèles de polymères périodiquement inhomogènes, comprenant en particulier le copolymère périodique en proximité d'une interface sélective, et le Théorème 1b.1 sera une conséquence de nos résultats principaux (voir Théorème 4.5 dans le Chapitre 4).

**2.5. Le diagramme de phase dans le cas aléatoire.** Dorénavant quand on traitera le cas aléatoire on omettra la dépendance en $\omega$ de l'énergie libre, qui sera indiquée simplement par $\mathrm{F}(\lambda, h)$. Dans le théorème suivant nous résumons ce qui a été prouvé dans la littérature concernant la courbe critique du modèle aléatoire (voir Figure 1b.3 pour une représentation graphique).

THÉORÈME 1b.3. *Pour tout $\lambda > 0$ on a les bornes suivantes :*

$$\underline{h}(\lambda) \;:=\; \frac{1}{4\lambda/3} \log \mathsf{M}\,(-4\lambda/3) \;\le\; h_c(\lambda) \;\le\; \frac{1}{2\lambda} \log \mathsf{M}\,(-2\lambda) \;=:\; \overline{h}(\lambda). \qquad \text{(1b.10)}$$

*En particulier, la pente à l'origine de $h_c(\cdot)$ est dans l'intervalle $[2/3, 1]$, dans le sens que la limite inférieure de $h_c(\lambda)/\lambda$ quand $\lambda \searrow 0$ n'est pas plus petite que $2/3$ et la limite supérieure n'est pas plus grande que $1$.*

On rappelle que $\mathsf{M}(\cdot)$ est la fonction génératrice des moments de $\omega_1$, voir (1b.2). Observons que la dernière affirmation découle immédiatement de (1b.10) en appliquant le développement asymptotique $\mathsf{M}(\alpha) = 1 + \alpha^2/2 + O(\alpha^3)$ pour $\alpha \to 0$ (on rappelle qu'on a imposé $\mathbb{E}[\omega_1^2] = 1$).

Remarquons que la différence principale par rapport au cas périodique, cf. Théorème 1b.2, est donnée par le comportement asymptotique de $h_c(\lambda)$ pour $\lambda \to 0$ : on pourrait dire que quand $\lambda$ est petit la localisation du polymère est plus simple dans le cas aléatoire que dans le cas périodique. On peut comprendre cela en

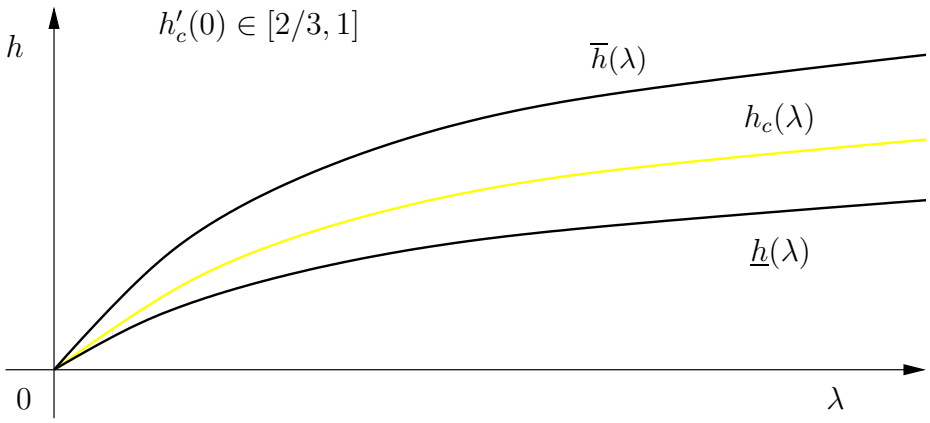


FIGURE 1B.3. Le diagramme de phase dans le cas aléatoire.

considérant que quand $\lambda$ est petit un rôle important est joué par les *grandes excursions* de la marche et en observant que la contribution énergétique d'une excursion de longueur $L$ est $O(\sqrt{L})$ dans le cas aléatoire (par le Théorème Central Limite), tandis que dans le cas périodique elle est bien sûr $O(1)$.

Giacomin et Toninelli ont prouvé dans [**36**], en utilisant des Inégalités de Concentration, que la limite de $h_c(\lambda)/\lambda$ quand $\lambda \to 0$ existe et *ne dépend pas de la loi de* $\omega_1$, pourvu que $\omega_1$ soit une variable symétrique bornée ou bien une variable Gaussienne standard. On verra que la pente à l'origine est aussi liée au diagramme de phase d'un modèle de copolymère Brownien, qui apparaît comme la limite d'échelle de notre modèle de copolymère quand $\lambda, h \to 0$, voir § 3.2. Ce *caractère universel* de la pente à l'origine rend cette quantité très intéressante.

Le Théorème 1b.3 est une simple généralisation des résultats prouvés dans [**12**] et [**9**] : l'extension consiste dans le fait que $\omega_1$ n'est pas nécessairement symétrique et cela ne demande que de changements minimaux. On notera que la *borne inférieure* $\underline{h}(\cdot)$ et la *borne supérieure* $\overline{h}(\cdot)$ diffèrent seulement par un facteur d'échelle : néanmoins leur origine est différente, comme on verra dans un instant. Remarquons que dans la littérature physique les deux conjectures $h_c(\cdot) = \underline{h}(\cdot)$ [**50, 64**] et $h_c(\cdot) = \overline{h}(\cdot)$ [**34, 69**] ont été proposées.

2.5.1. *La borne supérieure.* Nous reportons ici la preuve de la borne supérieure $h_c(\cdot) \le \overline{h}(\cdot)$ donnée dans [**12**], qui est tout à fait élémentaire. La limite dans (1b.8)

a lieu aussi dans $\mathbb{L}_1(\mathbb{P})$ et en appliquant l'inégalité de Jensen on peut écrire

$$\mathrm{F}(\lambda,h) \;=\; \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z^{\lambda,h}_{N,\omega} \;\le\; \limsup_{N\to\infty}\frac{1}{N}\log \mathbb{E}Z^{\lambda,h}_{N,\omega}\,,$$

et de (1b.7) on tire

$$\mathbb{E}Z^{\lambda,h}_{N,\omega} = \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}\big(\log \mathsf{M}(-2\lambda)-2\lambda h\big)\,\Delta_n\right)\right]. \tag{1b.11}$$

Comme pour $h \ge \overline{h}(\lambda)$ l'argument de l'exponentielle est négatif, $\mathbb{E}Z^{\lambda,h}_{N,\omega} \le 1$ et la borne supérieure suit immédiatement. De plus pour $h < \overline{h}(\lambda)$ il est facile de vérifier que $\lim_{N\to\infty}\big(\log \mathbb{E}Z^{\lambda,h}_{N,\omega}\big)/N > 0$ et $\overline{h}(\lambda)$ est donc effectivement la meilleure borne supérieure qu'on puisse extraire de (1b.11).

Cette manière d'obtenir une borne supérieure en effectuant l'intégration $\mathbb{E}$ par rapport aux variables du désordre *avant* de prendre le logarithme est un outil standard dans la Mécanique Statistique des systèmes désordonnés qui est appelé *borne annealed*. Néanmoins il faut observer que dans notre cas cette approche n'est pas complètement triviale : par exemple il est facile de voir qu'en suivant les mêmes étapes avec l'ancienne de partition $\widetilde{Z}^{\lambda,h}_{N,\omega}$ on obtient une borne inutile. La raison pour cela est que $Z^{\lambda,h}_{N,\omega}$ a été obtenue en ajoutant à l'Hamiltonien le terme $-\lambda\sum_{n=1}^{N}(\omega_n+h)$, qui d'une part ne dépend pas de $S$ (et donc ne modifie pas la mesure du copolymère) mais qui d'autre part présente une forte dépendance en $\omega$, capable d'influencer la borne annealed de façon drastique.

À ce point il est clair que l'on pourrait itérer la procédure, en ajoutant à l'Hamiltonien des termes supplémentaires qui dépendraient de $\omega$ afin d'améliorer la borne supérieure. Malheureusement l'application standard de cette technique à notre modèle, appelée *constrained annealing*, ne peut pas améliorer la borne annealed $\overline{h}(\cdot)$ sur la courbe critique : ce point est l'objet du Chapitre 3, où cette technique est expliquée en détail.

Bien entendu, les difficultés que l'on rencontre en essayant d'améliorer la borne supérieure $\overline{h}(\cdot)$ pourraient être dues au fait que $\overline{h}(\cdot)$ est la vraie courbe critique. Cependant, l'analyse numérique développée dans le Chapitre 2 semble indiquer que tel n'est pas le cas.

2.5.2. *La borne inférieure.* La preuve donnée dans [**9**] de la borne inférieure $h_c(\cdot) \ge \underline{h}(\cdot)$ consiste à calculer explicitement la contribution d'énergie/entropie à la fonction de partition donnée par une *stratégie* opportune pour le copolymère. Grossièrement, la stratégie choisie consiste à obliger le copolymère à passer la plus

grande partie de son temps dans le demi–plan supérieur, en le faisant descendre dans le demi–plan inférieur uniquement sur les *longues portions* de la suite des charges $\omega = \{\omega_n\}_n$ qui ont une moyenne empirique *atypiquement négative*. La statistique de ces longues portions est décrite par la fonction de taux des Grandes Déviations [**21**] pour des sommes de variables aléatoires IID distribuées comme $\omega_1$, qui n'est que la transformée de Legendre de $\log \mathsf{M}(\cdot)$ : c'est la raison pour laquelle cette forme fonctionnelle apparaît aussi dans la borne inférieure.

Nous ne reportons pas les détails de la preuve, puisque nous donnons une preuve alternative de la borne inférieure dans le Chapitre 2 : voir Section 3 pour un exposé des idées de la preuve et § 6.2 pour les détails. Nous soulignons que l'idée derrière la stratégie décrite ci-dessus (et aussi derrière notre preuve) prend son inspiration d'un *schéma de renormalisation* (non rigoureux) pour l'étude des systèmes désordonnés unidimensionnels, qui a été appliqué au modèle du copolymère par C. Monthus [**50**]. Cette approche a été proposée pour la première fois par D. S. Fisher dans le contexte du modèle d'Ising quantique avec un champ magnétique transverse [**30**] et appliquée ensuite à l'étude de la marche aléatoire dans un milieu aléatoire [**45**] avec un succès remarquable.

Nous soulignons que la borne inférieure $\underline{h}(\cdot)$ sur la courbe critique semble être très robuste : nous avons essayé en vain d'enrichir la stratégie décrite ci–dessus (c'est-à-dire de garder un nombre plus grand de trajectoires de la marche) pour améliorer la borne. Bien sûr il pourrait se trouver que $h_c(\cdot) = \underline{h}(\cdot)$, mais dans le Chapitre 2 nous présentons plusieurs observations numériques et un test statistique rigoureux qui indiquent clairement que $h_c(\cdot) > \underline{h}(\cdot)$.

**2.6. Le comportement des trajectoires.** Dans cette section on verra que la division du diagramme de phase dans les deux régions $\mathcal{L}$ et $\mathcal{D}$, qui ont été définies en fonction de l'énergie libre, correspond en effet à un comportement radicalement différent des trajectoires typiques de la mesure du copolymère, pour le moins si on ne considère pas le cas critique, c'est-à-dire si on considère le système pour $(\lambda, h) \in \mathcal{L}$ ou pour $(\lambda, h)$ dans l'intérieur de la région $\mathcal{D}$ (qui sera appelé dans la suite *région strictement délocalisée*). On verra que le régime localisé est bien compris, alors que le régime délocalisé reste plus élusif. Commençons par le cas périodique :

2.6.1. *Le cas périodique.* Des résultats très forts de localisation pour les trajectoires peuvent être obtenus en appliquant la technique utilisée dans [**62**] par Sinai pour étudier le cas aléatoire. Plus précisément, si $(\lambda, h) \in \mathcal{L}$ alors pour tout $\varepsilon > 0$ il

existe des constantes positives $N_0 > 0$, $L_0 > 0$ telles que pour tout $N \geq N_0$

$$\sup_{n=1,\dots,N} \mathbf{P}^{\lambda,h}_{N,\omega}\big(|S_n| > L\big) \;\leq\; \exp\big(-(\mathrm{F}_\omega(\lambda,h)-\varepsilon)\,L\big) \qquad \forall L \geq L_0\,. \tag{1b.12}$$

De plus, la méthode utilisant la théorie des Grandes Déviations exploitée dans [**11**] donne des informations détaillées sur les retours à zéro sous la mesure du copolymère, qui forment un ensemble de densité positive, voir [**11**, § 1.7].

Par contre, les résultats disponibles pour la phase délocalisée sont moins précis : le seul résultat connu en complète généralité est que dans le régime strictement délocalisé le polymère passe presque tout son temps au dessus de n'importe quel niveau préfixé, c'est-à-dire que pour tout $L > 0$,

$$\lim_{N\to\infty} \mathbf{E}^{\lambda,h}_{N,\omega}\left[\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{(S_n\geq L)}\right] = 1\,. \tag{1b.13}$$

Des résultats plus forts ont étés obtenus pour des exemples plus spécifiques : par exemple dans [**51**] le cas $\omega_n = (-1)^n$ a été traité pour un modèle de copolymère qui diffère du nôtre par la définition de sign(0) (nous renvoyons à [**11**, § 1.5] pour plus de détails sur l'implication de cette modification). Les auteurs calculent la loi des retours à zéro sous la mesure du copolymère, d'où on peut extraire les limites d'échelle Browniennes du modèle en utilisant les idées de [**40**] ou l'approche générale et plus robuste que nous adoptons dans le Chapitre 4. Plus précisément, on peut prouver que pour $(\lambda, h)$ dans l'intérieur de la région $\mathcal{D}$ la loi du processus $\{S_{\lfloor tN\rfloor}/\sqrt{N}\}_{t\in[0,1]}$ sous la mesure du copolymère $\mathbf{P}^{\lambda,h}_{N,\omega}$ converge faiblement vers la loi du méandre Brownien (c'est-à-dire la loi d'un mouvement Brownien standard conditionné à ne pas entrer dans le demi-plan inférieur, cf. [**60**]). On peut faire l'analyse aussi dans le cas critique, c'est-à-dire quand $h = h_c(\lambda)$, en montrant que dans ce cas le processus qui apparaît comme limite d'échelle est la valeur absolue d'un mouvement Brownien.

Nous soulignons que la preuve de ces résultats à été obtenue essentiellement par des calculs explicites : en effet en considérant les marginales d'un copolymère de période 2 sur les sites pairs on obtient un modèle d'accrochage/décrochage homogène qui est exactement résoluble [**40**] (voir aussi § 3.1 ci–dessous). Néanmoins, on pense que ces résultats devraient être vrais pour tout $\omega$ périodique.

Dans le Chapitre 4 nous prouverons en effet que les deux limites d'échelle Brownienne, dans le cas strictement délocalisé et critique, ont lieu en complète généralité pour une vaste classe de modèles de polymères périodiquement inhomogènes, y compris le copolymère à proximité d'une interface sélective (et aussi les modèles de

mouillage et d'accrochage/décrochage, qui seront étudiés dans § 3.1 ci–dessous). De plus nous donnerons aussi une description précise des *propriétés locales* des trajectoires de la mesure du copolymère dans la limite thermodynamique pour tous les régimes, y compris le régime localisé.

2.6.2. *Le cas aléatoire.* Dans le cas aléatoire il est établi de même que pour $(\lambda, h) \in \mathcal{L}$ les trajectoires du copolymère sont localisées dans un sens fort. L'analogue aléatoire de l'équation (1a.12) a été prouvé par Sinai dans [**62**] pour le cas $\lambda > 0, h = 0$ (mais la méthode peut être étendue à toute la région localisée, cf. [**35**]). Une certaine attention est nécessaire pour une formulation correcte : dans cette section on travaillera avec des suites de charges doublement infinies, c'est-à-dire qu'on suppose que $\omega = \{\omega_n\}_{n\in\mathbb{Z}}$ est un élément de l'espace $\Omega := \mathbb{R}^{\mathbb{Z}}$ et $\mathbb{P}$ est bien sûr la loi produit sur $\Omega$. On définit aussi pour $n \in \mathbb{N}$ la translation $\theta^n$ sur $\Omega$ par $(\theta^n\omega)_k := \omega_{n+k}$. On peut alors énoncer le résultat de Sinai : pour tout $\varepsilon > 0$ il existe deux variables aléatoires $N_0(\omega)$, $L_0(\omega) : \Omega \to \mathbb{N}$ telles que pour $\mathbb{P}$–presque tout $\omega$ et pour tout $N \geq N_0(\omega)$ on a la relation suivante :

$$\begin{aligned} \forall n \in \big\{ \log^\gamma N, \dots, N - \log^\gamma N \big\} \qquad \forall L \geq L_0\big(\theta^n(\omega)\big) \\ \mathbf{P}^{\lambda,h}_{N,\omega}\big(|S_n| > L\big) \;\leq\; \exp\big( - \big(\mathrm{F}_\omega(\lambda,h) - \varepsilon\big)\, L \big)\,, \end{aligned} \tag{1b.14}$$

où $\gamma > 0$ est une constante absolue (qui ne dépend ni de $\varepsilon$ ni de $\omega$).

Quelques observations sont nécessaires. La restriction sur les valeurs de $n$ n'a été imposée que par soucis de simplicité : il est possible d'étendre la relation ci–dessus à tout $n \leq N$, mais les notations deviennent plus compliquées. Le point clé est plutôt la condition $L \geq L_0\big(\theta^n(\omega)\big)$, qui signifie que le "rayon de localisation" dépend de $n$, et en effet celui-ci peut être arbitrairement grand puisque la variable aléatoire $L_0$ n'est pas bornée. Cette restriction peut faire apparaître l'estimation (1b.14) comme insatisfaisante, mais elle est en fait inévitable et la raison pour cela est due à la présence de *portions atypiques arbitrairement longues* dans la suite $\omega$ : en effet si un site $n$ est entouré par un segment $\{\omega_{n-k}, \dots, \omega_{n+k}\}$ avec une moyenne empirique atypiquement positive, cela produira un effet répulsif qui poussera $S_n$ à une hauteur d'ordre $\approx \sqrt{k}$ (dans le Chapitre 2 on verra qu'on peut aussi profiter des segments atypiques).

Quoi qu'il en soit, la situation n'est pas si mauvaise. D'un côté, la variable aléatoire $L_0$ peut être choisie telle que $\mathbb{E}\big[\exp(\alpha L_0)\big] < \infty$ pour un $\alpha > 0$ et donc pour $\mathbb{P}$–presque tout $\omega$ on a que $L_0(\theta^n\omega) \leq \alpha^{-1} \log n$ pour $n$ assez grand : par

conséquent le rayon de localisation est dans tous cas beaucoup plus petit que la taille du copolymère. De l'autre côté, le Théorème Ergodique de Birkhoff entraîne que pour tout $K > 0$, $\mathbb{P}(\mathrm{d}\omega)$–p.s.

$$\lim_{N\to\infty} \sum_{n=1}^{N} \mathbf{1}_{\{L_0(\theta^n\omega)>K\}} \;=\; \mathbb{P}\big(L_0 > K\big)\,,$$

donc en choisissant $K$ suffisamment grand on a que le rayon de localisation est plus petit que $K$ *pour la plus grande fraction du temps*.

On observe que des résultats de localisation très forts ont lieu aussi pour la limite thermodynamique de la mesure du copolymère : nous ne les reportons pas ici, et renvoyons à [**8, 1**] pour plus de détails.

Passons au régime délocalisé, en soulignant que pratiquement aucun résultat n'est pour l'instant disponible pour le cas critique. Dans la région strictement délocalisée la situation est dans un certain sens meilleure : par exemple on sait que (1b.13) est vraie pour $\mathbb{P}$–presque tout $\omega$. De toute façon, il s'agit d'une information plutôt faible sur le comportement des trajectoires, surtout par rapport aux résultats correspondants pour le cas périodique (et plus généralement pour les modèle non désordonnés, voir par exemple [**22**] et les référence citées), c'est-à-dire la limite d'échelle Brownienne.

La manière standard de prouver cette limite d'échelle pour le régime strictement délocalisé est de montrer que sous la mesure du copolymère $\mathbf{P}^{\lambda,h}_{N,\omega}$ l'instant de la dernière visite au demi–plan inférieur est $o(N)$. En fait pour les modèles non désordonnés on en sait beaucoup plus : dans la limite $N \to \infty$ le polymère devient transient et il ne visite le demi–plan inférieur (ou tout point au dessous de n'importe quel niveau préfixé) qu'un *nombre fini* de fois. La situation est différente pour le copolymère dans le cas aléatoire : en effet dans [**36**] il a été montré que pour $h < \overline{h}(\lambda)$ le nombre de visites au demi–plan inférieur sous la *mesure quenched moyennée* $\mathbb{E}\mathbf{E}^{\lambda,h}_{N,\omega}[\,\cdot\,]$ est $O(\log N)$. Cela ne suffit pas pour obtenir la limite d'échelle : on sait que le nombre de visites au demi–plan inférieur est $o(N)$ mais on ne sait pas si elles ont toutes lieu à proximité de l'origine (on renvoie à [**36**] pour une discussion plus détaillée de ce qu'il reste à prouver).

Soulignons que dans ces problèmes un rôle très important est joué par le comportement asymptotique de la fonction de partition $Z_{N,\omega}$ quand $N \to \infty$, dans l'intérieur de la région délocalisée. Dans le cas non désordonné on sait que $Z_{N,\omega} \approx N^{-1/2}$, voir par exemple le Théorème 4.5 du Chapitre 4. Par contre, on sait que

ce genre de comportement asymptotique n'est plus vrai dans le cas aléatoire : plus précisément pour tout $(\lambda, h)$ dans l'intérieur de la région $\mathcal{D}$ il existe $\varepsilon > 0$ et une sous–suite $\{\tau_N(\omega)\}_N$ telle que $N^{1/2-\varepsilon} Z_{\tau_N(\omega),\omega} \to \infty$ quand $N \to \infty$, voir Proposition 4.1 dans [**36**].

Le problème du comportement des trajectoires dans la région délocalisée pour le cas aléatoire est examiné sous d'autres aspects dans § 4.1 du Chapitre 2.

## 3. Autres modèles de polymères

**3.1. Accrochage à une interface et modèles de mouillage.** Un autre problème qui a été l'objet de beaucoup d'attention est la situation où une chaîne polymérique est attirée (ou repoussée) par une interface, qui peut être soit pénétrable soit impénétrable. On peut modéliser cette situation en donnant une récompense (ou une pénalisation) à chaque monomère situé sur l'interface, cette récompense/pénalisation peuvant varier d'un monomère à l'autre si la chaîne est hétérogène. Comme pour le modèle de copolymère analysé dans la section précédente, cette modification peut altérer considérablement les trajectoires de la marche en faisant apparaître une transition de localisation/délocalisation.

Nous allons définir un modèle probabiliste pour ces situations quand l'interface est plate (elle sera pour nous l'axe des abscisses). On commence par le cas où l'interface est pénétrable (*modèle d'accrochage/décrochage*) : comme dans la section précédente, on prend une marche aléatoire simple $\{S_n\}_n$ avec loi $\mathbf{P}$, et pour $N \in 2\mathbb{N}$, $\beta \in \mathbb{R}$ et $\omega = \{\omega_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ on définit une nouvelle loi $\mathbf{P}^{\beta}_{N,\omega}$ par

$$\frac{\mathrm{d}\mathbf{P}^{\beta}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) \;\propto\; \exp\left(\beta \sum_{n=1}^{N} \omega_n \mathbf{1}_{(S_n=0)}\right). \tag{1b.15}$$

Le cas de l'interface impénétrable est obtenu en interdisant aux trajectoires d'entrer dans le demi–plan inférieur jusqu'à l'instant $N$, c'est-à-dire en multipliant le membre de droite ci-dessus par $\mathbf{1}_{(S_1\geq 0,\ldots,S_N\geq 0)}$. Ce deuxième cas sera appelé un *modèle de mouillage*, dans la mesure où il peut être interprété comme un modèle pour une interface qui interagit avec un mur impénétrable.

Nous considérerons seulement les cas où la suite des charges $\omega$ est soit périodique (et déterministe) ou aléatoire (quenched), et notre but sera de comprendre le comportement de la mesure ci-dessus quand $N$ est grand. Les régimes de localisation/délocalisation peuvent être définis en fonction de l'énergie libre, exactement comme dans le cas du copolymère à proximité d'une interface sélective.

Un cas particulièrement simple est le cas homogène, quand la suite $\omega$ est constante : $\omega_n = \omega_1$ pour tout $n \in \mathbb{N}$, et à une redéfinission près du paramètre $\lambda$ on peut supposer $\omega_1 = 1$. Remarquons que dans ce cas les deux modèles d'accrochage/décrochage et de mouillage sont complètement résolubles, non seulement en ce qui concerne l'obtention du diagramme de phase (voir par exemple [**35**] pour une dérivation élémentaire de l'énergie libre) mais aussi l'analyse très détaillée du comportement des trajectoires du polymère [**40, 22**]. Nous ne voulons pas nous étendre sur cela, puisque dans le Chapitre 4 nous étudierons en détail le cas où la suite $\omega$ est périodique. Néanmoins, observons qu'avec des arguments de convexité il est facile de montrer que le diagramme de phase dans le cas homogène dépend d'un seul nombre $\beta_c$ tel que pour $\beta > \beta_c$ (resp. pour $\beta \leq \beta_c$) le polymère est localisé (resp. délocalisé). De plus :

- dans le modèle d'accrochage/décrochage $\beta_c = 0$ : cela signifie qu'une récompense arbitrairement petite est suffisante pour localiser le polymère ;
- dans le modèle de mouillage au contraire $\beta_c > 0$.

La raison pour laquelle le modèle de mouillage est plus difficilement localisé que le modèle d'accrochage/décrochage est que le conditionnement de la marche à rester non négative jusqu'au pas $N$ produit un effet de répulsion d'ordre $\sqrt{N}$ sur les trajectoires, un phénomène appelé *répulsion entropique*. Dans notre situation unidimensionnelle, une version plus précise de ce phénomène est donnée par le principe d'invariance suivant [**10**] : le processus $\{S_{\lfloor Nt \rfloor}/\sqrt{N}\}_{t\in[0,1]}$ conditionné à l'événement $\{S_1 \geq 0, \ldots, S_N \geq 0\}$ converge en loi quand $N \to \infty$ vers le méandre Brownien, c'est-à-dire vers un mouvement Brownien conditionné à rester non négatif [**60**]. Dans le Chapitre 6 nous prouverons une version locale de cette convergence en loi.

Nous avons déjà annoncé que nous étudierons dans le Chapitre 4 ces modèles dans le cas périodique. Par ailleurs, dans cette thèse nous ne considérerons pas le cas aléatoire. Mentionnons seulement que, de même que pour le copolymère à proximité d'un interface sélective, il n'y a pas de consensus dans la littérature physique sur le diagramme de phase du modèle, notamment sur les asymptotiques pour les petites

valeurs des paramètres : pour plus de détails sur ces questions et pour les résultats rigoureux disponibles nous renvoyons à [**2, 57**].

En conclusion, soulignons la pertinence des modèles basés sur une marche aléatoire pour la modélisation des molécules d'ADN. L'ADN a normalement une structure en forme de double hélice, mais il peut arriver que les deux brins composant cette double hélice se séparent, par exemple quand la température est suffisamment élevée (*transition de dénaturation*) ou sous l'effet d'une force externe (*pulling induced unzipping*). Comme l'interaction entre les deux brins peut être décrite (au moins en première approximation) par un Hamiltonien de la forme (1b.15), la compétition entre énergie et entropie qui conduit à de telles transitions de phase peut être comprise en étudiant les modèles d'accrochage/décrochage et de mouillage que nous venons de décrire.

**3.2. Un modèle Brownien : la limite d'échelle de l'énergie libre.** Un des principaux résultats de l'article de Bolthausen et Den Hollander [**12**] est que dans la limite de faible couplage le modèle de copolymère décrit dans la Section 2 peut être approximé par un modèle continu construit avec des mouvements Browniens en place des marches aléatoires. Ce modèle continu est défini d'une manière complètement analogue au modèle discret : on prend deux mouvements Browniens $B = \{B_t\}_{t\geq 0}$ (le polymère) et $\beta = \{\beta_t\}_{t\geq 0}$ (ses charges), de lois respectives $\widetilde{\mathbf{P}}$ et $\widetilde{\mathbb{P}}$, et pour $t > 0$, $\lambda, h \geq 0$ et $\{\beta_s\}_s$ trajectoire typique sous $\widetilde{\mathbb{P}}$, on introduit la mesure du polymère $\widetilde{\mathbf{P}}_{t,\beta}^{\lambda,h}$ sur les trajectoires de longueur $t$, définie par

$$\frac{\mathrm{d}\widetilde{\mathbf{P}}_{t,\beta}^{\lambda,h}}{\mathrm{d}\widetilde{\mathbf{P}}}(B) \;:=\; \frac{1}{Z_{t,\beta}^{\lambda,h}} \,\exp\left(\lambda \int_0^t \mathrm{sign}(B_s)\,\big(\mathrm{d}\beta_s + h\mathrm{d}s\big)\right),$$

où l'intégrale par rapport à $\beta_s$ est une intégrale d'Itô. La fonction de partition du modèle est donnée par

$$Z_{t,\beta}^{\lambda,h} \;=\; \widetilde{\mathbf{E}}\exp\left(\lambda \int_0^t \mathrm{sign}(\beta_s)\,\big(\mathrm{d}\beta_s + h\mathrm{d}s\big)\right),$$

et l'énergie libre $\widetilde{f}(\lambda, h)$ est définie par

$$\widetilde{f}(\lambda, h) \;:=\; \lim_{t\to\infty} \frac{1}{t}\,\log Z_{t,\beta}^{\lambda,h}\,,$$

où la limite a lieu à la fois $\widetilde{\mathbb{P}}$–p.s. et dans $\mathbb{L}_1(\widetilde{\mathbb{P}})$. Qui plus est, $\widetilde{f}(\lambda, h)$ n'est pas aléatoire, c'est-à-dire qu'elle ne dépend pas de $\beta$ (voir [**35**] pour une preuve détaillée

de l'existence d'une telle limite). Comme dans le cas discret, on a que

$$\widetilde{f}(\lambda, h) \;\geq\; \lambda h\,,$$

ce qui permet de distinguer entre un régime délocalisé ($\widetilde{f}(\lambda, h) = \lambda h$) et un régime localisé ($\widetilde{f}(\lambda, h) > \lambda h$). La propriété d'invariance par changement d'échelle (scaling) du mouvement Brownien entraîne que pour tout $a > 0$

$$\widetilde{f}(\lambda, h) \;=\; \frac{1}{a^2}\widetilde{f}(a\lambda, ah)\,,$$

d'où il suit immédiatement que la courbe critique de ce modèle est une *droite*, c'est-à-dire

$$\exists K_c > 0 : \quad \widetilde{f}(\lambda, h) \begin{cases} = \lambda h & \text{si } h \geq K_c\lambda \\ > \lambda h & \text{si } h < K_c\lambda \end{cases}.$$

En dépit de l'apparente simplicité du diagramme de phase, on peut dire que ce modèle continu contient dans la constante $K_c$ toute la complexité du modèle discret. Cette affirmation est précisée dans le théorème fondamental qui suit (cf. [**12**, Th. 5 et 6]), qui indique aussi dans quelle mesure le modèle continu est une approximation du modèle discret.

THÉORÈME 1b.4. *Soit $f(\lambda, h)$ l'énergie libre du modèle discret (voir éq.* (1b.3)*) dans le cas où $\mathbb{P}(\omega_1 = +1) = \mathbb{P}(\omega_1 = -1) = 1/2$, et $h = h_c(\lambda)$ la courbe critique correspondante (cf. Prop.* (1b.1)*). On a alors les relations suivantes :*

$$\lim_{a\to 0} \frac{1}{a^2} f(a\lambda, ah) \;=\; \widetilde{f}(\lambda, h) \quad \forall \lambda, h \geq 0 \tag{1b.16}$$

$$h_c'(0) \;:=\; \lim_{\lambda\to 0} \frac{h_c(\lambda)}{\lambda} \;=\; K_c\,. \tag{1b.17}$$

En particulier de l'équation (1b.10) il s'ensuit que $2/3 \leq K_c \leq 1$.

Remarquons que l'équation(1b.17) ne découle pas directement de (1b.16) : en effet la limite d'échelle de l'énergie libre exprimée par l'équation (1b.16) n'implique que la borne inférieure $h_c'(0) \geq K_c$. La preuve de (1b.17) s'obtient par des inégalités de comparaison très précises entre $f$ et $\widetilde{f}$ et demande des estimations très fines.

Soulignons d'autre part que le Théorème 1b.4 a été prouvé dans le cas où les charges ont une loi de Bernoulli symétrique, mais devrait être valable dans un cadre beaucoup plus général. L'explication intuitive serait que lorsque $\lambda, h \to 0$, seules les *grandes excursions* de la marche jouent un rôle fondamental, et par conséquence les détails microscopiques du modèle devraient être peu importants.

Ainsi, répétons que dans [**36**] il a été prouvé à l'aide d'Inégalités de Concentration que pour que (1b.17) ait lieu il suffit que $\omega_1$ soit bornée et symétrique (et telle que $\mathbb{E}[\omega_1] = 0$ et $\mathbb{E}[{\omega_1}^2] = 1$) ou bien que $\omega_1$ soit une variable Normale standard. Par ailleurs, la preuve originale du Théorème 1b.4 donnée dans [**12**] peut être adaptée pour montrer qu'en effet les deux équations (1b.16) et (1b.17) sont satisfaites pour n'importe quel choix de la loi de $\omega_1$ satisfaisant (1b.2) et telle que $\mathbb{E}[\omega_1] = 0$, $\mathbb{E}[{\omega_1}^2] = 1$.

Dans le Chapitre 5 nous introduirons un autre type de variante du modèle discret : plus précisément nous changerons la loi $\mathbf{P}$ de la marche aléatoire, en considérant des marches aléatoires réelles dont les accroissements sont bornés et de loi absolument continue. Intuitivement, cette modification ne devrait pas changer les conclusions du Théorème 1b.4 non plus, mais donner un preuve complète de ce fait n'est pas du tout trivial. Pour l'instant, nous avons réalisé quelques étapes dans la direction d'une preuve de l'équation (1b.16) : nous renvoyons au Chapitre 5 pour plus de détails sur ce problème.

## 4. Aperçu de la bibliographie

Le problème du copolymère à proximité d'une interface sélective a une longue histoire, notamment dans les domaines de la physique et de la chimie, mais le premier article qui a attiré l'attention des mathématiciens est probablement [**34**]. La première étude mathématique sur le sujet a été réalisée par Sinai dans [**62**] : le résultat principal de ce papier est que pour $h = 0$ et $\lambda > 0$ ($\lambda$ et $h$ sont les paramètres du modèle introduit dans la Section 2) les trajectoires typiques du copolymère couvert de charges aléatoire sont localisées dans un sens très fort (voir § 2.6 dessus). Dans le travail ultérieur de Albeverio et Zhou [**1**], on trouve une étude des trajectoires et une analyse détaillée de l'énergie libre (toujours pour le cas aléatoire et pour $h = 0$).

Comme on l'a vu, dans le cas des charges aléatoires nous avons étés principalement intéressés par le diagramme de phase, ce qui implique l'étude du copolymère pour $h > 0$. L'article fondamental dans cette direction est celui de Bolthausen et den Hollander [**12**], où l'existence et quelques propriétés basiques de la courbe critique $h_c(\cdot)$ (y compris la borne supérieure de (1b.10)) ont été établies. Mais le résultat le plus important de [**12**] est certainement la limite d'échelle de l'énergie libre donnée

par le Théorème 1b.4 dans § 3.2 ci-dessus. Le second résultat fondamental sur le diagramme de phase dans le cas aléatoire, la borne inférieure de (1b.10), a été prouvé par Bodineau et Giacomin dans [**9**].

La stratégie adoptée dans [**9**] prend son inspiration de l'article physique [**50**] de Monthus, où la borne inférieure $\underline{h}(\cdot)$ fut introduite pour la première fois (sous la forme d'une conjecture pour la vraie courbe critique). Remarquons que dans la littérature physique la conjecture $h_c(\cdot) = \underline{h}(\cdot)$ a été proposée aussi dans [**64**], motivée par des arguments à base de répliques, tandis que dans [**69**] et dans [**34**] la conjecture complémentaire $h_c(\cdot) = \overline{h}(\cdot)$ a été proposée.

Si nous revenons à la littérature mathématique, une analyse des trajectoires dans le cas aléatoire pour la région délocalisée $\mathcal{L}$ dans son entier a été obtenue par Biskup et den Hollander dans [**8**] : les mots-clés de leur approche sont la *limite thermodynamique* et les *mesures de Gibbs*. D'autre part, les résultats sur les trajectoires dans la région délocalisée semblent beaucoup plus difficiles à établir. Toutefois, des progrès récents dans cette direction ont été obtenus par Giacomin et Toninelli dans [**36**].

Dans le cas où la suite de charges est périodique, la question de la détermination du diagramme de phase a été résolue complètement dans l'article [**11**] de Bolthausen et Giacomin (voir § 2.4). Pour des références sur des travaux antérieurs sur le copolymère périodique, nous renvoyons aussi à cet article.

Dans la littérature on trouve aussi un grand nombre d'articles numérique sur les copolymères : on signale par exemple [**19, 65**]. Par rapport à l'approche numérique développée dans le Chapitre 2, l'attention est souvent centrée sur des aspects différents, notamment la détermination des exposants critiques et l'étude de modèles plus complexes basés sur des marches auto–évitantes plutôt que dirigées.

Pour terminer, nous mentionnons [**40**], [**22**], [**2**], [**71, 72**] et leur références en ce qui concerne les modèles d'accrochage/décrochage et de mouillage.

## 5. Organisation de la thèse

Notre exposé est organisé de la manière suivante :

- Dans le Chapitre 2 nous présentons une approche combinant des simulations et des calculs numériques couplés à des arguments rigoureux pour étudier le diagramme de phase et le comportement des trajectoires du modèle de copolymère à proximité d'une interface sélective, défini dans la Section 2. Nous considérons le cas où la suite des charges est *aléatoire*. Nous donnons

plusieurs indications sur le fait que la courbe critique est strictement comprise entre les deux bornes données dans l'équation (1b.10) et sur le fait que dans la région strictement délocalisée on a la limite d'échelle vers le méandre Brownien. En particulier la conjecture $h_c(\cdot) = \underline{h}(\cdot)$ peut être refusée avec un très grand niveau de confiance, grâce à un test statistique rigoureux, sur l'erreur duquel nous obtenons une borne explicite. Dans ce chapitre nous donnons aussi une preuve alternative de la borne inférieure $h_c(\cdot) \geq \underline{h}(\cdot)$.

L'article [**17**] a été tiré du contenu de ce chapitre.

- Dans le Chapitre 3 nous examinons la technique du *constrained annealing* qui consiste à rajouter à l'Hamiltonien d'un système désordonné des termes dépendant du désordre afin d'améliorer la borne annealed (voir § 2.5.1). Nous montrons que pour un grand nombre de modèles désordonnés de chaînes linéaires (y compris le copolymère à proximité d'une interface sélective et les modèles d'accrochage/décrochage et de mouillage décrits dans les sections précédentes) l'application standard de cette technique, basée sur l'utilisation de moyennes empiriques de fonctions locales, ne peut pas améliorer la borne annealed sur la courbe critique.

  L'article [**16**] a été tiré du contenu de ce chapitre.

- Dans le Chapitre 4 nous considérons un modèle général de polymère hétérogène à proximité d'une interface, qui inclut en particulier le copolymère à proximité d'une interface sélective et les modèles d'accrochage/décrochage et de mouillage, dans le cas où la suite des charges est *périodique*. Nous proposons une approche basée sur la Théorie du Renouvellement qui permet d'établir des estimations très précises de la fonction de partition du modèle dans tous les régimes, y compris le régime critique. À partir de ces résultats, nous obtenons une description très précise de la limite thermodynamique et des limites d'échelle de la mesure du polymère.

  La prépublication [**18**] a été tirée du contenu de ce chapitre.

- Dans le Chapitre 5 nous considérons une modification du modèle du copolymère à proximité d'une interface sélective où la mesure de référence $\mathbf{P}$ n'est plus la loi de la marche aléatoire simple sur $\mathbb{Z}$. Plus précisément,

nous considérons le cas où $\mathbf{P}$ est la loi d'une marche aléatoire réelle dont l'accroissement typique est centré, borné et de loi absolument continue. Nous nous concentrerons sur le cas des *charges aléatoires*. Nous donnons d'abord une preuve de l'existence de l'énergie libre et étudions ensuite le diagramme de phase du modèle, en soulignant les analogies avec le cas discret. En conclusion nous abordons la question d'étendre à ce modèle la limite d'échelle de l'énergie libre, cf. le Théorème 1b.4 (travail en cours), en donnant quelques résultats partiels dans cette direction et en décrivant les points à résoudre.

- Dans le Chapitre 6 nous prouvons un théorème local limite pour des marches aléatoires conditionnées à rester positives, dans un cadre très général (sous l'unique hypothèse que la marche se trouve dans le domaine d'attraction de la loi Normale). Ce théorème est un renforcement local de la convergence en loi vers le méandre Brownien. Ce résultat est intéressant non seulement en soi, mais constitue aussi un outil important pour les modèles de polymères construits sur des marches aléatoires plus générales que la marche aléatoire simple, comme pour le cas considéré dans le Chapitre 5.

  L'article [**15**] a été tiré du contenu de ce chapitre.

CHAPTER 2

# A numerical study of the phase diagram and path behavior of the copolymer model with random charges

In this chapter we study the copolymer near a selective interface model, defined in Section 2 of Chapter 1a, in the *random* case. We combine numerical computations with rigorous arguments to get to a better understanding of the phase diagram and of the path behavior. Our main aim is to provide evidences of the fact that the critical line of the model $h_c(\cdot)$ lies *strictly* in between the two bounds $\underline{h}(\cdot)$ and $\overline{h}(\cdot)$, defined in (1a.10) of Chapter 1a, and to numerically analyze the delocalization issues raised in § 2.6 of Chapter 1a. A detailed outline of the results is given in § 1.2.

The article [**17**] has been taken from the content of this chapter.

## 1. Introduction and results

**1.1. Prelimiaries.** The notations we will use are those introduced in Section 2 of Chapter 1a. We recall in particular the definitions (1a.7) and (1a.9) of the partition functions $Z^{\lambda,h}_{N,\omega}$ and $Z^{\lambda,h}_{N,\omega}(x)$ (we will be mainly interested in the case $x = 0$). Also remember that for the critical line $h_c(\cdot)$ of our model we have the bounds

$$\underline{h}(\lambda) \;\le\; h_c(\lambda) \;\le\; \overline{h}(\lambda), \tag{2.1}$$

see Theorem 1a.4 of Chapter 1a. For what follows we set

$$h^{(m)}(\lambda) \;=\; \frac{1}{2m\lambda} \log \mathsf{M}\left(-2m\lambda\right), \tag{2.2}$$

for $m > 0$, where we recall that $\mathsf{M}(\alpha) := \mathbb{E}[\exp(\alpha\omega_1)]$. Observe that the curves $\underline{h}(\cdot)$ and $\overline{h}(\cdot)$ correspond respectively to $m = 2/3$ and $m = 1$, and that for every $m$ we have $\frac{\mathrm{d}}{\mathrm{d}\lambda} h^{(m)}(\lambda)|_{\lambda=0} = m$.

Before proceeding, we present a different viewpoint on the process: this turns out to be useful for the intuition and it will be used in some technical steps. We call $\eta$ the first return time of the walk $S$ to 0, that is $\eta := \inf\{n \ge 1 : S_n = 0\}$, and set

$K(2n) := \mathbf{P}(\eta = 2n)$ for $n \in \mathbb{N}$. It is well known that $K(\cdot)$ is decreasing on the even natural numbers and

$$\lim_{x \in 2\mathbb{N}, x \to \infty} x^{3/2} K(x) = \sqrt{2/\pi}, \tag{2.3}$$

see e.g. [**28**, Ch. 3]. Let the IID sequence $\{\eta_j\}_{j=1,2,\ldots}$ denote the inter–arrival times at 0 for $S$, and we set $\tau_k := \eta_0 + \ldots + \eta_k$. If we introduce also $\ell_N = \max\{j \in \mathbb{N} \cup \{0\} : \tau_j \leq N\}$, then by exploiting the up–down symmetry of the excursions of $S$ we directly obtain

$$\begin{aligned} Z_{N,\omega}(0) \;&=\; \mathbf{E}\left[\prod_{j=1}^{\ell_N} \varphi\left(\lambda \sum_{n=\tau_{j-1}+1}^{\tau_j} \omega_n + \lambda h \eta_j\right); \tau_{\ell_N} = N\right] \\ &= \sum_{l=0}^{N} \sum_{\substack{x_0,\ldots,x_l \in 2\mathbb{N} \\ 0=:x_0<\ldots<x_l:=N}} \prod_{i=1}^{l} \varphi\left(\lambda \sum_{n=x_{i-1}+1}^{x_i} \omega_n + \lambda h (x_i - x_{i-1})\right) K(x_i - x_{i-1}), \end{aligned} \tag{2.4}$$

with $\varphi(t) := (1 + \exp(-2t))/2$. Of course the formula for $Z_{N,\omega}$ is just slightly different.

Formula (2.4) reflects the fact that what really matters for the copolymer are the return times to the interface.

**1.2. Outline of the results.** Formula (2.1) leaves an important gap, that hides the only partial understanding of the nature of this delocalization/localization transition. Our purpose is to go toward filling this gap: our results are both of theoretical and numerical nature. At the same time we address the delocalization issues raised in § 2.6 of Chapter 1a, which are intimately related with the precise asymptotic behavior of $Z_{N,\omega}$ and of $Z_{N,\omega}(0)$. More precisely:

(1) In Section 2 we present a statistical test with explicit error bounds, see Proposition 2.2, based on super–additivity and concentration inequalities, to state that a point $(\lambda, h)$ is localized. We apply this test to show that, with a very low level of error, the lower bound $h = \underline{h}(\lambda)$ does not coincide with the critical line.

(2) In Section 3 we give the outline of a new proof of the lower bound $h_c(\cdot) \geq \underline{h}(\cdot)$. The details of the proof are in § 6.2 and we point out in particular Proposition 2.10, that gives a necessary and sufficient condition for localization. This viewpoint on the transition, derived from [**36**, § 4], helps substantially in interpreting the *irregularities* in the behavior of $\{Z_{N,\omega}\}_N$ as $N \nearrow \infty$.

(3) In Section 4 we pick up the conjecture of Brownian scaling in the delocalized regime both in the intent of testing it and in trying to asses with reasonable confidence that $(\lambda, h)$ is in the interior of $\mathcal{D}$. In particular, we present quantitative evidences in favor of the fact that the upper bound $h = \overline{h}(\lambda)$ is strictly greater than the critical line. We stress that this is a very delicate issue, since delocalization, unlike localization, does not appear to be reducible to a finite volume issue.

(4) Finally, in Section 5, we report the results of a numerical attempt to determine the critical curve. While this issue has to be treated with care, mostly for the reasons raised in point 4 above, we observe a surprising phenomenon: the critical curve appears to be very close to $h^{(m)}(\cdot)$ for a suitable value of $m$. By the universality result proven in [**36**], building on the free energy Brownian scaling result proven in [**12**], the slope at the origin of $h_c(\cdot)$ does not depend on the law of $\omega$. Therefore if really $h^{(m)}(\cdot) = h_c(\cdot)$, since the slope at the origin of $h^{(m)}(\cdot)$ is $m$, $m$ is the universal constant we are looking for. We do not believe that the numerical evidence allows to make a clear cut statement, but what we observe is compatible with such a possibility.

We point out that our numerical results are based on a numerical computation of the partition function $Z_{N,\omega}$, exploiting the standard transfer–matrix approach (this item is discussed in more details in § 6.1).

## 2. A statistical test for the localized phase

**2.1. Checking localization at finite volume.** At an intuitive level one is led to believe that, when the copolymer is localized, it should be possible to detect it by looking at the system before the infinite volume limit. This intuition is due to the fact that in the localized phase the length of each excursion is finite, therefore for $N$ *much larger* that the *typical* excursion length one should already observe the localization phenomenon in a quantitative way. The system being disordered of course does not help, because it is more delicate to make sense of what *typicality* means in a non translation invariant set–up. However the translation invariance can be recovered by averaging and in fact it turns out to be rather easy to give a precise meaning to the intuitive idea we have just mentioned. The key word here is super–additivity of the averaged free energy.

In fact by considering only the $S$ trajectories such that $S_{2N} = 0$ and by applying the Markov property of $S$ one directly verifies that for any $N, M \in \mathbb{N}$

$$Z_{2N+2M,\omega}(0) \;\geq\; Z_{2N,\omega}(0)\, Z_{2M,\theta^{2N}\omega}(0), \tag{2.5}$$

$(\theta\omega)_n = \omega_{n+1}$, and therefore

$$\{\, \mathbb{E} \log Z_{2N,\omega}(0)\}_{N=1,2,\ldots} \tag{2.6}$$

is a super–additive sequence, which immediately entails the existence of the limit of $\mathbb{E}[\log Z_{2N,\omega}(0)]/2N$ and the fact that this limit coincides with the supremum of the sequence. Therefore from the existence of the quenched free energy we have that

$$\mathrm{F}(\lambda, h) \;=\; \sup_N \frac{1}{2N} \mathbb{E} \log Z_{2N,\omega}(0)\,. \tag{2.7}$$

In a more suggestive way one may say that:

$$(\lambda, h) \in \mathcal{L} \iff \text{there exists } N \in \mathbb{N} \text{ such that } \mathbb{E} \log Z_{2N,\omega}(0) > 0\,. \tag{2.8}$$

The price one pays for working with a disordered system is precisely in taking the $\mathbb{P}$–expectation and from the numerical viewpoint it is an heavy price: even with the most positive attitude one cannot expect to have access to $\mathbb{E} \log Z_{2N,\omega}(0)$ by direct numerical computation for $N$ above 10. Of course in principle small values of $N$ may suffice (and they do in some cases, see Remark 2.1), but they do not suffice to tackle the specific issue we are interested in. We elaborate at length on this interesting issue in § 2.4.

Remark 2.1. An elementary application of the localization criterion (2.8) is obtained for $N = 1$: $(\lambda, h) \in \mathcal{L}$ if

$$\mathbb{E}\left[\log\left(\frac{1}{2} + \frac{1}{2}\exp\left(-2\lambda\left(\omega_1 + \omega_2 + 2h\right)\right)\right)\right] > 0. \tag{2.9}$$

In the case $\mathbb{P}(\omega_1 = \pm 1) = 1/2$ from (2.9) we obtain that for $\lambda$ sufficiently large $h_c(\lambda) > 1 - c/\lambda$, with $c = (1/4)\log(2\exp(4) - 1) \approx 1.17$. From $\underline{h}(\cdot)$ we obtain the same type of bound, with $c = (3/4)\log 2 \approx 0.52$. This may raise some hope that for $\lambda$ large an explicit, possibly computer assisted, computation for small values of $N$ of $\mathbb{E} \log Z_{2N,\omega}(0)$ could lead to new estimates. This is not the case, as we show in § 2.4.

**2.2. Testing by using concentration.** In order to decide whether $\mathbb{E}\log Z_{2N,\omega}(0) > 0$ we resort to a Montecarlo evaluation of $\mathbb{E}\log Z_{2N,\omega}(0)$ that can be cast into a statistical test with explicit error bound by means of concentration of measure ideas. This procedure is absolutely general, but we have to choose a set–up for the computations and we take the simplest: $\mathbb{P}(\omega_1 = +1) = \mathbb{P}(\omega_1 = -1) = 1/2$. The reason for this choice is twofold:

- if $\omega_1$ is a bounded random variable, a Gaussian concentration inequality holds and if $\omega$ is symmetric and it takes only two values then one can improve on the explicit constant in such an inequality. This speeds up in a non negligible way the computations;
- generating *true randomness* is out of reach, but playing head and tail is certainly the most elementary case in such a far reaching task (the random numbers issue is briefly discussed in § 6.1 too).

A third reason to restrict testing to the Bernoulli case is explained at the end of the caption of Table 2.

We start the testing procedure by stating the null hypothesis:

$$\text{H0}: \quad \mathbb{E}\log Z_{2N,\omega}(0) \leq 0. \tag{2.10}$$

$N$ in H0 can be chosen arbitrarily. We stress that refusing H0 implies $\mathbb{E}\log Z_{2N,\omega}(0) > 0$, which by (2.8) implies localization.

The following concentration inequality for Lipschitz functions holds for the uniform measure on $\{-1,+1\}^N$: for every function $G_N : \{-1,+1\}^N \to \mathbb{R}$ such that $|G_N(\omega) - G_N(\omega')| \leq C_{\text{Lip}}\sqrt{(}\sum_{n=1}^N(\omega_n - \omega'_n)^2)$, where $C_{\text{Lip}}$ a positive constant and $G_N(\omega)$ is an abuse of notation for $G_N(\omega_1,\ldots,\omega_N)$, one has

$$\mathbb{E}\left[\exp\left(\alpha\left(G_N(\omega) - \mathbb{E}[G_N(\omega)]\right)\right)\right] \leq \exp\left(\alpha^2 C_{\text{Lip}}^2\right), \tag{2.11}$$

for every $\alpha$. Inequality (2.11) with an extra factor 4 at the exponent can be extracted from the proof of Theorem 5.9, page 100 in [**46**]. Such an inequality holds for variables taking values in $[-1,1]$: the factor 4 can be removed for the particular case we are considering (see [**46**, p. 110–111]). In our case $G_N(\omega) = \log Z_{2N,\omega}(0)$. By applying the Cauchy–Schwarz inequality one obtains that $G_N$ is Lipschitz with $C_{\text{Lip}} = 2\lambda\sqrt{N}$. Let us now consider an IID sequence $\{G_N^{(i)}(\omega)\}_i$ with $G_N^{(1)}(\omega) = G_N(\omega)$: if H0 holds

then we have that for every $n \in \mathbb{N}$, $u > 0$ and $\alpha = un/8\lambda^2 N$

$$\begin{aligned}\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} G_N^{(i)}(\omega) \geq u\right) &\leq \mathbb{E}\left[\exp\left(\frac{\alpha}{n}\left(G_N(\omega) - \mathbb{E}[G_N(\omega)]\right)\right)\right]^n \exp\left(-\alpha\left(u - \mathbb{E}[G_N(\omega)]\right)\right) \\ &\leq \exp\left(\frac{4\alpha^2\lambda^2 N}{n} - \alpha u\right) \\ &= \exp\left(-\frac{u^2 n}{16\lambda^2 N}\right).\end{aligned} \tag{2.12}$$

Let us sum up what we have obtained:

PROPOSITION 2.2. *Let us call $\widehat{u}_n$ the average of a sample of $n$ independent realizations of $\log Z^{\lambda,h}_{2N,\omega}(0)$. If $\widehat{u}_n > 0$ then we may refuse H0, and therefore $(\lambda, h) \in \mathcal{L}$, with a level of error not larger than $\exp\left(-\widehat{u}_n^2 n/16\lambda^2 N\right)$.*

**2.3. Numerical tests.** We report in Table 1 the most straightforward application of Proposition 2.2, obtained by a numerical computation of $\log Z_N$ for a sample of $n$ independent environments $\omega$. We aim at seeing how far above $\underline{h}(\cdot)$ one can go and still claim localization, keeping a reasonably small probability of error.

REMARK 2.3. One might be tempted to interpolate between the values in Table 1, or possibly to get results for small values of $\lambda$ in order to extend the result of the test to the slope of the critical curve in the origin. However the fact that $h_c(\lambda)$ is strictly increasing does not help much in this direction and the same is true for the finer result, proven in [**11**], that $h_c(\lambda)$ can be written as $U(\lambda)/\lambda$, $U(\cdot)$ a convex function.

**2.4. Improving on $\underline{h}(\cdot)$ is uniformly hard.** One can get much smaller $p$–values at little computational cost by choosing $h$ *just above* $\underline{h}(\lambda)$. As a matter of fact a natural choice is for example $h = h^{(0.67)}(\lambda) > \underline{h}(\lambda)$, recall (2.2), for a set of values of $\lambda$, and this is part of the content of Table 2: in particular $\mathbb{E}\log Z^{\lambda,h^{(0.67)}(\lambda)}_{2N_+,\omega}(0) > 0$ with a probability of error smaller than $10^{-5}$ for the values of $\lambda$ between 0.1 and 1. However we stress that for some of these $\lambda$'s we have a much smaller $p$–value, see the caption of Table 2, and that the content of this table is much richer and it approaches also the question of whether or not a symbolic computation or some other form of computer assisted argument could lead to $h_c(\lambda) > \underline{h}(\lambda)$ for some $\lambda$, and therefore

| $\lambda$ | 0.3 | 0.6 | 1 |
|---|---|---|---|
| $h$ | 0.22 | 0.41 | 0.58 |
| $p$–value | $1.5 \times 10^{-6}$ | $9.5 \times 10^{-3}$ | $1.6 \times 10^{-5}$ |
| $\underline{h}(\lambda)$ | 0.195 | 0.363 | 0.530 |
| $\overline{h}(\lambda)$ | 0.286 | 0.495 | 0.662 |
| $N$ | 300000 | 500000 | 160000 |
| $n$ | 225000 | 330000 | 970000 |
| C. I. 99% | $7.179 \pm 0.050$ | $9.011 \pm 0.045$ | $7.643 \pm 0.025$ |

TABLE 1. According to our numerical computations, the three pairs $(\lambda, h)$ are in $\mathcal{L}$ and this has been tested with the stated $p$–values (or probability/level of error). We report the values of $\overline{h}(\lambda)$ and $\underline{h}(\lambda)$ for reference. Of course in these tests there is quite a bit of freedom in the choice of $n$ and $N$: notice that $N$ enters in the evaluation of the $p$–value also because a larger value of $N$ yields a larger value of $\mathbb{E} \log Z^{\lambda,h}_{2N,\omega}(0)$. In the last line we report standard Gaussian 99% confidence intervals for $\mathbb{E} \log Z^{\lambda,h}_{2N,\omega}(0)$. Of course the $p$–value under the Gaussian assumption turns out to be totally negligible.

for $\lambda$ in an interval. Since such an argument would require $N$ to be *small*, intuitively the hope resides in large values of $\lambda$, recall also Remark 2.1. It turns out that one needs in any case $N$ larger than 700 in order to observe a localization phenomenon at $h^{(0.67)}(\lambda)$. We now give some details on the procedure that leads to Table 2.

First and foremost, the concentration argument that leads to Proposition 2.2 is symmetric and it works for deviations below the mean as well as above. So we can, in the very same way, test the null hypothesis $\mathbb{E} \log Z_{2N,\omega}(0) > 0$ and, possibly, refuse it if $\hat{u}_n < 0$, exactly with the same $p$–value as in Proposition 2.2. Of course an important part of Proposition 2.2 was coming from the finite volume localization condition (2.8): we do not have an analogous statement for delocalization (and we do not expect that there exists one). But, even if $\mathbb{E} \log Z_{2N,\omega}(0) \leq 0$ does not imply delocalization, it says at least that it is pointless to try to prove localization by looking at a system of that size.

In Table 2 we show two values of the system size $N$, $N_+$ and $N_-$, for which, at a given $\lambda$, one has that $\mathbb{E} \log Z_{2N_+,\omega}(0) > 0$ and $\mathbb{E} \log Z_{2N_-,\omega}(0) < 0$ with a fixed

| $\lambda$ | $0.05(\star)$ | 0.1 | 0.2 | 0.4 | 0.6 | 1 | $2(\star)$ | $4(\star\star)$ | $8(\star\star)$ |
|---|---|---|---|---|---|---|---|---|---|
| $N_+$ | 750000 | 190000 | 40000 | 9500 | 4250 | 1800 | 900 | 800 | 800 |
| $N_-$ | 600000 | 130000 | 33000 | 7500 | 3650 | 1550 | 750 | 700 | 700 |

TABLE 2. For a given $\lambda$, both $\mathbb{E}\log Z_{2N_+,\omega}^{\lambda,h^{(0.67)}(\lambda)}(0) > 0$ and $\mathbb{E}\log Z_{2N_-,\omega}^{\lambda,h^{(0.67)}(\lambda)}(0) < 0$ with a probability of error smaller than $10^{-5}$ (and in some cases much smaller than that). Instead for the two cases marked by a $(\star)$ the level of error is rather between $10^{-2}$ and $10^{-3}$. For large values of $\lambda$, the two cases marked with $(\star\star)$, it becomes computationally expensive to reach small $p$–values. However, above $\lambda = 3$ one observes that the values of $Z_{2N,\omega}(0)$ essentially do not depend anymore on the value of $\lambda$. This can be interpreted in terms of convergence to a limit ($\lambda \to \infty$) model, as it is explained in Remark 2.4. If we then make the hypothesis that this limit model sharply describes the copolymer along the curve $(\lambda, h^{(m)}(\lambda))$ for $\lambda$ sufficiently large and we apply the concentration inequality, then the given values of $N_+$ and $N_-$ are tested with a very small probability of error. Since the details of such a procedure are quite lengthy we do not report them here. We have constructed (partial) tables also for different laws of $\omega$, notably $\omega_1 \sim N(0,1)$, and they turned out to yield larger, at times substantially larger, values of $N_\pm(\lambda)$.

probability of error (specified in the caption of the Table). It is then reasonable to guess that the transition from negative to positive values of $\mathbb{E}\log Z_{\cdot,\omega}(0)$ happens for $N \in (N_-, N_+)$. There is no reason whastoever to expect that $\mathbb{E}\log Z_{N,\omega}(0)$ should be monotonic in $N$ but according to our numerical result it is not unreasonable to expect that monotonicity should set in for $N$ large or, at least, that for $N < N_-$ (respectively $N > N_+$) $\mathbb{E}\log Z_{2N,\omega}(0)$ is definitely negative (respectively positive).

REMARK 2.4. As pointed out in the caption of Table 2, from numerics one observes a very sharp convergence to a $\lambda$ independent behavior as $\lambda$ becomes large, along the line $h = h^{(m)}(\lambda)$. This is easily interpreted if one observes that $h^{(m)}(\lambda) =$

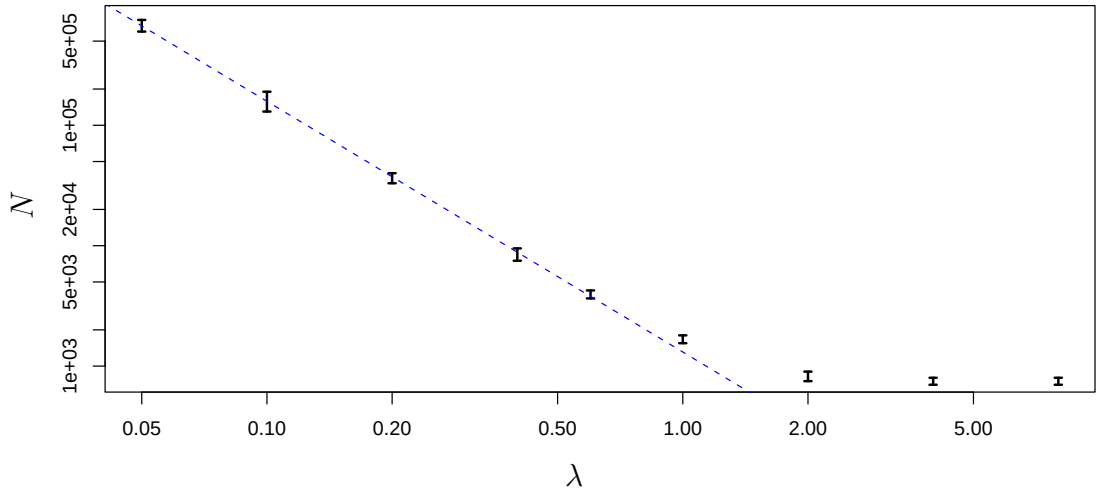


FIGURE 2.1. A graphical representation of Table 2. The plot is log–log, and a $\lambda^{-c}$ behavior is rather evident, $c$ is about 2.08. This can be nicely interpreted in terms of the coarse graining technique in the proof of the weak interaction scaling limit of the free energy in [**12**]: from that argument one extracts that if $\lambda$ is small the excursions that give a contribution to the free energy have *typical* length $\lambda^{-2}$ and that in the limit the polymer is just made up by this type of excursions. One therefore expects that it suffices a system of size $N(\lambda)$, with $\lim_{\lambda\searrow 0}\lambda^2 N(\lambda) = +\infty$, to observe localization if $m < h_c'(0)$, $h = h^{(m)}(\lambda) = m\lambda(1+o(1))$ and $\lambda$ is small.

$1-((\log 2)/2m\lambda)+O(\exp(-4m\lambda))$ so that

$$\lim_{\lambda\to\infty}\exp\left(-2\lambda\sum_{n=1}^{N}(\omega_n+h)\,\Delta_n\right) = \exp\left(\frac{\log 2}{m}\sum_{n=1}^{N}\Delta_n\right)\mathbf{1}_{\left\{\sum_{n=1}^{N}\Delta_n(1+\omega_n)=0\right\}}(S). \tag{2.13}$$

This corresponds to the model where a positive charge never enters the lower half-plane and where the energy of a configuration is proportional to the number of negative charges in the lower half-plane.

## 3. Lower bound strategies versus the true strategy

**3.1. An approach to lower bounds on the critical curve.** In this section we give an outline of a new derivation of the lower bound

$$\underline{h}(\lambda) \le h_c(\lambda), \tag{2.14}$$

with $\underline{h}(\lambda)$ defined in (1a.10) of Chapter 1a. The complete proof may be found in § 6.2. The argument takes inspiration from the ideas used in the proof of Proposition 3.1 in [**36**] and, even if it is essentially the proof of [**11**] in disguise, in the sense that the selection of the random walk trajectories that are kept and whose energy contribution is evaluated does not differ too much (in a word: the *strategy* of the polymer is similar), it is however conceptually somewhat different and it will naturally lead to some considerations on the precise asymptotic behavior of $Z_{N,\omega}$ in the delocalized phase and even in the localized phase close to criticality.

The first step in our proof of (2.14) is a different way of looking at localization. For any fixed positive number $C$ we introduce the stopping time (with respect to the natural filtration of the sequence $\{\omega_n\}$) $T^C = T^{C,\lambda,h}(\omega)$ defined by

$$T^{C,\lambda,h}(\omega) := \inf\{N \in 2\mathbb{N} : \ Z^{\lambda,h}_{N,\omega}(0) \ge C\}\,. \tag{2.15}$$

The key observation is that if $\mathbb{E}[T^C] < \infty$ for some $C > 1$, then the polymer is localized. Let us sketch a proof of this fact (for the details, see Proposition 2.10 of § 6.2): notice that by the very definition of $T^C$ we have $Z_{T^C(\omega),\omega}(0) \ge C$. Now the polymer that is in zero at $T^C(\omega)$ is equivalent to the original polymer, with a translated environment $\omega' = \theta^{T^C}\omega$, and setting $T_2(\omega) := T^C(\omega')$ we easily get $Z_{T_1(\omega)+T_2(\omega),\omega}(0) \ge C^2$ (we have put $T_1(\omega) := T^C(\omega)$). Notice that the new environment $\omega'$ is still typical, since $T^C$ is a stopping time, so that $T_2$ is independent of $T_1$ and has the same law. This procedure can be clearly iterated, yielding an IID sequence $\{T_i(\omega)\}_{i=1,2,\dots}$ that gives the following lower bound on the partition function:

$$Z_{T_1(\omega)+\dots+T_n(\omega),\omega}(0) \ge C^n\,. \tag{2.16}$$

From this bound one easily obtains that

$$\mathrm{F}(\lambda,h) \overset{\text{a.s.}}{=} \lim_{n\to\infty} \frac{\log Z_{T_1(\omega)+\dots+T_n(\omega),\omega}(0)}{T_1(\omega)+\dots+T_n(\omega)} \ge \frac{\log C}{\mathbb{E}[T^C]}\,, \tag{2.17}$$

where we have applied the strong law of large numbers, and localization follows since by hypothesis $C > 1$ and $\mathbb{E}[T^C] < \infty$.

REMARK 2.5. It turns out that also the reciprocal of the claim just proved holds true, that is *the polymer is localized if and only if* $\mathbb{E}[T^C] < \infty$, with an arbitrary choice of $C > 1$, see Proposition 2.10. In fact the case $\mathbb{E}[T^C] = \infty$ may arise in two different ways:

(1) the variable $T^C$ is defective, $\mathbb{P}[T^C = \infty] > 0$: in this case with positive probability $\{Z_{N,\omega}(0)\}_N$ is a bounded sequence, and delocalization follows immediately;
(2) the variable $T^C$ is proper with infinite mean, $\mathbb{P}[T^C = \infty] = 0$, $\mathbb{E}[T^C] = \infty$: in this case we can still build a sequence $\{T_i(\omega)\}_{i=1,2,\dots}$ defined as above and this time the lower bound (2.16) has *subexponential* growth. Moreover it can be shown that in this case the lower bound (2.16) gives the true free energy, cf. Lemma 2.9, which therefore is zero, so that delocalization follows also in this case.

As a matter of fact, it is highly probable that in the interior of the delocalized phase $Z_{N,\omega}(0)$ vanishes $\mathbb{P}(\mathrm{d}\omega)$–a.s. when $N \to \infty$ and this would rule out the scenario (2) above, saying that for $C > 1$ the random variable $T^C$ must be either integrable or defective. We take up again this point in Sections 4 and 5: we feel that this issue is quite crucial in order to fully understand the delocalized phase of disordered models.

REMARK 2.6. Dealing directly with $T^C$ may be difficult. Notice however that if one finds a random time (by this we mean simply an integer–valued random variable) $T = T(\omega)$ such that

$$Z_{T(\omega),\omega}(0) \geq C > 1\,, \qquad \text{with } \mathbb{E}[T] < \infty\,, \tag{2.18}$$

then localization follows. This is simply because this implies $T^C \leq T$ and hence $\mathbb{E}[T^C] < \infty$. Therefore localization is equivalent to the condition $\log Z_{T(\omega),\omega}(0) > 0$ for an *integrable* random time $T(\omega)$: we would like to stress the analogy between this and the criterion for localization given in § 2.1, see (2.8).

Now we can turn to the core of our proof: we are going to show that for every $(\lambda, h)$ with $h < \underline{h}(\lambda)$ we can build a random time $T = T(\omega)$ that satisfies (2.18). The construction of $T$ is based on the idea that for $h > 0$ if localization prevails is because of rare $\omega$–stretches that invite the polymer to spend time in the lower half–plane in spite of the action of $h$.

The strategy we use consists in looking for $q$–atypical stretches of length at least $M \in 2\mathbb{N}$, where $q < -h$ is the average charge of the stretch. Rephrased a bit more precisely, we are looking for the smallest $n \in 2\mathbb{N}$ such that $\sum_{i=n-k+1}^{n} \omega_i/k < q$ for some even integer $k \geq M$. It is well known that such a random variable grows, in the sense of Laplace, as $\exp(\Sigma(q)M)$ for $M \to \infty$, where $\Sigma(q)$ is the Cramer functional

$$\Sigma(q) := \sup_{\alpha \in \mathbb{R}} \{\alpha q - \log \mathsf{M}(\alpha)\} . \tag{2.19}$$

One can also show without much effort that the length of such a stretch cannot be much longer than $M$. Otherwise stated, this is the familiar statement that the longest $q$–atypical sub–stretch of $\omega_1, \ldots, \omega_N$ is of typical length $\sim \log N/\Sigma(q)$. So $T(\omega)$ is for us the end–point of a $q$–atypical stretch of length approximately $(\log T(\omega))/\Sigma(q)$: by looking for sufficiently long $q$–atypical stretches we have always the freedom to choose $T(\omega) \gg 1$, in such a way that also $\log T(\omega) \ll T(\omega)$ and this is helpful for the estimates. So let us bound $Z_{T(\omega),\omega}$ from below by considering only the trajectories of the walk that stay in the upper half–plane up to the beginning of the $q$–atypical stretch and that are negative in the stretch, coming back to zero at step $T(\omega)$ (see Fig. 2.2: the polymer is cut at the first dashed vertical line). The contribution of these trajectories is easily evaluated: it is approximately

$$\left(\frac{1}{T(\omega)^{3/2}}\right) \exp\left(-2\lambda(q+h)\frac{\log T(\omega)}{\Sigma(q)}\right) . \tag{2.20}$$

For such an estimate we have used (2.3) and $\log T(\omega) \ll T(\omega)$ both in writing the probability that the first return to zero of the walk is at the beginning of the $q$–atypical stretch and in neglecting the probability that the walk is negative inside the stretch. It is straightforward to see that if

$$\frac{4\lambda}{3}h < -\frac{4\lambda}{3}q - \Sigma(q), \tag{2.21}$$

and if $T(\omega)$ is large, then also the quantity in (2.20) is large. We can still optimize this procedure by choosing $q$ (which must be sufficently negative, i.e. $q < -h$). By playing with (2.19) one sees that one can choose $q_0 \in \mathbb{R}$ such that for $q = q_0$ the right–hand side in (2.21) equals $\log \mathsf{M}(-4\lambda/3)$ and if $h < \log \mathsf{M}(-4\lambda/3)/(4\lambda/3) = \underline{h}(\lambda)$ then $q_0 < -h$. This argument therefore is saying that there exists $C > 1$ such that

$$Z_{T(\omega),\omega}(0) \geq C, \tag{2.22}$$

for every $\omega$. It only remains to show that $\mathbb{E}[T] < \infty$: this fact, together with a detailed proof of the argument just presented can be found in § 6.2.

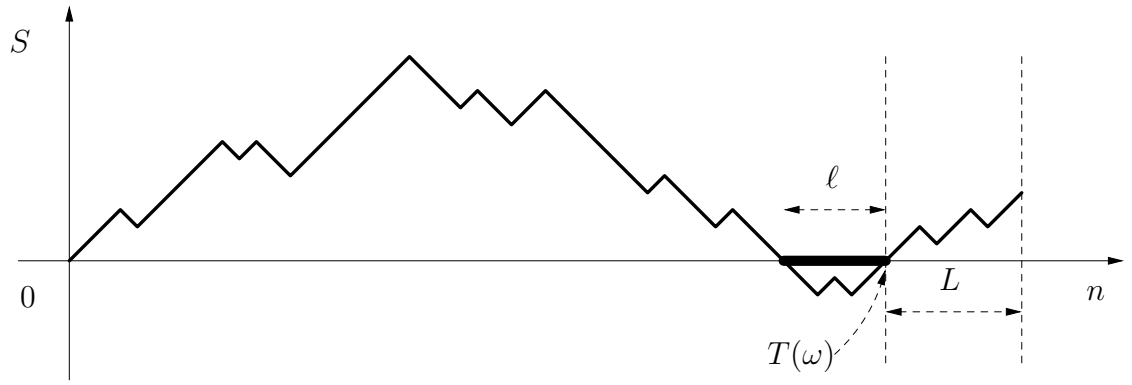


FIGURE 2.2. Inequality (2.23) comes simply from restricting the evaluation of $Z_{T(\omega)+L,\omega}$ to the trajectories visiting the *q–atypical* stretch of length $\ell$ and by staying away from the unfavorable solvent after that.

**3.2. Persistence of the effect of rare stretches.** As pointed out in the previous section, there is strong evidence that $h_c(\lambda) > \underline{h}(\lambda)$. At this stage Fig. 2.3 is of particular interest. Notice first of all that in spite of being substantially above $\underline{h}(\cdot)$ the copolymer appears to be still localized, see in particular case A.

The rigorous lower bounds that we are able to prove cannot establish localization in the region we are considering. All the same, notice that if one does not cut the polymer at $T(\omega)$, as in the argument above, but at $T(\omega) + L$, a lower bound of the following type

$$Z_{T(\omega)+L,\omega} \overset{\text{roughly}}{\geq} \text{const.} \frac{1}{T(\omega)^{3/2}} \exp\left(-2\lambda(q+h)\frac{\log T(\omega)}{\Sigma(q)}\right) \frac{1}{L^{1/2}}, \tag{2.23}$$

is easily established. Of course we are being imprecise, but we just want to convey the idea, see also Fig. 2.2, that after passing through an atypically *negative* stretch of environment ($q > 0$), the effect of this stretch decays at most like $L^{-1/2}$, that is the probability that a walk stays positive for a time $L$.

At this point we stress that the argument outlined in § 3.1 and re–used for (2.23) may be very well applied to $h > \underline{h}(\lambda)$, except that this time it does not suffice for (2.22). But it yields nevertheless that for $h \in \big(\underline{h}(\lambda), \overline{h}(\lambda)\big)$ the statement $Z_{N,\omega} \sim N^{-1/2}$, something a priori expected (for example [**14**]) in the delocalized regime and true for non disordered systems, is violated. More precisely, one can find a sequence of random times $\{\tau_j\}_j$, $\lim_j \tau_j = \infty$ such that $Z_{\tau_j,\omega} \geq {\tau_j}^{-1/2+a}$, $a = a(\lambda, h) > 0$ (see Proposition 4.1 in [**36**]). These random times are constructed

exactly by looking for $q$–atypical stretches as above and one can appreciate such an irregular decay for example in case B of Fig. 2.3, and this in spite of the fact that the data have been strongly coarse grained.

Therefore the lower bound (2.23), both in the localized and in the delocalized regime, yields the following picture: the lower bound we found on $Z_{N,\omega}$ grows suddenly in correspondence of atypical stretches and after that it decays with an exponent $1/2$, up to another atypical stretch. This matches Fig. 2.3, at least at a qualitative level, see the caption of the figure.

## 4. The delocalized phase: a path analysis

Let us start with a qualitative observation: if we set the parameters $(\lambda, h)$ of the copolymer to $(\lambda, h^{(m)}(\lambda))$ with $m = 0.9$, then the observed behavior of $\{Z^{\lambda,h}_{N,\omega}(0)\}_N$ –suitably averaged over blocks in order to eliminate local fluctuations– is somewhat close to $(\text{const})/N^{3/2}$. This is true for all the numerically accessible values of $N$ (up to $N \sim 10^8$), at once for a number of values of $\lambda$ and for a great number of typical environments $\omega$. Of course this is suggesting that for $m = 0.9$ the curve $h^{(m)}(\lambda)$ lies in the delocalized region, but it is not easy to convert this qualitative observation into a precise statement, because we do not have a rigorous finite–volume criterion to state that a point $(\lambda, h)$ belongs to the delocalized phase (the contrast with the localized phase, see (2.8), is evident). In other words, we cannot exclude the possibility that the system is still localized but with a characteristic size much larger than the one we are observing.

Nevertheless, the aim of this section is to give an empirical criterion, based on an analysis of the path behavior of the copolymer, that will allow us to provide some more quantitative argument in favor of the fact that the curve $h^{(m)}(\lambda)$ lies in the delocalized region even for values of $m < 1$. This of course would entail that the upper bound $\overline{h}(\lambda)$ defined in (2.1) is not strict.

**4.1. Known and expected path behavior.** We want to look at the whole *profile* $\{Z^{\lambda,h}_{N,\omega^r}(x)\}_{x\in\mathbb{Z}}$ rather than only at $Z^{\lambda,h}_{N,\omega^r}(0)$, where by $\omega^r$ we mean the environment $\omega$ in the *backward direction*, that is $(\omega^r)_n := \omega_{N+1-n}$ (the reason for this choice is explained in Remark 2.7 below). The link with the path behavior of the

copolymer, namely the law of $S_N$ under the polymer measure $\mathbf{P}^{\lambda,h}_{N,\omega^r}$, is given by

$$\frac{Z^{\lambda,h}_{N,\omega^r}(x)}{Z^{\lambda,h}_{N,\omega^r}} = \mathbf{P}^{\lambda,h}_{N,\omega^r}(S_N = x)\,. \tag{2.24}$$

We have already remarked in § 2.6 of Chapter 1a that, although the localized and delocalized phases have been defined in terms of free energy, they do correspond to sharply different path behaviors. In the localized phase it is known [**62, 8**] that the laws of $S_N$ under $\mathbf{P}^{\lambda,h}_{N,\omega^r}$ are *tight*, which means that the polymer is essentially at $O(1)$ distance from the $x$–axis. The situation is completely different in the (interior of the) delocalized phase, where one expects that $S_N = O(\sqrt{N})$: in fact the conjectured path behavior (motivated by the analogy with the known results for non disordered models, see in particular [**51**], [**22**] and [**18**]) should be weak convergence under diffusive scaling to the *Brownian meander process* (that is Brownian motion conditioned to stay positive on the interval $[0, 1]$, see [**60**]). Therefore in the (interior of the) delocalized phase the law of $S_N/\sqrt{N}$ under $\mathbf{P}^{\lambda,h}_{N,\omega^r}$ should converge weakly to the corresponding marginal of the Brownian meander, whose law has density $x\exp(-x^2/2)\mathbf{1}_{(x\geq 0)}$.

In spite of the lack of precise rigorous results, the analysis we are going to describe is carried out under the hypothesis that, in the interior of the delocalized phase, the scaling limit towards Brownian meander holds true (as it will be seen, the numerical results provide a sort of *a posteriori* confirmation of this hypothesis).

Remark 2.7. From a certain point of view attaching the environment backwards does not change too much the model: for example it is easy to check that if one replaces $\omega$ by $\omega^r$ in (1a.8), the limit still exists $\mathbb{P}(\mathrm{d}\omega)$–a.s. and in $\mathbb{L}_1(\mathrm{d}\mathbb{P})$. Therefore the free energy is the same, because $\{\omega^r_n\}_{1\leq n\leq N}$ has the same law as $\{\omega_n\}_{1\leq n\leq N}$, for any fixed $N$.

However, if one focuses on the law of $S_N$ as a function of $N$ *for a fixed environment* $\omega$, the behavior reveals to be much smoother under $\mathbf{P}^{\lambda,h}_{N,\omega^r}$ than under $\mathbf{P}^{\lambda,h}_{N,\omega}$. For instance, under the original polymer measure $\mathbf{P}^{\lambda,h}_{N,\omega}$ it is no more true that in the localized region the laws of $S_N$ are tight (it is true only *most of the time*, see [**35**] for details). The reason for this fact is to be sought in the presence of long *atypical* stretches in every typical $\omega$ (this fact has been somewhat quantified in [**36**, Section 4] and it is at the heart of the approach in Section 3) that are encountered along

the copolymer as $N$ becomes larger. Of course the effect of these stretches is very much damped with the backward environment.

A similar and opposite phenomenon takes place also in the delocalized phase. In fancier words, we could say that for fixed $\omega$ and as $N$ increases, the way $S_N$ approaches its *limiting behavior* is faster when the environment is attached backwards: it is for this reason that we have chosen to work with $\mathbf{P}^{\lambda,h}_{N,\omega^r}$.

**4.2. Observed path behavior: a numerical analysis.** In view of the above considerations, we choose as a measure of the delocalization of the polymer the $\ell_1$ distance $\triangle^{\lambda,h}_N(\omega)$ between the numerically computed profile for a polymer of size $2N$ under $\mathbf{P}^{\lambda,h}_{2N,\omega^r}$, and the conjectured asymptotic delocalized profile:

$$\triangle^{\lambda,h}_N(\omega) := \sum_{x\in 2\mathbb{Z}} \left| \frac{Z^{\lambda,h}_{N,\omega^r}(x)}{Z^{\lambda,h}_{N,\omega^r}} \; - \; \frac{1}{\sqrt{2N}}\,\varphi^+\!\left(\frac{x}{\sqrt{2N}}\right)\right|, \qquad \varphi^+(x) := x\, e^{-x^2/2}\mathbf{1}_{(x\geq 0)}\,. \tag{2.25}$$

Loosely speaking, when the parameters $(\lambda, h)$ are in the interior of the the delocalized region we expect $\triangle_N$ to decrease to 0 as $N$ increases, while this certainly will not happen if we are in the localized phase.

The analysis has been carried out at $\lambda = 0.6$: we recall that the lower and upper bound of (2.1) give respectively $\underline{h}(0.6) \simeq 0.36$ and $\overline{h}(0.6) \simeq 0.49$, while the lower bound we derived with our test for localization is $h = 0.41$, see Table 1. However, as observed in Section 3, Fig. 2.3, there is numerical evidence that $h = 0.43$ is still localized, and for this reason we have analyzed the values of $h = 0.44, 0.45, 0.46, 0.47$ (see below for an analysis on smaller values of $h$).

For each couple $(\lambda, h)$ we have computed $\triangle^{\lambda,h}_N(\omega)$ for the sizes $N = a \times 10^6$ with $a = 1, 2, 5, 10$ and for 500 independent environments. Of course some type of statistical analysis must be performed on the data in order to decide whether there is a decay of $\triangle$ or not. The most direct strategy would be to look at the sample mean of a family of IID variables distributed like $\triangle_N(\omega)$, but it turns out that the fluctuations are too big to get reasonable confidence intervals for this quantity (in other words, the sample variance does not decrease fast enough), at least for the numerically accessible sample sizes. A more careful analysis shows that the variance is essentially due to a *very small* fraction of data that have *large* deviations from the mean, while the most of the data mass is quite concentrated.

For this reason we have chosen to focus on the *sample median* rather than on the sample mean. Table 3 contains the results of the analysis (see also Fig. 2.4 for a graphical representation): for each value of $h$ we have reported the standard 95% confidence interval for the sample median (see Remark 2.8 below for details) for the four different values of $N$ analyzed. While for $h = 0.44$ the situation is not clear, we see that for the values of $h$ greater than 0.45 there are quantitative evidences for a decrease in $\triangle_N$: this leads us to the conjecture that the points $(\lambda, h)$ with $\lambda = 0.6$ and $h \geq 0.45$ (equivalently, the points $(\lambda, h^{(m)}(\lambda))$ with $m \gtrsim 0.876$) lie in the delocalized region.

| $h \backslash N (\times 10^6)$ | 1 | 2 | 5 | 10 |
|---|---|---|---|---|
| 0.44 | [.0603, .0729] | [.0574, .0682] | [.0572, .0689] | [.0570, .0695] |
| 0.45 | [.0258, .0286] | [.0207, .0232] | [.0170, .0190] | [.0149, .0171] |
| 0.46 | [.0140, .0154] | [.0108, .0116] | [.00792, .00869] | [.00647, .00731] |
| 0.47 | [.00905, .00963] | [.00676, .00711] | [.00475, .00508] | [.00364, .00398] |

TABLE 3. The table contains the standard 95% confidence interval for the median of a sample $\{\triangle_N^{\lambda,h}(\omega)\}_\omega$ of size 500, where $\lambda = 0.6$ and $h, N$ take the different values reported in the table. For the values of $h \geq 0.45$ the decreasing behavior of $\triangle_N$ is quite evident (the confidence intervals do not overlap), see also Fig. 2.4.

As already remarked, these numerical observations cannot rule out the possibility that the system is indeed localized, but the system size is too small to see it. For instance, we have seen that there are evidences for $h = 0.43$ to be localized (see case C of Fig. 2.3). In any case, the exponential increasing of $Z_N(0)$ is detectable only at sizes of order$\sim 10^8$, while for smaller system sizes (up to$\sim 10^7$) the qualitative observed behavior of $Z_N(0)$ is rather closer to $(const)/N^{3/2}$, thus apparently suggesting delocalization (see case D of Fig. 2.3).

For this reason it is interesting to look at $\triangle_N^{0.6,\,h}$ for $h = 0.42, 0.43$ and for $N \ll 10^8$. For definiteness we have chosen $N = a \times 10^6$ with $a = 1, 2, 5, 10$, performing the computations for 3000 independent environments: the results are reported in Table 4 (see also Fig. 2.4). As one can see, this time there are clear evidences for an *increasing* behavior of $\triangle_N$. On the one hand this fact gives some more confidence

on the data of Table 3, on the other hand it suggests that looking at $\{\triangle_N\}_N$ is a more reliable criterion for detecting (de)localization than looking at $\{Z_N(0)\}_N$.

| $h\backslash N(\times 10^5)$ | 1 | 2 | 5 | 10 |
|---|---|---|---|---|
| 0.42 | [.351, 0.382] | [.480, 0.517] | [.751, 0.794] | [1.01, 1.06] |
| 0.43 | [.143, 0.155] | [.165, 0.180] | [.197, 0.215] | [.236, 0.264] |

TABLE 4. The table contains the standard 95% confidence interval for the median of a sample $\{\triangle_N^{\lambda,h}(\omega)\}_\omega$ of size 3000, where $\lambda = 0.6$ and $h, N$ take the values reported in the table. For both values of $h$ an increasing behavior of $\triangle_N$ clearly emerges, see also Fig. 2.4 for a graphical representation.

REMARK 2.8. A confidence interval for the sample median can be obtained in the following general way (the steps below are performed under the assumption that the median is unique, which is, strictly speaking, not true in our case, but it will be clear that a finer analysis would not change the outcome). Let $\{Y_k\}_{1\le k\le n}$ denote a sample of size $n$, that is the variables $\{Y_k\}_k$ are independent with a common distribution, whose median we denote by $\xi_{1/2}$: $\mathbf{P}\left(Y_1 \le \xi_{1/2}\right) = 1/2$. Then the variable

$$\mathcal{N}_n := \#\{i \le n :\ Y_i \le \xi_{1/2}\} \tag{2.26}$$

has a binomial distribution $\mathcal{N}_n \sim B(n, 1/2)$ and when $n$ is large (for us it will be at least 500) we can approximate $\mathcal{N}_n/n \approx 1/2 + Z/(2\sqrt{n})$, where $Z \sim N(0,1)$ is a standard gaussian. Let us denote the sample quantiles by $\Xi_q$, defined for $q \in (0,1)$ by

$$\#\{i \le n :\ Y_i \le \Xi_q\} = \lfloor qn \rfloor . \tag{2.27}$$

If we set $a := |\Phi^{-1}(0.025)|$ ($\Phi$ being the standard gaussian distribution function) then the random interval

$$\left[\Xi_{\frac12 - \frac{a}{2\sqrt{n}}},\ \Xi_{\frac12 + \frac{a}{2\sqrt{n}}}\right] \tag{2.28}$$

is a 95% confidence interval for $\xi_{1/2}$, indeed

$$\begin{aligned} 0.95 = \mathbf{P}\big(Z \in [-a,a]\,\big) &= \mathbf{P}\left(\frac12 + \frac{1}{2\sqrt{n}} Z \ \in\ \left[\frac12 - \frac{a}{2\sqrt{n}}\ ,\ \frac12 + \frac{a}{2\sqrt{n}}\right]\right) \\ &\approx \mathbf{P}\left(\frac{\mathcal{N}_n}{n} \in \left[\frac12 - \frac{a}{2\sqrt{n}}\ ,\ \frac12 + \frac{a}{2\sqrt{n}}\right]\right) = \mathbf{P}\left(\Xi_{\frac12 - \frac{a}{2\sqrt{n}}} \le \xi_{1/2} \le \Xi_{\frac12 + \frac{a}{2\sqrt{n}}}\right). \end{aligned} \tag{2.29}$$

## 5. An empirical observation on the critical curve

The key point of this section is that, from a numerical viewpoint, $h_c(\cdot)$ seems very close to $h^{(m)}(\cdot)$, for a suitable value of $m$. Of course any kind of statement in this direction requires first of all a procedure to estimate $h_c(\cdot)$ and we explain this first.

Our analysis is based on the following conjecture:

$$(\lambda, h) \in \overset{\circ}{\mathcal{D}} \implies \lim_{N\to\infty} Z^{\lambda,h}_{2N,\omega}(0) = 0, \ \mathbb{P}(\mathrm{d}\omega) - \text{a.s.}. \tag{2.30}$$

The arguments in Section 3 suggest the validity of such a conjecture, which is comforted by the numerical observation. Since, if $(\lambda, h) \in \mathcal{L}$, $Z^{\lambda,h}_{2N,\omega}(0)$ diverges (exponentially fast) $\mathbb{P}(\mathrm{d}\omega)$–almost surely and since $Z^{\lambda,h}_{2N,\omega}(0)$ is decreasing in $h$, we define $\hat{h}_{N,\omega}(\lambda)$ as the only $h$ that solves $Z^{\lambda,h}_{2N,\omega}(0) = 1$. We expect that $\hat{h}_{N,\omega}(\lambda)$ converges to $h_c(\lambda)$ as $N$ tends to infinity, for typical $\omega$'s. Of course setting the threshold to the value 1 is rather arbitrary, but it is somewhat suggested by (2.8) and by the idea behind the proof of (2.14) (Proposition 2.10 and equation (2.15)).

What we have observed numerically, see Figures 2.5 and 2.6, may be summed up by the statement

$$\text{there exists } m \text{ such that } \hat{h}_{N,\omega}(\lambda) \approx h^{(m)}(\lambda). \tag{2.31}$$

Practically this means that $\hat{h}_{N,\omega}(\lambda)$, for a set of $\lambda$ ranging from 0.05 to 4, may be fitted with remarkable precision by the one parameter family of functions $\left\{h^{(m)}(\cdot)\right\}_m$. The fitting value of $m =: \hat{m}_{N,\omega}$ does depend on $N$ and it is essentially increasing. This is of course expected since localization requires a sufficiently large system (recall in particular Table 2 and Fig. 2.1 – see the caption of Fig. 2.5 for the fitting criterion). We stress that we are presenting results that have been obtained for one fixed sequence of $\omega$: based on what we have observed for example in Section 2.1 for different values of $\lambda$ one does expect that for smaller values of $\lambda$ one should use larger values of $N$, but changing $N$ corresponds to selecting a longer, or shorter, stretch of $\omega$, that is a different sequence of charges and this may have a rather strong effect on the value of $\hat{m}_{N,\omega}$. Moreover there is the problem of deciding which $\lambda$-dependence to choose. This may explain the deviations from (2.31) that are observed for small values of $\lambda$, but these are in any case rather moderate (see Fig. 2.6).

A source of stronger (and unavoidable) deviations arises in the cases of unbounded charges: of course if

$$h \geq h_{\mathrm{sat}} := \max_{n\in\{1,\dots,N\}} \left(-(\omega_{2n-1}+\omega_{2n})/2\right), \tag{2.32}$$

then $Z^{\lambda,h}_{2N,\omega}(0) < 1$, regardless of the value of $\lambda$. Moreover it is immediate to verify that $\lim_{\lambda\to\infty} Z^{\lambda,h}_{2N,\omega}(0) = +\infty$ for $h < h_{\mathrm{sat}}$ and therefore $\hat{h}_{N,\omega}(\lambda) \nearrow h_{\mathrm{sat}}$ as $\lambda \nearrow \infty$. We refer to the captions of Fig. 2.6 for more on this saturation effect.

We have tried also alternative definitions of $\hat{h}_{N,\omega}(\lambda)$, namely:

(1) the value of $h$ such that $Z^{\lambda,h}_{2N,\omega} = 1$ (or a different fixed value);
(2) the value of $h$ such that the $\ell_1$ distance between the distribution of the endpoint and the distribution of the meander, cf. Section 4, is smaller than a fixed threshold, for example 0.05.

What we have observed is that (2.31) still holds. What is not independent of the criterion is $\hat{m}_{N,\omega}$. Of course believing deeply in (2.31) entails the expectation that $\hat{m}_{N,\omega}$ converges to the non random quantity $h_c'(0)$. The results reported in this section suggest a value of $h_c'(0)$ larger than 0.83 and the cases presented in Section 4 suggest that it should be smaller than 0.86.

## 6. Appendix

**6.1. The algorithm for computing $Z_{N,\omega}$.** We are going to briefly illustrate the algorithm we used in the numerical computation of the partition function $Z_N = Z^{\lambda,h}_{N,\omega}$. We recall its definition:

$$Z_N = \mathbf{E}\left[\exp\left(-2\lambda\sum_{n=1}^{N}(\omega_n+h)\Delta_n\right)\right], \tag{2.33}$$

where $\Delta_n := (1-\mathrm{sign}(S_n))/2$ and the convention for sign(0) described in the introduction.

Observe that a direct computation of $Z_N$ from (2.33) would require to sum the contributions of $2^N$ random walk trajectories, making the problem numerically intractable. However, here we can make profitably use of the *additivity* of our Hamiltonian: loosely speaking, if we join together two (finite) random walk segments, the energy of the resulting path is the sum of the energies of the building segments.

We can exploit this fact to derive a simple recurrence relation for the sequence of functions $\left\{\mathcal{Z}_M(y) := Z_{2M}(2y),\ y\in\mathbb{Z}\right\}_{M\in\mathbb{N}}$, where $Z_N(x) = Z^{\lambda,h}_{N,\omega}(x)$, the latter

defined in (1a.9), and we recall that we work with even values of $N$. Conditioning on $S_{2M}$ and using the Markov property one easily finds

$$\mathcal{Z}_{M+1}(y) = \begin{cases} \frac{1}{4}\mathcal{Z}_M(y+1) \ + \ \frac{1}{2}\mathcal{Z}_M(y) \ + \ \frac{1}{4}\mathcal{Z}_M(y-1) & y > 0 \\ \frac{1}{4}\Big[\mathcal{Z}_M(1) + \mathcal{Z}_M(0)\Big] \ + \ \frac{1}{4}\alpha_M\Big[\mathcal{Z}_M(0) + \mathcal{Z}_M(-1)\Big] & y = 0 \\ \alpha_M\Big[\frac{1}{4}\mathcal{Z}_M(y+1) \ + \ \frac{1}{2}\mathcal{Z}_M(y) \ + \ \frac{1}{4}\mathcal{Z}_M(y-1)\Big] & y < 0 \end{cases} , \quad (2.34)$$

where we have put $\alpha_M := \exp\big(-2\lambda\,(\omega_{2M+1} + \omega_{2M+2} + 2h)\big)$.

From equation (2.34) and from the trivial observation that $\mathcal{Z}_M(y) = 0$ for $|y| > M$, it follows that $\{\mathcal{Z}_{M+1}(y),\ y \in \mathbb{Z}\}$ can be obtained from $\{\mathcal{Z}_M(y),\ y \in \mathbb{Z}\}$ with $O(M)$ computations. This means that we can compute $Z_N$ in $O(N^2)$ steps.[1]

We point out that sometimes one is satisfied with *lower bounds* on $Z_N$, for instance in the statistical text for localization described in Section 2.1. In this case the algorithm can be further speeded up by restricting the computation to a suitable set of random walk trajectories. In fact when the system size is $N$ the polymer is at most at distance $O(\sqrt{N})$ (we recall the discussion in Section 4 on the path behavior), hence a natural choice to get a lower bound on $Z_N$ is to only take into account the contribution coming from those random walk paths $\{s_n\}_{n\in\mathbb{N}}$ for which

$$-A\sqrt{n} \le s_n \le B\sqrt{n} \qquad \text{for } n \ge N_0\,, \quad (2.35)$$

where $A, B, N_0$ are positive constants. Observe that this is easily implemented in the algorithm described above: it suffices to apply relation (2.34) only for $y \in [-A\sqrt{M}, B\sqrt{M}]$, while setting $\mathcal{Z}_{M+1}(y) = 0$ for the other values of $y$. In this way the number of computations needed to obtain $Z_N$ is reduced to $O(N^{3/2})$.

The specific values of $A, B, N_0$ we used in our numerical computations are $3, 8, 1000$, and we would like to stress that the lower bound on $Z_N$ we got coincides up to the $8^{\text{th}}$ decimal digit with the *true value* obtained applying the complete algorithm.

A final important remark is that for the results we have reported we have used the Mersenne–Twister [**48**] pseudo–random number generator. However we

[1]The algorithm just described can be implemented in a standard way: the code we used, written in C, is available on the web page: `http://www.proba.jussieu.fr/pageperso/giacomin/C/prog.html`. Graphic representations and standard statistical procedures have been performed with R [**58**].

have also tried other pseudo–random number generators and *true randomness* from `www.random.org`: the results appear not to depend on the generator.

**6.2. Proof of the lower bound on $h_c$.** We are going to give a detailed proof of the lower bound (2.14) on the critical curve, together with some related result. We stress that this appendix can be made substantially lighter if one is interested only in the *if* part of Proposition 2.10. In this case the first part of this appendix is already contained in the first part of § 3.1, up to (2.17), and it suffices to look at the proof of the lower bound starting from page 80.

We recall that $Z^{\lambda,h}_{N,\omega}(0)$ is the partition function corresponding to the polymer pinned at its right endpoint, see (1a.9), and $T^C = T^C(\omega)$ is the first $N$ for which $Z_{N,\omega}(0) \geq C$, see (2.15). In particular, for all $\omega$ such that $T^C(\omega) < \infty$ we have

$$Z^{\lambda,h}_{T^C(\omega),\omega}(0) \geq C\,. \tag{2.36}$$

We will also denote by $\mathcal{F}_n := \sigma(\omega_1, \dots, \omega_n)$ the natural filtration of the sequence $\{\omega_n\}_{n\in\mathbb{N}}$.

6.2.1. *A different look at (de)localization.* We want to show that (de)localization can be read from $T^C$. We introduce some notation: given an increasing, $2\mathbb{N}$–valued sequence $\{t_i\}_{i\in\mathbb{N}}$, we set $t_0 := 0$ and $\zeta_N := \max\{k : t_k \leq N\}$. Then we define

$$\begin{aligned}\widehat{Z}_{N,\omega}(0) = \widehat{Z}^{\{t_i\},\lambda,h}_{N,\omega}(0) \;&:=\; \mathbf{E}\left[e^{-2\lambda\sum_{n=1}^N(\omega_n+h)\Delta_n};\, S_{t_1}=0,\,\dots,\,S_{t_{\zeta_N}}=0,\,S_N=0\right]\\ &=\; \prod_{i=0}^{\zeta_N-1} Z^{\lambda,h}_{t_{i+1}-t_i,\theta^{t_i}\omega}(0)\;\cdot\; Z^{\lambda,h}_{N-t_{\zeta_N}(\omega),\theta^{t_{\zeta_N}}\omega}(0)\,,\end{aligned} \tag{2.37}$$

and we recall that $\theta$ denotes the translation on the environment. One sees immediately that $\widehat{Z}_{N,\omega}(0) \leq Z_{N,\omega}(0)$. We first establish a preliminary result.

LEMMA 2.9. *If the sequence $\{t_i\}_i$ is such that $\zeta_N/N \to 0$ as $N \to \infty$, then*

$$\lim_{N\to\infty}\frac{1}{N}\log\widehat{Z}^{\{t_i\},\lambda,h}_{N,\omega}(0) \;=\; \mathrm{F}(\lambda,h)\,, \tag{2.38}$$

*both $\mathbb{P}(\mathrm{d}\omega)$–a.s. and in $\mathbb{L}_1(\mathbb{P})$.*

**Proof.** By definition we have $Z_{N,\omega}(0) \geq \widehat{Z}_{N,\omega}(0)$. On the other hand, we are going to show that

$$Z^{\lambda,h}_{N,\omega}(0) \;\leq\; 4^{\zeta_N}\,A^{2\zeta_N}\,\left(\prod_{i=1}^{\zeta_N}(t_i-t_{i-1})\cdot(N-t_{\zeta_N})\right)^3\,\widehat{Z}^{\{t_i\},\lambda,h}_{N,\omega}(0)\,, \tag{2.39}$$

where $A$ is a positive constant. To derive this bound, we resort to the equation (2.4) that expresses $Z_{N,\omega}(0)$ in terms of random walk excursions. We recall that $K(2n)$ is the discrete probability density of the first return time of the walk $S$ to 0, and that $K(t) \geq 1/(A\, t^{3/2})$, $t \in 2\mathbb{N}$, for some positive constant $A$: it follows that for $a_1, \ldots, a_k \in 2\mathbb{N}$

$$K(a_1 + \ldots + a_k) \;\leq\; 1 \;\leq\; A^k \, (a_1 \cdot \ldots \cdot a_k)^{3/2} \, K(a_1) \cdot \ldots \cdot K(a_k) \, . \tag{2.40}$$

This gives us an upper bound to the entropic cost needed to split a random walk excursion of length $(a_1 + \ldots + a_k)$ into $k$ excursions of lengths $a_1, \ldots, a_k$.

Now let us come back to the second line of (2.4), that can be rewritten as

$$Z_{N,\omega}(0) \;=\; \sum_{\{x_i\} \subseteq \{0,\ldots,N\} \cap 2\mathbb{N}} G(\{x_i\}) \, . \tag{2.41}$$

A first observation is that if we restrict the above sum to the $\{x_i\}$ such that $\{x_i\} \supseteq \{t_i\}$, then we get $\widehat{Z}^{\{t_i\}}_{N,\omega}(0)$. Now for each $\{x_i\}$ we aim at finding an upper bound on the term $G(\{x_i\})$ of the form $c \cdot G(\{x_i\} \cup \{t_i\})$ for some $c > 0$ not depending on $\{x_i\}$. Each term $G(\{x_i\})$, see (2.4), is the product of two terms: an entropic part depending on $K(\cdot)$ and an energetic part depending on $\varphi(\cdot)$. Replacing the entropic part costs no more than

$$c_{\text{ent}} \;:=\; A^{2\zeta_N} \left( \prod_{i=1}^{\zeta_N} (t_i - t_{i-1}) \cdot (N - t_{\zeta_N}) \right)^3 \, , \tag{2.42}$$

thanks to (2.40). On the other hand, the cost for replacing the energetic part is easily bounded above by

$$c_{\text{energy}} \;:=\; 2^{\zeta_N} \, , \tag{2.43}$$

so that the bound $G(\{x_i\}) \leq c \cdot G(\{x_i\} \cup \{t_i\})$ holds true with $c := c_{\text{ent}}\, c_{\text{energy}}$. Replacing in this way each term in the sum in the r.h.s. of (2.41), we are left with a sum of terms $G(\{y_i\})$ corresponding to sets $\{y_i\}$ such that $\{y_i\} \supseteq \{t_i\}$. It remains to count the *multiplicity* of any such $\{y_i\}$, that is how many original sets $\{x_i\}$ are such that $\{x_i\} \cup \{t_i\} = \{y_i\}$. Sets $\{x_i\}$ satisfying this last condition must differ only for a subset of $\{t_i\}$, hence the sought multiplicity is $2^{\zeta_N}$ (the cardinality of the parts of $\{t_i\}$) and the bound (2.39) follows.

Therefore we get

$$\left|\frac{\log \widehat{Z}^{\{t_i\},\lambda,h}_{N,\omega}(0)}{N} - \frac{\log Z^{\lambda,h}_{N,\omega}(0)}{N}\right| \;\le\; (2\log 2A)\frac{\zeta_N}{N} \;+\; 3\,\frac{1}{N}\,\log\left(\prod_{i=1}^{\zeta_N}(t_i - t_{i-1})\cdot(N - t_{\zeta_N})\right) \tag{2.44}$$

$$\le\; (2\log 2A)\frac{\zeta_N}{N} \;+\; 3\,\frac{\zeta_N+1}{N}\,\log\left(\frac{N}{\zeta_N+1}\right),$$

where in the second inequality we have made use of the elementary fact that once the sum of $k$ positive numbers is fixed, their product is maximal when all the numbers coincide (for us $k = \zeta_N + 1$). Since by hypothesis $\zeta_N/N \to 0$ as $N \to \infty$, the Lemma is proved. □

Now we are ready to prove the characterization of $\mathcal{L}$ and $\mathcal{D}$ in terms of $T^C$. Fix any $C > 1$.

PROPOSITION 2.10. *A point $(\lambda, h)$ is localized, that is $h < h_c(\lambda)$, if and only if $\mathbb{E}[T^C] < \infty$.*

**Proof.** We set $\mathcal{A} := \{\omega : T^C(\omega) < \infty\}$. Observe that for $\omega \in \mathcal{A}^{\complement}$ we have $Z_{N,\omega}(0) \le C$ for every $N \in 2\mathbb{N}$, and consequently $\log Z^{\lambda,h}_{N,\omega}(0)/N \to 0$ as $N \to \infty$.

Consider first the case when the random variable $T^C$ is defective, that is $\mathbb{P}[\mathcal{A}^{\complement}] > 0$ (this is a particular case of $\mathbb{E}[T^C] = \infty$). Since we know that $\log Z^{\lambda,h}_{N,\omega}(0)/N \to \mathrm{F}(\lambda, h)$, $\mathbb{P}(\mathrm{d}\omega)$–a.s., from the preceding observation it follows that $\mathrm{F}(\lambda, h) = 0$ and the Proposition is proved in this case.

Therefore in the following we can assume that $T^C$ is proper, that is $\mathbb{P}(\mathcal{A}) = 1$, so that equation (2.36) holds for almost every $\omega$. Setting $\theta^{-1}\mathcal{A} := \{\omega : \theta\omega \in \mathcal{A}\}$, we have $\mathbb{P}\left(\theta^{-1}\mathcal{A}\right) = 1$ since $\mathbb{P}$ is $\theta$–invariant, and consequently $\mathbb{P}\left(\cap_{k=0}^{\infty}\theta^{-k}\mathcal{A}\right) = 1$, which amounts to saying that (2.36) can be actually strengthened to

$$Z^{\lambda,h}_{T^C(\theta^k\omega),\theta^k\omega}(0) \ge C \qquad \forall k \ge 0,\ \mathbb{P}(\mathrm{d}\omega)\text{–a.s.}\,. \tag{2.45}$$

Observe that the sequence $\{(\theta^{T^C(\omega)}\omega)_n\}_{n\in\mathbb{N}}$ has the same law as $\{\omega_n\}_{n\in\mathbb{N}}$ and it is independent of $\mathcal{F}_{T^C}$. We can define inductively an increasing sequence of stopping times $\{T_n\}_{n\in\mathbb{N}}$ by setting $T_0 := 0$ and $T_{k+1}(\omega) - T_k(\omega) := T^C(\theta^{T_k(\omega)}\omega) =: S_k(\omega)$. We also set $\zeta_N(\omega) := \max\{n :\ T_n(\omega) \le N\}$. Since $\{S_k\}_{k\in\mathbb{N}}$ is an IID sequence, by the strong law of large numbers we have that, $\mathbb{P}(\mathrm{d}\omega)$–a.s., $T_n(\omega)/n \to \mathbb{E}[T^C]$ as

$n \to \infty$, and consequently $\zeta_N(\omega)/N \to 1/\mathbb{E}[T^C]$ as $N \to \infty$ (with the convention that $1/\infty = 0$).

Now let us consider the lower bound $\widehat{Z}_{N,\omega}(0)$ corresponding to the sequence $\{t_i\} = \{T_i(\omega)\}$: from (2.37) and (2.45) we get that $\mathbb{P}(\mathrm{d}\omega)$–a.s.

$$\begin{aligned} \widehat{Z}_{N,\omega}^{\{T_i(\omega)\},\lambda,h}(0) \;&=\; \prod_{i=0}^{\zeta_N(\omega)-1} Z^{\lambda,h}_{T^C(\theta^{T_i}\omega),\theta^{T_i}\omega}(0) \,\cdot\, Z^{\lambda,h}_{N-T_{\zeta_N(\omega)}(\omega),\theta^{T_{\zeta_N(\omega)}}\omega}(0) \\ &\geq\; C^{\zeta_N(\omega)} \cdot \frac{c}{N^{3/2}}\,, \end{aligned} \tag{2.46}$$

where $c$ is a positive constant (to estimate the last term we have used the lower bound $Z_k(0) \geq c/k^{3/2}$, cf. (1a.5)), and consequently

$$\mathrm{F}(\lambda,h) \;=\; \lim_{N\to\infty} \frac{\log Z^{\lambda,h}_{N,\omega}(0)}{N} \;\geq\; \liminf_{N\to\infty} \frac{\log \widehat{Z}^{\{T_i(\omega)\},\lambda,h}_{N,\omega}(0)}{N} \;\geq\; \frac{\log C}{\mathbb{E}[T^C]}\,. \tag{2.47}$$

It follows that if $\mathbb{E}[T^C] < \infty$ then $\mathrm{F}(\lambda,h) > 0$, that is $(\lambda,h)$ is localized.

It remains to consider the case $\mathbb{E}[T^C] = \infty$, and we want to show that this time $\widehat{Z}_{N,\omega}(0)$, defined in (2.46), gives a null free energy. In fact, as $T^C(\eta)$ is defined as the *first* $N$ such that $Z_{N,\eta}(0) \geq C$, it follows that $Z_{T^C(\eta),\eta}(0)$ cannot be much greater than $C$. More precisely, one has that

$$Z_{T^C(\eta),\eta}(0) \leq C\, \exp(2\lambda|\eta_{T^C(\eta)-1} + \eta_{T^C(\eta)}|)\,, \tag{2.48}$$

and from the first line of (2.46) it follows that

$$\frac{1}{N}\log \widehat{Z}_{N,\omega}(0) \;\leq\; \frac{\zeta_N(\omega)+1}{N}\log C \;+\; \frac{2\lambda}{N}\sum_{i=1}^{\zeta_N(\omega)} \Big(|\omega_{T_i(\omega)}| + |\omega_{T_i(\omega)-1}|\Big)\,. \tag{2.49}$$

We estimate the second term in the r.h.s. in the following way:

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^{\zeta_N(\omega)} \Big(|\omega_{T_i(\omega)}| + |\omega_{T_i(\omega)-1}|\Big) &= \frac{1}{N}\sum_{k=1}^{N} \mathbf{1}_{\{\exists i:\, T_i(\omega)=k\}} \Big(|\omega_k| + |\omega_{k-1}|\Big) \\ &\leq \left(\frac{1}{N}\sum_{k=1}^{N} \mathbf{1}_{\{\exists i:\, T_i(\omega)=k\}}\right)^{1/2} \left(\frac{1}{N}\sum_{k=1}^{N} \Big(|\omega_k| + |\omega_{k-1}|\Big)^2\right)^{1/2} \\ &\leq \sqrt{\frac{\zeta_N(\omega)}{N}} \cdot 2\sqrt{\frac{1}{N}\sum_{k=1}^{N} |\omega_k|^2} \leq A\sqrt{\frac{\zeta_N(\omega)}{N}}\,, \end{aligned} \tag{2.50}$$

for some positive constant $A = A(\omega)$ and eventually as $N \to \infty$, having used the Cauchy–Schwartz inequality and the law of large numbers for the sequence

$\{|\omega_k|^2\}_{k\in\mathbb{N}}$. Therefore

$$\frac{1}{N}\log \widehat{Z}_{N,\omega}(0) \;\le\; \frac{\zeta_N(\omega)+1}{N}\log C \;+\; 4\lambda A\sqrt{\frac{\zeta_N(\omega)}{N}}\,, \tag{2.51}$$

and since $\mathbb{E}[T^C]=\infty$ implies $\zeta_N(\omega)/N\to 0$, $\mathbb{P}(\mathrm{d}\omega)$–a.s., we have $\log \widehat{Z}_{N,\omega}(0)/N\to 0$, $\mathbb{P}(\mathrm{d}\omega)$–a.s.. Then Lemma 2.9 allows us to conclude that $\mathrm{F}(\lambda,h)=0$, and the proof of the Proposition is completed. □

6.2.2. *Proof of the lower bound on $h_c$.* To prove equation (2.14), we are going to build, for every $(\lambda,h)$ such that $h<\underline{h}(\lambda)$, a random time $T$ such that $\mathbb{E}[T]<\infty$ and $Z^{\lambda,h}_{T(\omega),\omega}(0)\ge C$, for some $C>1$. It follows that $T^C\le T$, yielding that $\mathbb{E}[T^C]<\infty$ and by Proposition 2.10 $(\lambda,h)$ is localized, that is, $\underline{h}(\lambda)\le h_c(\lambda)$.

Given $M\in 2\mathbb{N}$ and $q<-h$, we start defining the stopping time

$$\tau_M(\omega)=\tau_{M,q}(\omega):=\inf\left\{n\in 2\mathbb{N}:\ \exists k\in 2\mathbb{N},\ k\ge M:\ \frac{\sum_{i=n-k+1}^{n}\omega_i}{k}\le q\right\}. \tag{2.52}$$

This is the first instant at which a $q$–atypical stretch of length at least $M$ appears along the sequence $\omega$. The asymptotic behavior of $\tau_M$ is given by Theorem 3.2.1 in [**21**, § 3.2] which says that $\mathbb{P}(\mathrm{d}\omega)$–a.s.

$$\frac{\log\tau_M(\omega)}{M}\to\Sigma(q)\qquad \text{as } M\to\infty\,, \tag{2.53}$$

where $\Sigma(q)$ is Cramer's Large Deviations functional for $\omega$, (2.19). We also give a name to the shortest of the terminal stretches in the definition of $\tau_M$:

$$R_M(\omega)=R_{M,q}(\omega):=\inf\left\{k\in 2\mathbb{N},\ k\ge M:\ \frac{\sum_{i=\tau_M-k+1}^{\tau_M}\omega_i}{k}\le q\right\}, \tag{2.54}$$

and it is not difficult to realize that $R_M\le 2M$.

We are ready to give a simple lower bound on the partition function of size $\tau_{M,q}$ (for any $M\in 2\mathbb{N}$ and $q<-h$): it suffices to consider the contribution of the trajectories that are negative in correspondence of the last (favorable) stretch of size $R_M$, and stay positive the rest of the time. Recalling that we use $K(\cdot)$ for the discrete density of the first return time to the origin and that by (2.3) we have $K(2n)\ge c/n^{-3/2}$ for a constant $c>0$, we estimate

$$\begin{aligned} Z^{\lambda,h}_{\tau_M(\omega),\omega}(0) &\ge \frac{1}{4}\,K\left(\tau_M-R_M\right)\,K\left(R_M\right)\,e^{-2\lambda(q+h)R_M} \ge \frac{c^2}{4\tau_M^{3/2}(2M)^{3/2}}e^{-2\lambda(q+h)M} \\ &\ge c'\,\exp\left\{\frac{3}{2}M\left[(-4\lambda/3)q-\frac{\log\tau_M}{M}-(4\lambda/3)h-\frac{\log M}{M}\right]\right\}, \end{aligned} \tag{2.55}$$

where $c' := c^2/(8\sqrt{2})$.

Having in mind (2.53), we define a random index $\ell = \ell_{A,\varepsilon,q}$ depending on the two parameters $A \in 2\mathbb{N},\ \varepsilon > 0$ and on $q$:

$$\ell(\omega) = \ell_{A,\varepsilon,q}(\omega) := \inf\left\{k \in 2\mathbb{N},\ k \geq A :\ \frac{\log \tau_{k,q}(\omega)}{k} \leq \Sigma(q) + \varepsilon\right\}, \tag{2.56}$$

and we finally set

$$T(\omega) = T_{A,\varepsilon,q}(\omega) := \tau_{\ell(\omega)}(\omega)\,. \tag{2.57}$$

Then for the partition function of size $T(\omega)$ we get

$$Z^{\lambda,h}_{T(\omega),\omega}(0) \geq c' \exp\left\{\frac{3}{2}A\left[(-4\lambda/3)q - \Sigma(q) - (4\lambda/3)h - \frac{\log A}{A} - \varepsilon\right]\right\}. \tag{2.58}$$

The fact that $\mathbb{E}[T_{A,\varepsilon,q}] < \infty$ for any choice of $A, \varepsilon, q$ (with $q < -h$) is proved in Lemma 2.11 below. It only remains to show that for every fixed $(\lambda, h)$ such that $h < \underline{h}(\lambda)$, or equivalently

$$(4\lambda/3)h < \log \mathsf{M}(-4\lambda/3)\,, \tag{2.59}$$

the parameters $A, \varepsilon, q$ can be chosen such that the right–hand side of equation (2.58) is greater than 1.

The key point is the choice of $q$. Note that the generating function $\mathsf{M}(\cdot)$ is smooth, since finite on the whole real line. Moreover for all $\lambda \in \mathbb{R}$ there exists some $q_0 \in \mathbb{R}$ such that

$$\log \mathsf{M}(-4\lambda/3) = (-4\lambda/3)q_0 - \Sigma(q_0)\,, \tag{2.60}$$

and from (2.59) it follows that $q_0 < -h$. Therefore we can take $q = q_0$, and equation (2.58) becomes

$$Z^{\lambda,h}_{T(\omega),\omega}(0) \geq c' \exp\left\{\frac{3}{2}A\left[\log \mathsf{M}(-4\lambda/3) - (4\lambda/3)h - \frac{\log A}{A} - \varepsilon\right]\right\}. \tag{2.61}$$

It is now clear that for every $(\lambda, h)$, such that (2.59) holds, by choosing $\varepsilon$ sufficiently small and $A$ sufficiently large, the right–hand side of (2.61) is greater than 1, and the proof of (2.14) is complete.

LEMMA 2.11. *For every $A \in 2\mathbb{N}$, $\varepsilon > 0$ and $q < -h$ the random variable $T(\omega) = T_{A,\varepsilon,q}(\omega)$ defined below (2.56) is integrable: $\mathbb{E}[T] < \infty$.*

**Proof.** By the definition (2.56) of $\ell = \ell_{A,\varepsilon,q}$ we have

$$T_{A,\varepsilon,q} \leq \exp\left((\Sigma(q)+\varepsilon)\,\ell_{A,\varepsilon,q}\right), \tag{2.62}$$

so it suffices to show that for any $\beta > 0$ the random variable $\exp(\beta\,\ell_{A,\varepsilon,q})$ is integrable.

For any $l \in 2\mathbb{N}$, we introduce the IID sequence of random variables $\{Y_n^l\}_{n\in\mathbb{N}}$ defined by

$$Y_n^l := \frac{1}{l}\sum_{i=(n-1)l+1}^{nl} \omega_i\,. \tag{2.63}$$

By Cramer's Theorem [**21**] we have that for any fixed $q < 0$ and $\varepsilon > 0$ there exists $l_0$ such that $\mathbb{P}\left(Y_1^l \leq q\right) \geq e^{-l(\Sigma(q)+\varepsilon/2)}$ for every $l \geq l_0$. By (2.56) have that

$$\{\ell > l\} \subseteq \{\tau_l > \exp((\Sigma(q)+\varepsilon)l)\} \subseteq \bigcap_{i=1}^{\lfloor M/l\rfloor} \{Y_i^l > q\}\,, \tag{2.64}$$

with $M := \exp((\Sigma(q)+\varepsilon)l)$, so that

$$\begin{aligned}\mathbb{P}\left(\ell > l\right) \leq \left(1-e^{-l(\Sigma(q)+\varepsilon/2)}\right)^{\lfloor M/l\rfloor} &\leq \exp\left(-\lfloor M/l\rfloor e^{-l(\Sigma(q)+\varepsilon/2)}\right)\\ &\leq \exp\left(-\exp\left(l\varepsilon/4\right)\right),\end{aligned} \tag{2.65}$$

where the last step holds if $l$ is sufficiently large (we have also used $1-x \leq e^{-x}$). Therefore

$$\mathbb{P}\left(\exp(\beta\,\ell) > N\right) = \mathbb{P}\left(\ell > (\log N)/\beta\right) \leq \exp\left(-N^{\varepsilon/4\beta}\right), \tag{2.66}$$

when $N$ is large, and the proof is complete. □

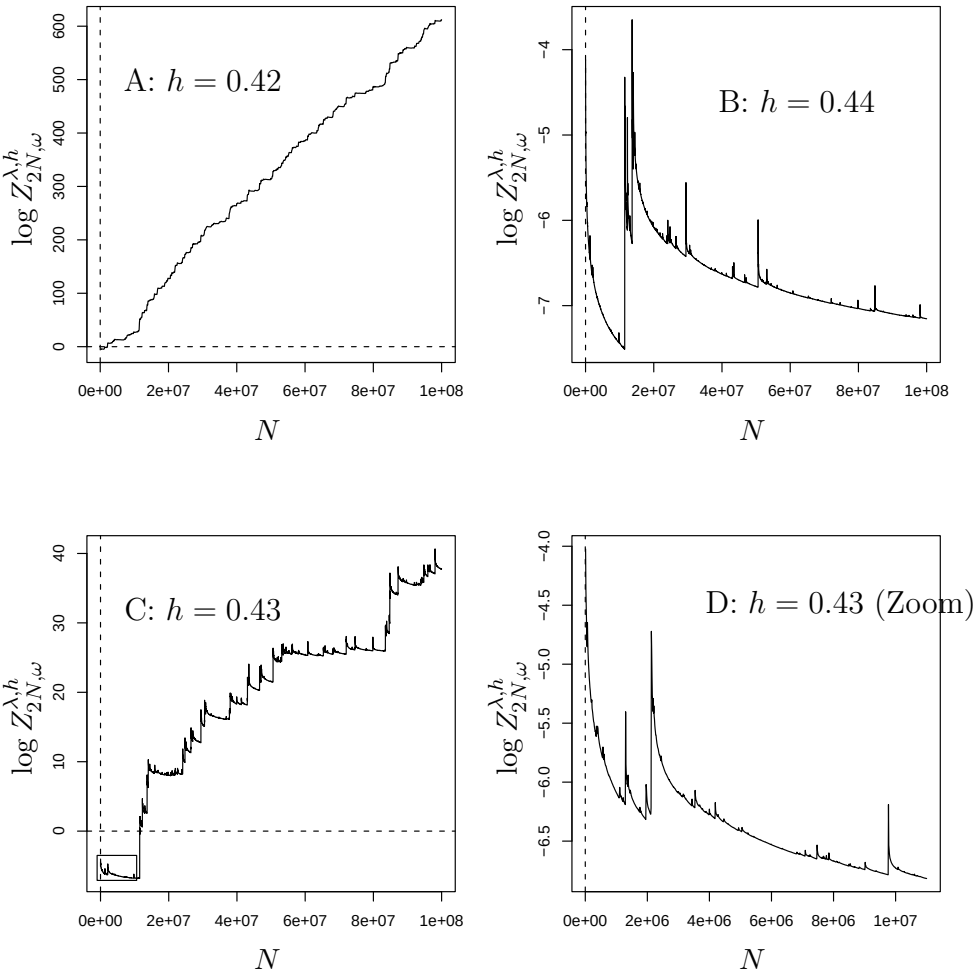


FIGURE 2.3. For $\lambda = 0.6$ ($\underline{h}(0.6) \simeq 0.36$ and $\overline{h}(0.6) \simeq 0.49$) , the behavior of $\log Z_{2N,\omega}$ for $h = 0.42$ (A), 0.43 (C,D) and 0.44 (B). In case A, the polymer is localized with free energy approximately $3 \cdot 10^{-6}$: the linear growth is quite clear, but a closer look shows sudden jumps, which correspond to atypically negative stretches of charges. Getting closer to the critical point, case C, the linear growth is still evident, but it is clearly the result of sudden growths followed by slow decays (approximately polynomial with exponent $-1/2$). Case B suggests delocalization: a closer analysis reveals a decay of the type $N^{-1/2}$, but sharp deviations are clearly visible. Case D is the zoom of the rectangle in the left corner of C. The similarity between B and D make clear that claiming delocalization looking at the behaviour of the partition function is difficult.

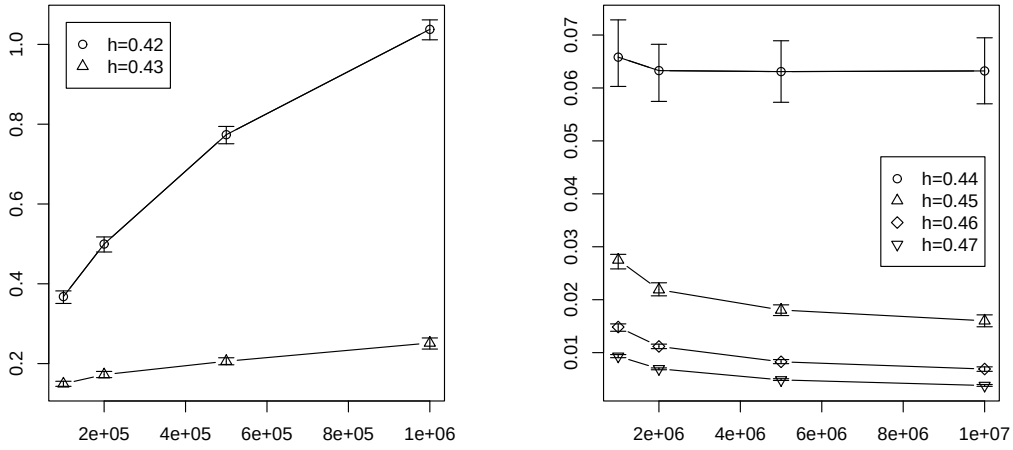


FIGURE 2.4. Graphical representation of the data of Tables 3 (on the right) and 4 (on the left). The plotted points are the sample medians against the sample size, the error bars correspond to the confidence intervals given in Tables 3 and 4.

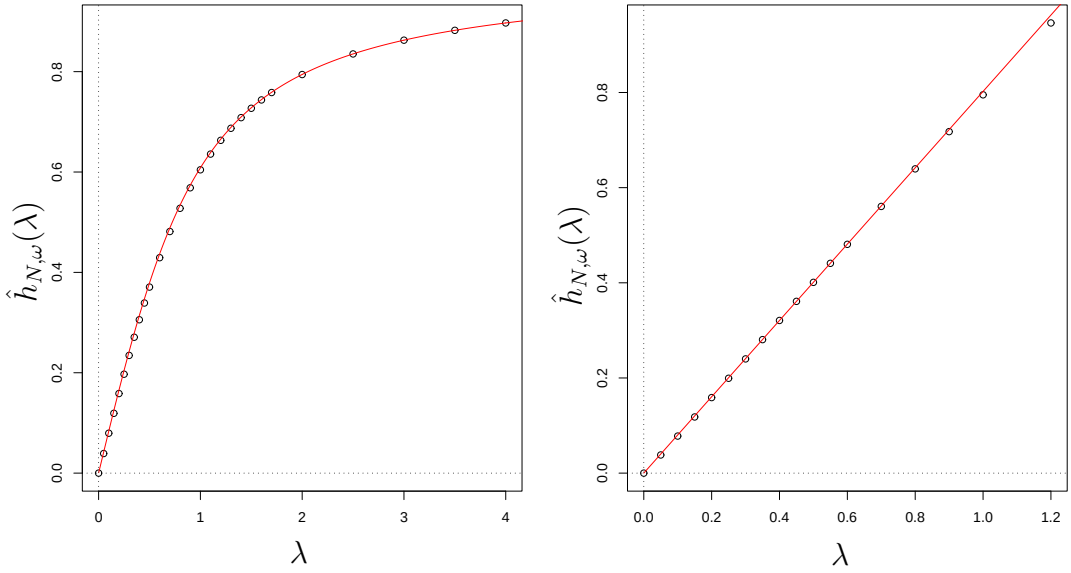


FIGURE 2.5. On the left the case of binary symmetric $\omega_1$ and on the right the case of $\omega_1 \sim N(0,1)$, boths for $N = 3.2 \cdot 10^7$. The small circles represent the computed values: the errors on $\hat{h}_{N,\omega}(\lambda)$ are negligible and the plotted points are at the centers of the circles. The continuous line is instead the curve $h^{(m)}(\cdot)$. In the binary case $m = 0.841$ and it has been chosen by solving $h^{(m)}(4) = \hat{h}_{N,\omega}(4)$. In the Gaussian case $m = 0.802$, the maximum of $\hat{h}_{N,\omega}(\lambda)/\lambda$ for the plotted values of $\lambda(> 0)$. The rather different values of $\hat{m}_{N,\omega}$ may be somewhat understood both by considering that these two curves have been obtained for a fixed realization of $\omega$ and by taking into account the remark at the end of the caption of Table 2: it appears that for Gaussian charges one needs longer systems in order to get closer to the values of $m$ observed in the binary case (in particular: for the prolongation, with the same random number generator, of the Gaussian $\omega$ sample used here up to $N = 5 \cdot 10^7$ one obtains $\hat{m}_{N,\omega} = 0.812$).

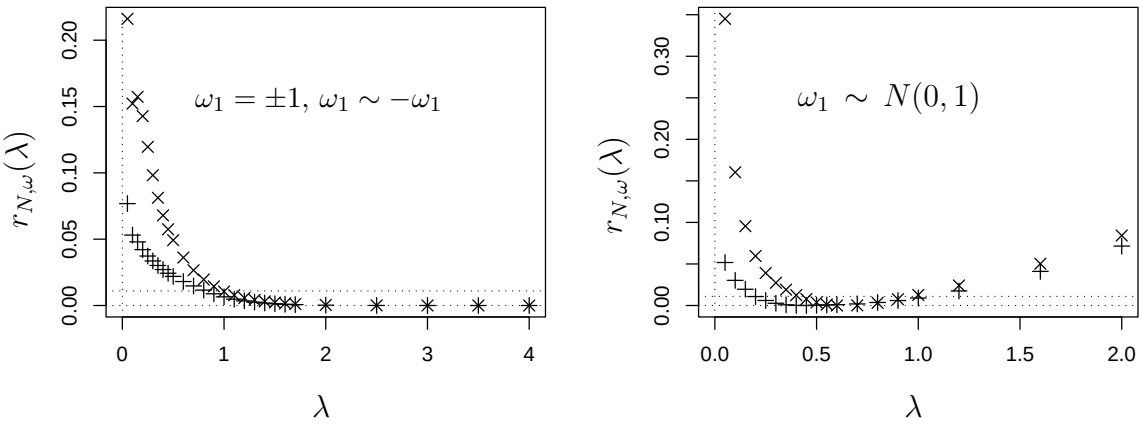


FIGURE 2.6. Relative errors $r_{N,\omega}(\lambda) := \left(h^{(m)}(\lambda) - \hat{h}_{N,\omega}(\lambda)\right)/\hat{h}_{N,\omega}(\lambda)$, for the value $m = \hat{m}_{N,\omega}$ explained in the caption of Fig. 2.5 and for the cases of $N = 2.5 \cdot 10^5$ ($\times$ dots), and $N = 3.2 \cdot 10^7$ (+ dots). Notice that in the binary case the error is more important for small values of $\lambda$ (recall Table 2 and Fig. 2.1). Instead for the Gaussian case there is a deviation both for small and large values of $\lambda$: the deviation for large values is due to the saturation effect explained in the text. Given the fact that $h_{\mathrm{sat}}$, cf. (2.32), behaves almost surely and to leading order for $N \to \infty$ as $\sqrt{\log N}$ one understand why the slow disappearing of the saturation effect has to be expected. In both graphs the dotted line above the axis is at level 0.01. The fitted values for $\hat{m}_{N,\omega}$, $N = 2.5 \cdot 10^5$, are 0.821 in the binary case and 0.778 in the Gaussian case.

CHAPTER 3

# On improving the annealed bound for polymer chains with random charges

In this chapter we address the issue of improving the anneal bound on the critical line $h_c(\cdot)$ of the random copolymer via the so–called constrained annealing, that means nothing but applying the annealing procedure (which is just Jensen's inequality) after having added to the Hamiltonian a disorder–dependent term (sometimes interpreted as a Lagrange multiplier) in a way that the quenched expressions are left unchanged, remember § 2.5 of Chapter 1a.

A popular class of multipliers is the one consisting of empirical averages of local functions of the disorder. These multipliers are particularly suitable for computations, and it is often believed that in this class one can approximate arbitrarily well the quenched free energy.

We are going to prove that *this is not the case* for a wide family of polymer models, including the copolymer near a selective interface and the pinning/wetting models defined in Chapter 1a. More precisely we show that the multipliers in the above class cannot improve on the basic annealed bound from the viewpoint of characterizing the phase diagram. For simplicity the proof has been carried out under the assumption that the random variable $\omega_1$ takes only a finite number of values, however the statement remains true also the general case, provided one makes some suitable boundedness assumptions on the multiplier.

The article [**16**] has been taken from the content of this chapter.

## 1. The framework and the main result

**1.1. The general set–up.** A number of disordered models of linear chains undergoing localization or pinning effects can be put into the following general framework. Let $S := \{S_n\}_{n=0,1,\dots}$ be a process with $S_n$ taking values in $\mathbb{Z}^d$, $d \in \mathbb{N} := \{1, 2, \dots\}$ and law $\mathbf{P}$.

The disorder in the system is given by a sequence $\omega := \{\omega_n\}_n$ of IID random variables taking values in a finite set $\Gamma$ with law $\mathbb{P}$, acting on the path of $S$ via an Hamiltonian that, for a system of size $N$, is a function $H_{N,\omega}$ of the trajectory $S$, but depending only on $S_0, S_1, \ldots, S_N$. One is interested in the properties of the probability measures $\mathbf{P}_{N,\omega}$ defined by giving the density with respect to $\mathbf{P}$:

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{Z_{N,\omega}} \exp\left(H_{N,\omega}(S)\right), \tag{3.1}$$

where $Z_{N,\omega} := \mathbf{E}\left[\exp\left(H_{N,\omega}(S)\right)\right]$ is the normalization constant. Our attention focuses on the asymptotic behavior of $\log Z_{N,\omega}$.

In the sequel we will assume:

ASSUMPTION 3.1. There exists a sequence $\{D_n\}_n$ of subsets of $\mathbb{Z}^d$ such that $\mathbf{P}(S_n \in D_n \text{ for } n = 1, 2, \ldots, N) \overset{N\to\infty}{\asymp} 1$, namely

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbf{P}\left(S_n \in D_n \text{ for } n = 1, 2, \ldots, N\right) = 0, \tag{3.2}$$

and $H_{N,\omega}(S) = 0$ if $S_n \in D_n$ for $n = 1, 2, \ldots, N$.

One sees directly that this hypothesis implies

$$\liminf_{N\to\infty} \frac{1}{N} \log Z_{N,\omega} \geq \lim_{N\to\infty} \frac{1}{N} \log \mathbf{P}\left(S_n \in D_n \text{ for } n = 1, 2, \ldots, N\right) = 0, \tag{3.3}$$

$\mathbb{P}(\mathrm{d}\omega)$–a.s.. We will assume that $\{(1/N) \log Z_{N,\omega}\}_N$ is a sequence of integrable random variables that converges in the $L^1(\mathbb{P}(\mathrm{d}\omega))$ sense and $P(\mathrm{d}\omega)$–almost surely to a constant, the free energy, that we will call $f$. These assumptions are verified in the large majority of the interesting situations, for example whenever super/sub–additivity tools are applicable.

Of course (3.3) says that $f \geq 0$ and one is lead to the natural question of whether $f = 0$ or $f > 0$. In the instances that we are going to consider the free energy may be zero or positive according to some parameters from which the $H_{N,\omega}$ depends: $f = 0$ and $f > 0$ are associated to sharply different behaviors of the system.

In order to establish upper bounds on $f$ one may apply directly Jensen inequality (*annealed bound*) obtaining

$$f \leq \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left[Z_{N,\omega}\right] =: \widetilde{f}, \tag{3.4}$$

and, in our context, if $\widetilde{f} = 0$ then $f = 0$. The annealed bound may be improved by adding to $H_{N,\omega}(S)$ an integrable function $A_N : \Gamma^{\mathbb{N}} \to \mathbb{R}$ such that $\mathbb{E}\left[A_N(\omega)\right] = 0$:

while the left–hand side is unchanged, $\widetilde{f}$ may depend on the choice of $\{A_N\}_N$. We stress that not only $f$ is left unchanged by $H_{N,\omega}(S) \to H_{N,\omega}(S) + A_N(\omega)$, but $\mathbf{P}_{N,\omega}$ itself is left unchanged (for every $N$). Notice that the choice $A_N(\omega) = -\log Z_{N,\omega} + \mathbb{E}\left[\log Z_{N,\omega}\right]$ yields the equality in (3.4).

In the sequel when we refer to $\widetilde{f}$ we mean that $Z_{N,\omega}$ is defined with respect to $H_{N,\omega}$ satisfying the Basic Hypothesis (no $A_N$ term added).

**1.2. The result.** What we prove in this note is that

PROPOSITION 3.2. *If $\widetilde{f} > 0$ then for every local function $F : \Gamma^{\mathbb{N}} \longrightarrow \mathbb{R}$ such that $\mathbb{E}\left[F(\omega)\right] = 0$ one has*

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathbf{E}\left[\exp\left(H_{N,\omega}(S) + \sum_{n=0}^{N} F(\theta_n\omega)\right)\right] > 0, \tag{3.5}$$

*where $(\theta_n\omega)_m = \omega_{n+m}$.*

We can sum up this result by saying that when $f = 0$ but $\widetilde{f} > 0$ it is of no use modifying the Hamiltonian by adding the empirical average of a (centered) local function.

On a mathematical level it is clear that we are playing with an exchange of limits and that it is not obvious that the free energy, recall the optimal choice of $A_N$ above, may be approximated via empirical averages of a local function of the disorder. But we remark that in the physical literature the approach of approximating the free energy via what can be viewed as a constrained annealed computation, the term $\sum_{n=0}^{N} F(\theta_n\omega)$ being interpreted as a Lagrange multiplier, is often considered as an effective way of approximating the quenched free energy. Here we mention in particular [**52**] and [**44**] in which this approach is taken up in a systematic way: the aim is to approach the quenched free energy by constrained annealing via local functions $F$ that are more and more complex, the most natural example being linear combinations of correlations of higher and higher order.

The proof of Proposition 3.2 is based on the simple observation that whenever $A_N$ is centered

$$\frac{1}{N}\log \mathbb{E}\mathbf{E}\left[\exp\left(H_{N,\omega}(S)+A_N(\omega)\right)\right] \geq \frac{1}{N}\log \mathbb{E}\left[\exp\left(A_N(\omega)\right)\right]+\frac{1}{N}\log \mathbf{P}\left(S_n\in D_n \text{ for } n=1,2,\dots,N\right) =: Q_N+P_N. \tag{3.6}$$

By hypothesis $P_N = o(1)$ so one has to consider the asymptotic behavior of $Q_N$. If $\liminf_N Q_N > 0$ there is nothing to prove. So let us assume that $\liminf_N Q_N = 0$: in this case the inferior limit of the left–hand side of (3.6) may be zero and we want to exclude this possibility when $\widetilde{f} > 0$ and $A_N(\omega) = \sum_{n=0}^N F(\theta_n\omega)$, $F$ local and centered (of course in this case $\lim_N Q_N$ does exist). And in Theorem 3.5 below in fact we show that if $\log \mathbb{E}\left[\exp\left(A_N(\omega)\right)\right] = o(N)$, then there exists a local function $G$ such that $F(\omega) = G(\theta_1\omega) - G(\omega)$ so that $\{\sum_{n=0}^N F(\theta_n\omega)\}_N$ is just a boundary term and the corresponding constrained annealing is just the standard annealing.

Notice that having chosen $\Gamma$ finite frees us from integrability conditions.

Remark 3.3. We stress that our Basic Hypothesis is more general than it may look at first. As already observed, one has the freedom of adding to the Hamiltonian $H_{N,\omega}(S)$ any term that does not depend on $S$ (but possibly does depend on $\omega$ and $N$) without changing the model $\mathbf{P}_{N,\omega}$. It may therefore happen that the *natural* formulation of the Hamiltonian does not satisfy our Basic Hypothesis, but it does after a suitable additive correction. This happens for example for the Copolymer near a selective interface model, as we have seen in § 2.3 of Chapter 1a (see also § 1.5 below): the additive correction in this case is linear in $\omega$ and it corresponds to what in [**55**] is called *first order* Morita approximation. In these terms, Proposition 3.2 is saying that *higher order* Morita approximations cannot improve the bound on the critical curve found with the first order computation.

Let us now look at applications of Proposition 3.2.

**1.3. Random rewards or penalties at the origin.** Let $S$, $S_0 = 0 \in \mathbb{Z}^d$, be a random walk with centered IID non degenerate increments $\{X_n\}_n$, $(X_n)_j \in \{-1,0,1\}$ for $j = 1,2,\dots,d$, and

$$H_{N,\omega} = \beta\sum_{n=1}^N (1+\varepsilon\omega_n)\,\mathbf{1}_{\{S_n=0\}}. \tag{3.7}$$

for $\beta \geq 0$ and $\varepsilon \geq 0$. This model is a $d$–dimensional version (with somewhat different notations) of the pinning model introduced in § 3.1 of Chapter 1a. The random variable $\omega_1$ is chosen such that $\mathbb{E}[\exp(\lambda\omega_1)] < \infty$ for every $\lambda \in \mathbb{R}$, and centered. We write $f(\beta, \varepsilon)$ for $f$: by super–additive arguments $f$ exists and it is self–averaging (this observation is valid for all the models we consider and will not be repeated). As we already remarked in Chapter 1a, for $\varepsilon = 0$ the model can be solved, see e.g. [**35**], and in particular $f(\beta, 0) = 0$ if and only if $\beta \leq \beta_c(d) := -\log(1 - \mathbf{P}(S \text{ never comes back to } 0))$. Adding the disorder makes this model much more complex: the annealed bound yields $f(\beta, \varepsilon) = 0$ if $\beta \leq \beta_c(d) - \log \mathbb{E}\left[\exp(\varepsilon\omega_1)\right] =: \widetilde{\beta}_c$. It is an open question whether $\widetilde{\beta}_c$ coincides with the quenched critical value or not, that is whether $f(\beta, \varepsilon) = 0$ implies $\beta \leq \widetilde{\beta}_c$ or not. For references about this issue we refer to [**2, 57**], see however also the next paragraph: the model we are considering can in fact be exactly mapped to the wetting problem ([**2**], [**35**]). Proposition 3.2 applies to this context with $D_n = \{0\}^{\complement}$ for every $n$ [**28**, Ch. 3] and says that one cannot answer this question via constrained annealed bounds.

**1.4. Wetting models in $1 + d$ dimensions.** Let $S$ and $\omega$ as in the previous example and

$$H_{N,\omega} = \begin{cases} \beta \sum_{n=1}^{N} (1 + \varepsilon\omega_n) \, \mathbf{1}_{\{(S_n)_d = 0\}} & \text{it } (S_n)_d \geq 0 \text{ for } n = 1, 2, \ldots, N \\ -\infty & \text{otherwise.} \end{cases} \tag{3.8}$$

with $\beta \geq 0$ and $\varepsilon \geq 0$. If one takes the directed walk viewpoint, that is if one considers the walk $\{(n, S_n)\}_n$, then this is a model of a walk constrained above the (hyper–)plane $x_d = 0$ and rewarded $\beta$, on the average, when touching this plane. If $d = 1$ then this is an effective model for a (1+1)–dimensional interface above a wall which mostly attracts it. As a matter of fact in this case there is no loss of generality in considering $d = 1$, since in the directions parallel to the wall the model is just the original walk. Once again if $\varepsilon = 0$ the model can be solved in detail, see e.g. [**35**]. Computing the critical $\beta$ and deciding whether the annealed bound is sharp, at least for small $\varepsilon$, is an unresolved and disputed question in the physical literature, see e.g. [**31**], [**20**] and [**68**]. Proposition 3.2 applies with the choice $D_n = \mathbb{Z}^{d-1} \times \mathbb{N}$.

**1.5. Copolymer and adsorption models.** Choose $S$ as above and take the directed walk viewpoint. Imagine that above the axis ($x_d > 0$) is filled of a solvent $A$, while below ($x_d < 0$) there is a solvent $B$. At $x_d = 0$ there is the interface. We

choose $\omega = \{A, B\}$ and for example

$$H^{AB}_{N,\omega}(S) \;=\; \sum_{n=1}^{N} \left( a\mathbf{1}_{\{\mathrm{sign}(S_n)=+1,\, \omega_n=A\}} + b\mathbf{1}_{\{\mathrm{sign}(S_n)=-1,\, \omega_n=B\}} + c\mathbf{1}_{\{S_n=0\}} \right) \tag{3.9}$$

with $a$, $b$ and $c$ real parameters, $\mathrm{sign}(S_n) := \mathrm{sign}\left((S_n)_d\right)$ and the convention $\mathrm{sign}(S_n) = \mathrm{sign}(S_{n-1})$ if $(S_n)_d = 0$ already used in Chapter 1a. In order to apply Proposition 3.2 one has to subtract a disorder dependent term, cf. Remark 3.3: if $a \geq b$ we change the Hamiltonian

$$H_{N,\omega}(S) \;:=\; H^{AB}_{N,\omega}(S) - \sum_{n=1}^{N} a\mathbf{1}_{\{\omega_n=A\}}. \tag{3.10}$$

without changing the measure $\mathbf{P}_{N,\omega}$ while the free energy has the trivial shift from $f$ to $f - a\mathbb{P}(\omega_1 = A)$. One can therefore choose $D_n = \mathbb{Z}^{d-1} \times \mathbb{N}$ and Proposition 3.2 applies. This model has been considered for example in [**55**].

Note that if $c = 0$ and $d = 1$ the model is nothing but the copolymer model introduced in Chapter 1a, that is we can cast (3.9) in the form

$$H_{N,\omega}(S) \;=\; \lambda \sum_{n=1}^{N} (\omega_n + h)\, \mathrm{sign}(S_n)\,, \tag{3.11}$$

with $\omega$ taking values in $\mathbb{R}$. Once again the Hamiltonian has to be corrected by subtracting the term $\lambda \sum_n (\omega_n + h)$ (which is exactly what was done in § 2.3 of Chapter 1a) in order to apply Proposition 3.2. One readily sees that (3.10) and (3.11) are the same model when in the second case $\omega$ takes only the values $\pm 1$, $A = +1$ and $B = -1$, and $h = (a-b)/(a+b)$, $\lambda = (a+b)/4$.

Proposition 3.2 acquires some interest in this context: in fact we have already remarked that the physical literature is rather split on the precise value of the critical curve and on whether the annealed bound is sharp or not. We recall that the numerical analysis performed in Chapter 2 is suggesting that the annealed curve does not coincide with the quenched one, and in view of Proposition 3.2 this would mean that constrained annealing via local functions cannot capture the phase diagram of the quenched system.

**1.6. Further models and observations.** In spite of substantial numerical evidence that in several instances $f = 0$ but $\widetilde{f} > 0$, we are unaware of an *interesting* model for which this situation is rigorously known to happen. Consider however the

case $\mathbb{P}(\omega_n = \pm 1) = 1/2$ and

$$H_{N,\omega}(S) = \beta \sum_{n=1}^{N} (1 + \varepsilon\omega_n)\, \mathbf{1}_{\{S_n = n\}}, \tag{3.12}$$

with $\beta$ and $\varepsilon$ real numbers and $S$ the simple random walk on $\mathbb{Z}$. We observe that Proposition 3.2 applies to this case with $D_n = \{n\}^{\complement}$ and that the model is solvable in detail. In particular $f(\beta, \varepsilon) = (\beta - \log 2) \vee 0$, regardless of the value of $\varepsilon$. The annealed computation instead yields $\widetilde{f}(\beta, \varepsilon) = (\beta + \log\cosh(\varepsilon) - \log 2) \vee 0$. Notice in particular that the critical values of $\beta$, respectively $\log 2$ and $\log 2 - \log\cosh(\varepsilon)$, differ as long as there is disorder in the system ($\varepsilon \neq 0$). It is interesting to see in this toy model how $A_N$ has to be chosen *very non local* in order to improve on the annealed bound.

Remark 3.4. We point out that we restricted our examples only to cases in which $S$ is a simple random walk, but in principle our approach goes through for much more general models, like walks with correlated increments or self–interacting walks, see [**56**] for an example. And of course $S_n$ takes values in $\mathbb{Z}^d$ only for ease of exposition and can be easily generalized. It is however unclear whether our argument applies to the disordered wetting problem in $d+1$ dimensions, $d > 1$. In this case $S$ is a random interface, the Hamiltonian is like in (3.8), but $n \in \{0, 1, 2, \ldots\}^d$, $S_n \in \mathbb{Z}$ or $\mathbb{R}$. We set for example $S_n = 0$ when one of the coordinates of $n$ is zero. The missing ingredient is an analog of Theorem 3.5 in higher dimensions.

## 2. On cocycles with null free energy

Let $\{\boldsymbol{\omega}_n\}_{n\in\mathbb{N}}$ be an IID sequence of random variables under the probability measure $\mathbb{P}$, taking values in a finite space $\Gamma$ (we have switched the notation $\omega \to \boldsymbol{\omega}$ for clarity). The law of $\boldsymbol{\omega}_1$ on $\Gamma$ is denoted by $\nu$: we will assume that $\nu(\alpha) > 0$ for all $\alpha \in \Gamma$.

We are interested in families $A = \{A_N\}_{N\in\mathbb{N}}$ of random variables of the form of empirical averages of a centered local function $F$, that is

$$A_N = \sum_{n=1}^{N} F(\boldsymbol{\omega}_n, \ldots, \boldsymbol{\omega}_{n+k}), \tag{3.13}$$

where $k \in \{0\} \cup \mathbb{N}$ and $F$ is a real function defined on $\Gamma^{k+1}$ such that $\int F \mathrm{d}\nu^{*(k+1)} = 0$. We will call $A = \{A_N\}_{N\in\mathbb{N}}$ a centered *cocycle*, and with some abuse of notation we will speak of *the cocycle* $F$ to mean the cocycle $\{A_N\}_{N\in\mathbb{N}}$ defined by (3.13).

A cocycle $F : \Gamma^{k+1} \to \mathbb{R}$ is said to be a *coboundary* if (when $k \geq 1$) there exists a function $G : \Gamma^k \to \mathbb{R}$ such that

$$F(\alpha_1, \ldots, \alpha_{k+1}) = G(\alpha_2, \ldots \alpha_{k+1}) - G(\alpha_1, \ldots, \alpha_k) \tag{3.14}$$

for all $\alpha_1, \ldots, \alpha_{k+1} \in \Gamma$. When $k = 0$, we say that $F$ is a *coboundary* if it is identically zero: $F(\alpha) = 0$ for every $\alpha \in \Gamma$.

For $\beta \in \mathbb{R}$ we define the free energy $L^F(\beta)$ of a cocycle $F$ as

$$L^F(\beta) := \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\Big[e^{\beta A_N}\Big]\,. \tag{3.15}$$

The limit above is easily seen to exist by a standard superadditive argument, and Jensen's inequality yields immediately $L^F(\beta) \geq 0$. Of course, if $F$ is a coboundary then the corresponding free energy vanishes for all $\beta \in \mathbb{R}$. That also the converse is true is the object of the following theorem.

THEOREM 3.5. *Let $F$ be a centered cocycle, and let $L^F(\beta)$ be the corresponding free energy, defined by (3.15). The following conditions are equivalent:*

(1) *$F$ is a coboundary;*
(2) *$L^F(\beta) = 0$ for all $\beta \in \mathbb{R}$;*
(3) *$L^F(\beta_0) = 0$ for some $\beta_0 \in \mathbb{R} \setminus \{0\}$.*

The proof is obtained combining convexity ideas with the following combinatorial reformulation of the condition that a function be a coboundary.

LEMMA 3.6. *A function $F : \Gamma^{k+1} \to \mathbb{R}$ is a coboundary if and only if for every $N \in \mathbb{N}$ and for every $(\eta_1, \ldots, \eta_N) \in \Gamma^N$ the following relation holds:*

$$\sum_{i=1}^{N} F(\eta_i, \eta_{i\oplus_N 1}, \ldots, \eta_{i\oplus_N k}) = 0\,, \tag{3.16}$$

*where for $a, b \in \mathbb{N}$ we have set $a \oplus_N b := (a + b) \mod N$.*

**Proof.** The *if* part trivially follows from the definition of a coboundary (see (3.14)), so we can focus on the *only if* part. As a matter of fact, we will use the hypothesis of the Lemma only for two values of $N$, namely $N = 2k$ and $N = 2k + 1$.

Let us take $k$ elements $\gamma_1, \ldots, \gamma_k \in \Gamma$, arbitrarily chosen, that will be kept *fixed* throughout the proof; moreover, let $\alpha_1, \ldots, \alpha_{k+1}$ denote generic elements of $\Gamma$. We

start rewriting equation (3.16) for $N = 2k+1$, with $(\eta_1, \dots, \eta_N) = (\alpha_1, \dots, \alpha_{k+1}, \gamma_1, \dots, \gamma_k)$, as

$$F(\alpha_1, \dots, \alpha_{k+1}) = -\sum_{i=1}^{k} F(\alpha_{i+1}, \dots, \alpha_{k+1}, \gamma_1, \dots, \gamma_i) - \sum_{i=1}^{k} F(\gamma_i, \dots, \gamma_k, \alpha_1, \dots, \alpha_i)\,. \tag{3.17}$$

In order to determine an alternative expression for the second sum in the r.h.s., we use again equation (3.16), this time with $N = 2k$ and $(\eta_1, \dots, \eta_N) = (\alpha_1, \dots, \alpha_k, \gamma_1, \dots, \gamma_k)$, getting

$$\sum_{i=1}^{k} F(\gamma_i, \dots, \gamma_k, \alpha_1, \dots, \alpha_i) = -\sum_{i=1}^{k} F(\alpha_i, \dots, \alpha_k, \gamma_1, \dots, \gamma_i)\,. \tag{3.18}$$

If now we introduce a function $G : \Gamma^k \to \mathbb{R}$, defined by

$$G(\zeta_1, \dots, \zeta_k) := -\sum_{i=1}^{k} F(\zeta_i, \dots, \zeta_k, \gamma_1, \dots, \gamma_i)\,,$$

we can combine equations (3.17) and (3.18) to get

$$F(\alpha_1, \dots, \alpha_{k+1}) = G(\alpha_2, \dots \alpha_{k+1}) - G(\alpha_1, \dots \alpha_k)\,,$$

so that the proof is completed. □

**Proof of Theorem 3.5.** It has already been remarked that (1) ⇒ (2), and of course (2) ⇒ (3) holds trivially. In the following we are going to prove that (3) ⇒ (2) ⇒ (1).

We start determining an explicit expression for the free energy. For this, we define a slight modification of the cocycle $A$ defined by (3.13), by setting

$$\widetilde{A}_N := \sum_{n=1}^{N} F(\boldsymbol{\omega}_n, \boldsymbol{\omega}_{n\oplus_N 1}, \dots, \boldsymbol{\omega}_{n\oplus_N k})\,, \tag{3.19}$$

where by $\oplus_N$ we mean addition modulo $N$. Of course, only the last $k$ addends in the sum are really changed: as $F$ is a bounded function (the space $\Gamma$ is finite), it easily follows that the free energies of $A$ and $\widetilde{A}$ are the same, so that we can write

$$L^F(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta) \qquad \text{where} \qquad Z_N(\beta) = Z_N^F(\beta) = \mathbb{E}\Big[e^{\beta \widetilde{A}_N}\Big]\,. \tag{3.20}$$

Now we introduce the $\Gamma^{k+1} \times \Gamma^{k+1}$ matrix $A_\beta$, defined for $\alpha_i, \gamma_i \in \Gamma,\ i = 1, \dots, k+1$ by

$$A_\beta\big[(\alpha_1, \dots, \alpha_{k+1}), (\gamma_1, \dots, \gamma_{k+1})\big] := \delta_{\gamma_1,\alpha_2} \cdots \delta_{\gamma_k,\alpha_{k+1}} \cdot e^{\beta F(\gamma_1,\dots,\gamma_{k+1})} \cdot \nu(\gamma_{k+1})\,. \tag{3.21}$$

Developing the expectation defining $Z_N(\beta)$ we get

$$\begin{aligned} Z_N(\beta) &= \sum_{\zeta_1,\dots,\zeta_N\in\Gamma} e^{\beta\sum_{i=1}^N F(\zeta_i,\zeta_{i\oplus_N 1},\dots,\zeta_{i\oplus_N k})}\cdot\nu(\zeta_1)\cdots\nu(\zeta_N) \\ &= \operatorname{Tr}\big[A_\beta^N\big] = \sum_{i=1}^{|\Gamma|^{2(k+1)}} e_i(\beta)^N\,, \end{aligned} \tag{3.22}$$

where $\{e_i(\beta),\ i=1,\dots,|\Gamma|^{2(k+1)}\}$ are the (possibly complexes) eigenvalues of the matrix $A_\beta$ (counted repeatedly according to their algebraic multiplicity). It's immediate to check that $A_\beta$ is an irreducible, aperiodic matrix, and since its entries are nonnegative we can apply Perron–Frobenius theory [**5**]: there exists a real positive simple eigenvalue, say $e_1(\beta)$, such that $|e_i(\beta)| < e_1(\beta)$ for every $i \geq 2$. To lighten the notation, from now on we will let $e(\beta) := e_1(\beta)$. Combining (3.20) with (3.22) we get

$$Z_N(\beta) = e(\beta)^N\cdot\left(1+\sum_{i=2}^{|\Gamma|^{2(k+1)}}\left(\frac{e_i(\beta)}{e(\beta)}\right)^N\right), \tag{3.23}$$

so that

$$Z_N(\beta)\cdot e(\beta)^{-N}\to 1 \quad \text{as } N\to\infty\,.$$

From this *sharp asymptotics* for $Z_N(\beta)$ we obtain in particular the explicit expression of $L^F(\beta)$ we were looking for:

$$L^F(\beta) = \log e(\beta)\,. \tag{3.24}$$

This equation shows that $L^F(\beta)$ is a *real analytic* function of $\beta\in\mathbb{R}$, since $e(\beta)$ is so: this is because the Perron–Frobenius eigenvalue is a simple root of the characteristic polynomial and the entries of $A_\beta$ are real–analytic functions of $\beta\in\mathbb{R}$.

From (3.20) it is clear that $\log Z_N(\beta)$ is a convex function of $\beta\in\mathbb{R}$, for every $N\in\mathbb{N}$. Moreover, we have $Z_N(\beta)\geq 1$ for every $\beta\in\mathbb{R}$ by Jensen's inequality, and trivially $Z_N(0)=1$. It follows immediately that $L^F(\beta)$ is a convex function too, being the pointwise limit of $\log Z_N(\beta)/N$, that $L^F(\beta)\geq 0$ for every $\beta\in\mathbb{R}$, and $L^F(0)=0$.

Let's assume that condition (3) in the statement of the theorem holds, that is $L^F(\beta_0)=0$ for some $\beta_0>0$ (the case $\beta_0<0$ is completely analogous): the preceding observations yield $L^F(\beta)=0$ for every $\beta\in[0,\beta_0]$, and by analyticity we conclude that indeed $L^F(\beta)=0$ for every $\beta\in\mathbb{R}$. We have thus shown that (3) $\Rightarrow$ (2).

Now we assume that condition (2) holds: by (3.24) this means $e(\beta) = 1$ for every $\beta \in \mathbb{R}$, and (3.23) we have that

$$|Z_N(\beta)| \leq e(\beta)^N \cdot |\Gamma|^{2(k+1)} = |\Gamma|^{2(k+1)} \qquad \forall N \in \mathbb{N}\,,\ \forall \beta \in \mathbb{R}\,. \tag{3.25}$$

Since $\log Z_N(\beta)$ is a convex function, $Z_N(\beta)$ is convex too; furthermore, we have already remarked that $Z_N(\beta) \geq 1$ for every $\beta \in \mathbb{R}$ and that $Z_N(0) = 1$. Since (3.25) shows that $|Z_N(\beta)|$ is bounded, by elementary convex analysis it follows that $Z_N$ must be constant, therefore $Z_N(\beta) = 1$ for all $\beta \in \mathbb{R}$ and $N \in \mathbb{N}$. This means that for every $\beta \in \mathbb{R}$ Jensen's inequality for $Z_N(\beta)$ it's not strict: since for any $\beta > 0$ the function $\{x \mapsto e^{\beta x}\}$ is a strictly convex function, this can happen if and only if $\widetilde{A}_N$ is $\mathbb{P}$–a.s. constant, for every $N \in \mathbb{N}$. Recalling (3.19) and the fact that by hypothesis $\nu(\alpha) > 0$ for every $\alpha \in \Gamma$, this amounts to saying that

$$\sum_{i=1}^{N} F(\eta_i, \eta_{i\oplus_N 1}, \ldots, \eta_{i\oplus_N k}) = 0\,,$$

for every $N \in \mathbb{N}$ and for every $\eta_1, \ldots, \eta_N \in \Gamma$: applying Lemma 3.6 we conclude that $F$ is a coboundary, and the proof is complete. □

CHAPTER 4

# A renewal theory approach to polymers with periodic distribution of charges

In this chapter we consider a general model of an heterogeneous polymer chain in the proximity of an interface between two selective solvents, which includes as special cases the copolymer near a selective interface and the pinning model introduced in Chapter 1a. The heterogeneous character of the model comes from the fact that the interaction of each *monomer unit* is governed by a *charge* that it carries. We consider the model in the *periodic setting*, that is when the charges repeat themselves along the chain in a periodic fashion. The main question is of course whether the polymer remains tightly close to the interface (*localization*) or there is a marked preference for one solvent (*delocalization*).

We propose an approach based on renewal theory that yields sharp estimates on the partition function of the model in all the regimes (localized, delocalized and critical). This in turn allows to get a very precise description of the polymer measure, both in a local sense (*thermodynamic limit*) and in a global sense (*scaling limits*): see § 1.3 for an outline of our results and § 1.5 for a detailed exposition. A key point, but also a byproduct, of our analysis is the closeness of the polymer measure to a suitable *Markov Renewal Process*.

The preprint [**18**] has been taken from the content of this chapter.

## 1. Introduction and main results

**1.1. Two motivating models.** We slightly enlarge our setting with respect to Chapter 1a, namely we work with a random walk $S := \{S_n\}_{n=0,1,\dots}$ with IID symmetric increments $\{X_j\}_{j\geq 1}$ taking values in $\{-1, 0, +1\}$. Hence the law of the walk is identified by $p := \mathbf{P}(X_1 = 1) (= \mathbf{P}(X_1 = -1))$, and we assume that $p \in (0, 1/2)$. Note that we have excluded the case $p = 1/2$ and this has been done in order to lighten the exposition: all the results we present have a close analog in the case $p = 1/2$, however the statements require a minimum of notational care because

of the periodicity of the walk. We also consider a sequence $\omega := \{\omega_n\}_{n\in\mathbb{N}=\{1,2,\ldots\}}$ of real numbers with the property that $\omega_n = \omega_{n+T}$ for some $T \in \mathbb{N}$ and for every $n$: we denote by $T(\omega)$ the minimal value of $T$.

Before defining the general model that will be the object of our analysis, we recall the two motivating models that were introduced in Chapter 1a.

(1) *Pinning and wetting models.* For $\lambda \geq 0$ consider the probability measure $\mathbf{P}_{N,\omega}$ defined by

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) \propto \exp\left(\lambda \sum_{n=1}^{N} \omega_n \mathbf{1}_{\{S_n=0\}}\right). \tag{4.1}$$

The walk receives a *pinning reward*, which may be negative or positive, each time it visits the origin. By considering the directed walk viewpoint, that is $\{(n, S_n)\}_n$, one may interpret this model in terms of a directed linear chain receiving an energetic contribution when it touches an interface. In this context it is natural to introduce the asymmetry parameter $h := \sum_{n=1}^{T} \omega_n/T$, so that one isolates a constant drift term from the *fluctuating* behavior of $\omega$. The question is whether for large $N$ the measure $\mathbf{P}_{N,\omega}$ is rather attracted or repelled by the interface (there is in principle the possibility for the walk to be essentially indifferent of such a change of measure, but we anticipate that this happens only in trivially degenerate cases while in *critical* situations a more subtle scenario shows up).

By multiplying the right–hand side of (4.1) by $\mathbf{1}_{\{S_n\geq 0:\, n=1,\ldots,N\}}$ one gets to a so called *wetting model*, that is the model of an interface interacting with an impenetrable wall. The *hard–wall* condition induces a repulsion effect of purely entropic origin which is in competition with attractive energy effects: one expects that in this case $h$ needs to be positive for the energy term to overcome the entropic repulsion effect, but quantitative estimates are not a priori obvious.

There is an extensive literature on periodic pinning and wetting models, the majority of which is restricted to the $T = 2$ case, we mention for example [**33, 54**].

(2) *Copolymer near a selective interface.* Much in the same way we introduce

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) \propto \exp\left(\lambda \sum_{n=1}^{N} \omega_n \,\mathrm{sign}\,(S_n)\right), \tag{4.2}$$

where if $S_n = 0$ we set $\mathrm{sign}(S_n) := \mathrm{sign}(S_{n-1})\,\mathbf{1}_{\{S_{n-1}\neq 0\}}$. This convention for defining sign(0), that will be kept throughout the chapter, has the following simple interpretation: $\mathrm{sign}(S_n) = +1, 0, -1$ according to whether the bond joining $S_{n-1}$ and $S_n$ lies above, on, or below the $x$–axis.

Also in this case we take a directed walk viewpoint and then $\mathbf{P}_{N,\omega}$ may be interpreted as a polymeric chain in which the monomer units, the bonds of the walk, are charged. An interface, the $x$–axis, separates two solvents, say oil above and water below: positively charged monomers are hydrophobic and negatively charged ones are instead hydrophilic. In this case one expects a competition between three possible scenarios: polymer preferring water, preferring oil or undecided between the two and choosing to fluctuate in the proximity of the interface. We will therefore talk of delocalization in water (or oil) or of localization at the interface. Critical cases are of course of particular interest.

We select [**51, 63**] from the physical literature on periodic copolymers, keeping however in mind that periodic copolymer modeling has a central role in applied chemistry and material science.

**1.2. A general model.** We point out that the models presented in § 1.1 are particular examples of the polymer measure with Hamiltonian

$$\mathcal{H}_N(S) = \sum_{i=\pm 1}\sum_{n=1}^{N}\omega_n^{(i)}\mathbf{1}_{\{\mathrm{sign}(S_n)=i\}} + \sum_{n=1}^{N}\omega_n^{(0)}\mathbf{1}_{\{S_n=0\}} + \sum_{n=1}^{N}\widetilde{\omega}_n^{(0)}\mathbf{1}_{\{\mathrm{sign}(S_n)=0\}}, \quad (4.3)$$

where $\omega^{(\pm 1)}$, $\omega^{(0)}$ and $\widetilde{\omega}^{(0)}$ are periodic sequences of real numbers. Observe that, by our conventions on sign(0), the last term gives an energetic contribution (of pinning/depinning type) to the bonds lying on the interface.

Besides being a natural model, generalizing and interpolating between pinning and copolymer models, the general model we consider is the one considered at several instances, see e.g. [**65**] and references therein.

Remark 4.1. The *copolymer* case corresponds to $\omega^{(+1)} = -\omega^{(-1)} = \lambda\,\omega$ and $\omega^{(0)} = \widetilde{\omega}^{(0)} = 0$, while the *pinning* case corresponds to $\omega^{(0)} = \lambda\,\omega$ and $\omega^{(+1)} = \omega^{(-1)} = \widetilde{\omega}^{(0)} = 0$. We stress that the *wetting* case can be included too, with the choice $\omega^{(0)} = \lambda\omega$, $\omega_n^{(-1)} = -\infty$ for every $n$ and $\omega^{(+1)} = \widetilde{\omega}^{(0)} = 0$. Of course plugging $\omega_n^{(-1)} = -\infty$ into the Hamiltonian (4.3) is a bit formal, but it simply corresponds to

a constraint on $S$ in the polymer measure associated to $\mathcal{H}_N$, see (4.4) below. For ease of exposition we will restrict to finite values of the charges $\omega$, but the generalization is straightforward.

REMARK 4.2. We take this occasion for stressing that, from an applied viewpoint, the interest in periodic models of the type we consider appears to be at least two–fold. On one hand periodic models are often chosen as caricatures of the *quenched disordered* models, like the ones in which the charges are a typical realization of a sequence of independent random variables (e.g. [**2, 12, 35, 65**] and references therein). In this respect and taking a mathematical standpoint, the relevance of periodic models, which may be viewed as *weakly inhomogeneous*, for understanding the strongly inhomogeneous quenched set–up is at least questionable and the approximation of quenched models with periodic ones, in the limit of large period, poses very interesting and challenging questions. In any case, the precise description of the periodic case that we have obtained in this work highlights limitations and perspectives of periodic modeling for strongly inhomogeneous systems. One the other hand, as already mentioned above, periodic models are absolutely natural and of direct relevance for application, for example when dealing with *molecularly engineered* polymers [**53, 63**].

Starting from the Hamiltonian (4.3), for $a = \mathrm{c}$ (*constrained*) or $a = \mathrm{f}$ (*free*) we introduce the *polymer measure* $\mathbf{P}^a_{N,\omega}$ on $\mathbb{Z}^{\mathbb{N}}$, defined by

$$\frac{\mathrm{d}\mathbf{P}^a_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) \;=\; \frac{\exp\left(\mathcal{H}_N(S)\right)}{\widetilde{Z}^a_{N,\omega}} \left(\mathbf{1}_{\{a=\mathrm{f}\}} + \mathbf{1}_{\{a=\mathrm{c}\}}\mathbf{1}_{\{S_N=0\}}\right), \tag{4.4}$$

where $\widetilde{Z}^a_{N,\omega} := \mathbf{E}[\exp(\mathcal{H}_N)\,(\mathbf{1}_{\{a=\mathrm{f}\}} + \mathbf{1}_{\{a=\mathrm{c}\}}\mathbf{1}_{\{S_N=0\}})]$ is the *partition function*, that is the normalization constant. Here $\omega$ is a shorthand for the four periodic sequences appearing in the definition (4.3) of $\mathcal{H}_N$, and we will use $T = T(\omega)$ to denote the smallest common period of the sequences.

The Laplace asymptotic behavior of $\widetilde{Z}_{N,\omega}$ plays an important role and the quantity

$$f_\omega \;:=\; \lim_{N\to\infty} \frac{1}{N} \log \widetilde{Z}^{\mathrm{c}}_{N,\omega}, \tag{4.5}$$

is usually called *free energy*. The existence of the limit above follows from a direct super–additivity argument, and it is easy to check that $\widetilde{Z}^{\mathrm{c}}_{N,\omega}$ can be replaced by $\widetilde{Z}^{\mathrm{f}}_{N,\omega}$

without changing the value of $f_\omega$, see e.g. [**35**]. The standard free energy approach to this type of models starts from the observation that

$$\begin{aligned} f_\omega \;&\geq\; \lim_{N\to\infty} \frac{1}{N} \log \mathbf{E}\Big[ \exp\big(\mathcal{H}_N(S)\big)\,;\, S_n > 0 \text{ for } n = 1,\dots,N \Big] \\ &=\; \frac{1}{T(\omega)} \sum_{n=1}^{T(\omega)} \omega_n^{(+1)} \;+\; \lim_{N\to\infty} \frac{1}{N} \log \mathbf{P}\big(S_n > 0 \text{ for } n = 1,\dots,N\big)\,. \end{aligned} \tag{4.6}$$

It is a classical result [**29**, Ch. XII.7] that $\mathbf{P}(S_n > 0 \text{ for } n = 1,\dots,N) \sim cN^{-1/2}$, as $N \to \infty$, for some $c \in (0,\infty)$ (by $a_N \sim b_N$ we mean $a_N/b_N \to 1$). Hence the limit of the last term of (4.6) is zero and one easily concludes that

$$f_\omega \;\geq\; f_\omega^{\mathcal{D}} := \max_{i=\pm 1} h_\omega(i), \qquad h_\omega(i) := \frac{1}{T(\omega)} \sum_{n=1}^{T(\omega)} \omega_n^{(i)}. \tag{4.7}$$

Having in mind the steps in (4.6), one is led to the following basic

Definition 4.3. *The polymer chain defined by* (4.4) *is said to be:*

- *localized (at the interface) if* $f_\omega > f_\omega^{\mathcal{D}}$*;*
- *delocalized above the interface if* $f_\omega = h_\omega(+1)$*;*
- *delocalized below the interface if* $f_\omega = h_\omega(-1)$*.*

Notice that, with this definition, if $h_\omega(+1) = h_\omega(-1)$ and the polymer is delocalized, it is delocalized both above and below the interface.

Remark 4.4. Observe that the polymer measure $\mathbf{P}_{N,\omega}^a$ is invariant under the joint transformation $S \to -S$, $\omega^{(+1)} \to \omega^{(-1)}$, hence by symmetry we may (and will) assume that

$$h_\omega := h_\omega(+1) - h_\omega(-1) \geq 0\,. \tag{4.8}$$

It is also clear that we can add to the Hamiltonian $\mathcal{H}_N$ a constant term (with respect to $S$) without changing the polymer measure. Then we set

$$\mathcal{H}'_N(S) := \mathcal{H}_N(S) - \sum_{n=1}^{N} \omega_n^{(+1)}\,,$$

which amounts to redefining $\omega_n^{(+1)} \to 0$, $\omega_n^{(-1)} \to (\omega_n^{(-1)} - \omega_n^{(+1)})$ and $\widetilde{\omega}_n^{(0)} \to (\widetilde{\omega}_n^{(0)} - \omega_n^{(+1)})$, and we can write

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}^a}{\mathrm{d}\mathbf{P}}(S) \;=\; \frac{\exp\big(\mathcal{H}'_N(S)\big)}{Z_{N,\omega}^a} \left(\mathbf{1}_{\{a=\mathrm{f}\}} + \mathbf{1}_{\{a=\mathrm{c}\}}\mathbf{1}_{\{S_N=0\}}\right), \tag{4.9}$$

where $Z^a_{N,\omega}$ is a new partition function which coincides with $\widetilde{Z}^a_{N,\omega}\exp(-\sum_{n=1}^N \omega_n^{(+1)})$. The corresponding free energy $\mathrm{F}_\omega$ is given by

$$\mathrm{F}_\omega \;:=\; \lim_{N\to\infty}\frac{1}{N}\log Z^a_{N,\omega} \;=\; f_\omega - f^{\mathcal{D}}_\omega\,, \tag{4.10}$$

and notice that in terms of $\mathrm{F}_\omega$ the condition for localization (resp. delocalization) becomes $\mathrm{F}_\omega > 0$ (resp. $\mathrm{F}_\omega = 0$). From now on, speaking of partition function and free energy we will always mean $Z^a_{N,\omega}$ and $\mathrm{F}_\omega$.

**1.3. From free energy to path behavior.** In order to understand the spirit of our approach, let us briefly outline our results (complete results are given in § 1.5 below).

Our first goal is to give necessary and sufficient *explicit conditions* in terms of the charges $\omega$ for the (de)localization of the polymer chain, see Theorem 4.5. We point out that the content of this theorem is in fact much richer, as it gives the *sharp asymptotic behavior* (and not only the Laplace one [**11**]) as $N\to\infty$ of the constrained partition function $Z^{\mathrm{c}}_{N,\omega}$. In particular we show that when the polymer is delocalized ($\mathrm{F}_\omega = 0$) the constrained partition function $Z^{\mathrm{c}}_{N,\omega}$ is actually vanishing as $N\to\infty$. Moreover the rate of the decay induces a further distinction in the delocalized regime between a *strictly delocalized regime* ($Z^{\mathrm{c}}_{N,\omega}\sim c_1 N^{-3/2}$, $c_1\in(0,\infty)$) and a *critical regime* ($Z^{\mathrm{c}}_{N,\omega}\sim c_2 N^{-1/2}$, $c_2\in(0,\infty)$).

These asymptotic results are important because they allow to address further interesting issues. For example, it has to be admitted that defining (de)localization in terms of the free energy is not completely satisfactory, because one would like to characterize the polymer path properties. In different terms, given a polymer measure which is (de)localized according to Definition 4.3, to what extent are its typical paths really (de)localized? Some partial answers to this question are known, at least in some particular instances: we mention here the case of $T(\omega)=2$ copolymers [**51**] and the case of homogeneous pinning and wetting models [**22, 40, 70**].

Our main aim is to show that, for the whole class of models we are considering, free energy (de)localization does correspond to a *strong form* of path (de)localization. More precisely, we look at path behavior from two different viewpoints.

- *Thermodynamic limit.* We show that the measure $\mathbf{P}^a_{N,\omega}$ converges weakly as $N\to\infty$ toward a measure $\mathbf{P}_\omega$ on $\mathbb{Z}^{\mathbb{N}}$, of which we give an explicit construction, see Section 3. It turns out that the properties of $\mathbf{P}_\omega$ are radically

different in the three regimes (localized, strictly delocalized and critical), see Theorem 4.7. It is natural to look at these results as those characterizing the *local* structure of the polymer chain.

- *Brownian scaling limits.* We prove that the diffusive rescaling of the polymer measure $\mathbf{P}^a_{N,\omega}$ converges weakly in $C([0,1])$ as $N \to \infty$. Again the properties of the limit process, explicitly described in Theorem 4.8, differ considerably in the three regimes. Moreover we stress that scaling limits describe *global* properties of the chain.

We insist on the fact that the path analysis just outlined has been obtained exploiting heavily the sharp asymptotic behavior of $Z^{\mathrm{c}}_{N,\omega}$ as $N \to \infty$. In this sense our results are the direct sharpening of the Large Deviations approach taken in [**11**], where a formula for $\mathrm{F}_\omega$ was obtained for periodic copolymers (but the method of course directly extends to the general case considered here). Such a formula (see § 2.3), that reduces the problem of computing the free energy to a finite dimensional problem connected to a suitable Perron–Frobenius matrix, in itself suggests the new approach taken here since it makes rather apparent the link between periodic copolymers and the class of *Markov renewal processes* [**5**]. On the other hand, with respect to [**11**], we leave aside any issue concerning the phase diagram (except for § 1.6 below).

**1.4. The order parameter $\delta^\omega$.** It is a remarkable fact that the dependence of our results on the charges $\omega$ is essentially encoded in one single parameter $\delta^\omega$, that can be regarded as the *order parameter* of our models. For the definition of this parameter, we need some preliminary notation. We start with the law of the first return to zero of the original walk:

$$\tau_1 := \inf\{n > 0 : \ S_n = 0\} \qquad K(n) \ := \ \mathbf{P}\big(\tau_1 = n\big)\,. \tag{4.11}$$

It is a classical result [**29**, Ch. XII.7] that

$$\exists \lim_{n\to\infty} n^{3/2}\, K(n) \ =: \ c_K \in (0,\infty)\,. \tag{4.12}$$

Then we introduce the Abelian group $\mathbb{S} := \mathbb{Z}/(T\mathbb{Z})$ and to indicate that an integer $n$ is in the equivalence class $\beta \in \mathbb{S}$ we write equivalently $[n] = \beta$ or $n \in \beta$. Notice that the charges $\omega_n$ are functions of $[n]$, and with some abuse of notation we can write $\omega_{[n]} := \omega_n$. The key observation is that, by the $T$–periodicity of the charges $\omega$

and by the definition (4.8) of $h_\omega$, we can write

$$\sum_{n=n_1+1}^{n_2} (\omega_n^{(-1)} - \omega_n^{(+1)}) \;=\; -(n_2 - n_1)\, h_\omega \;+\; \Sigma_{[n_1],[n_2]}\,.$$

Thus we have decomposed the above sum into a drift term and a more fluctuating term, where the latter has the remarkable property of depending on $n_1$ and $n_2$ only through their equivalence classes $[n_1]$ and $[n_2]$. Now we can define three basic objects:

- for $\alpha, \beta \in \mathbb{S}$ and $\ell \in \mathbb{N}$ we set

$$\Phi^\omega_{\alpha,\beta}(\ell) \;:=\; \begin{cases} \omega_\beta^{(0)} \;+\; \left(\widetilde{\omega}_\beta^{(0)} - \omega_\beta^{(+1)}\right) & \text{if } \ell = 1,\ \ell \in \beta - \alpha \\ \omega_\beta^{(0)} + \log\left[\dfrac{1}{2}\Big(1 + \exp\big(-\ell\, h_\omega + \Sigma_{\alpha,\beta}\big)\Big)\right] & \text{if } \ell > 1,\ \ell \in \beta - \alpha \\ 0 & \text{otherwise} \end{cases}, \tag{4.13}$$

  which is a sort of integrated version of our Hamiltonian;

- for $x \in \mathbb{N}$ we introduce the $\mathbb{S} \times \mathbb{S}$ matrix $M^\omega_{\alpha,\beta}(x)$ defined by

$$M^\omega_{\alpha,\beta}(x) \;:=\; e^{\Phi^\omega_{\alpha,\beta}(x)}\, K(x)\, \mathbf{1}_{(x \in \beta - \alpha)}\,; \tag{4.14}$$

- summing the entries of $M^\omega$ over $x$ we get a $\mathbb{S} \times \mathbb{S}$ matrix that we call $B^\omega$:

$$B^\omega_{\alpha,\beta} := \sum_{x \in \mathbb{N}} M^\omega_{\alpha,\beta}(x)\,. \tag{4.15}$$

The meaning and motivation of these definitions, that at this point might appear artificial, are explained in detail in § 2.2. For the moment we only stress that the above quantities are *explicit functions* of the charges $\omega$ and of the law of the underlying random walk (to lighten the notation, the $\omega$–dependence of these quantities will be often dropped in the following).

We can now define our order parameter $\delta^\omega$. Observe that $B_{\alpha,\beta}$ is a finite dimensional matrix with nonnegative entries, hence the Perron–Frobenius (P–F) Theorem (see e.g. [**5**]) entails that $B_{\alpha,\beta}$ has a unique real positive eigenvalue, called the Perron–Frobenius eigenvalue, with the property that it is a simple root of the characteristic polynomial and that it coincides with the spectral radius of the matrix. This is exactly our parameter:

$$\delta^\omega \;:=\; \text{Perron–Frobenius eigenvalue of } B^\omega\,. \tag{4.16}$$

**1.5. The main results.** Now we are ready to state our results. We start characterizing the (de)localization of the polymer chain in terms of $\delta^\omega$.

THEOREM 4.5 (**Sharp asymptotics**). *The polymer chain is localized if and only if $\delta^\omega > 1$. More precisely, the asymptotic behavior of $Z^{\mathrm{c}}_{N,\omega}$ as $N \to \infty$, $[N] = \eta$ is given by*

(1) *for $\delta^\omega > 1$ (localized regime)* $Z^{\mathrm{c}}_{N,\omega} \sim C^{>}_{\omega,\eta} \exp\left(\mathrm{F}_\omega N\right)$ ;

(2) *for $\delta^\omega < 1$ (strictly delocalized regime)* $Z^{\mathrm{c}}_{N,\omega} \sim C^{<}_{\omega,\eta} / N^{3/2}$ ;

(3) *for $\delta^\omega = 1$ (critical regime)* $Z^{\mathrm{c}}_{N,\omega} \sim C^{=}_{\omega,\eta} / \sqrt{N}$ ,

*where $\mathrm{F}_\omega > 0$ is the free energy and its explicit definition in terms of $\omega$ is given in § 2.3, while $C^{>}_{\omega,\eta}$, $C^{<}_{\omega,\eta}$ and $C^{=}_{\omega,\eta}$ are explicit positive constants, depending on $\omega$ and $\eta$, whose value is given in Section 2.*

REMARK 4.6. Theorem 4.5 is the building block of all the path analysis that follows. It is therefore important to stress that, in the quenched disordered case, cf. Remark 4.2, such a strong statement in general does not hold, see [**36**, Section 4].

Next we investigate the thermodynamic limit, that is the weak limit as $N \to \infty$ of the sequence of measures $\mathbf{P}^a_{N,\omega}$ on $\mathbb{Z}^{\mathbb{N}}$ (endowed with the standard product topology). The next theorem provides a first connection between free energy (de)localization and the corresponding path properties.

Before stating the result, we need a notation: we denote by $\mathcal{P}$ the set of $\omega$ such that:

$$\mathcal{P} := \{\omega : \delta^\omega \le 1, \quad h_\omega = 0, \quad \exists\, \alpha, \beta : \Sigma_{\alpha,\beta} \neq 0\}, \tag{4.17}$$
$$\mathcal{P}^< := \mathcal{P} \cap \{\delta^\omega < 1\}, \qquad \mathcal{P}^= := \mathcal{P} \cap \{\delta^\omega = 1\}.$$

Here $\mathcal{P}$ stands for *problematic*, or *pathologic*. Indeed, we shall see that for $\omega \in \mathcal{P}$ the results are weaker and more involved than for $\omega \notin \mathcal{P}$. We stress however that these restrictions do not concern localized regime, because $\mathcal{P} \subset \{\omega : \delta^\omega \le 1\}$. We also notice that for the two motivating models of § 1.1, the pinning and the copolymer models, $\omega$ *never* belongs to $\mathcal{P}$. This is clear for the pinning case, where by definition $\Sigma \equiv 0$. On the other hand, in the copolymer case it is known that if $h_\omega = 0$ and $\exists\, \alpha, \beta : \Sigma_{\alpha,\beta} \neq 0$ then $\delta^\omega > 1$: see § 5.4 or [**11**]. In reality the *pathological* aspects observed for $\omega \in \mathcal{P}$ may be understood in statistical mechanics terms and we sketch

an interpretation in § 1.6 below: this goes rather far from the point of view adopted here, since it is an issue tightly entangled with the analysis of the free energy. It will therefore be taken up in a further work.

THEOREM 4.7 (**Thermodynamic limit**). *If $\omega \notin \mathcal{P}^<$, then both the polymer measures $\mathbf{P}^{\mathrm{f}}_{N,\omega}$ and $\mathbf{P}^{\mathrm{c}}_{N,\omega}$ converge as $N \to \infty$ to the same limit $\mathbf{P}_\omega$, law of an irreducible Markov process on $\mathbb{Z}$ which is:*

(1) *positive recurrent if $\delta^\omega > 1$ (localized regime);*

(2) *transient if $\delta^\omega < 1$ (strictly delocalized regime);*

(3) *null recurrent if $\delta^\omega = 1$ (critical regime).*

*If $\omega \in \mathcal{P}^<$ (in particular $\delta^\omega < 1$), for all $\eta \in \mathbb{S}$ and $a = \mathrm{f}, \mathrm{c}$ the measure $\mathbf{P}^a_{N,\omega}$ converges as $N \to \infty$, $[N] = \eta$ to $\mathbf{P}^{a,\eta}_\omega$, law of an irreducible transient Markov chain on $\mathbb{Z}$.*

We stress that in all regimes the limit law $\mathbf{P}_\omega$ or $\mathbf{P}^{a,\eta}_\omega$ has an explicit construction in terms of $M^\omega_{\alpha,\beta}(x)$, see Section 3 for details.

We finally turn to the analysis of the *diffusive rescaling* of the polymer measure $\mathbf{P}^a_{N,\omega}$. More precisely, let us define the map $X^N : \mathbb{R}^N \mapsto C([0,1])$:

$$X^N_t(x) \;=\; \frac{x_{\lfloor Nt \rfloor}}{\sigma N^{1/2}} + (Nt - \lfloor Nt \rfloor)\, \frac{x_{\lfloor Nt \rfloor + 1} - x_{\lfloor Nt \rfloor}}{\sigma N^{1/2}}, \qquad t \in [0,1],$$

where $\lfloor \cdot \rfloor$ denotes the integer part and $\sigma^2 := 2p$ is the variance of $X_1$ under the original random walk measure $\mathbf{P}$. Notice that $X^N_t(x)$ is nothing but the linear interpolation of $\{x_{\lfloor Nt \rfloor}/(\sigma\sqrt{N})\}_{t \in \frac{\mathbb{N}}{N} \cap [0,1]}$. For $a = \mathrm{f}, \mathrm{c}$ we set:

$$Q^a_{N,\omega} \;:=\; \mathbf{P}^a_{N,\omega} \circ (X^N)^{-1},$$

Then $Q^a_{N,\omega}$ is a measure on $C([0,1])$, the space of real continuous functions defined on the interval $[0,1]$, and we want to study the behavior as $N \to \infty$ of this sequence of measures.

We start fixing a notation for the following standard processes:

- the Brownian motion $\{B_\tau\}_{\tau \in [0,1]}$;
- the Brownian bridge $\{\beta_\tau\}_{\tau \in [0,1]}$ between 0 and 0;

- the Brownian motion *conditioned to stay non-negative on* $[0,1]$ or, more precisely, the Brownian meander $\{m_\tau\}_{\tau\in[0,1]}$, see [**60**];
- the Brownian bridge *conditioned to stay non-negative on* $[0,1]$ or, more precisely, the normalized Brownian excursion $\{e_\tau\}_{\tau\in[0,1]}$, also known as the Bessel bridge of dimension 3 between 0 and 0, see [**60**] .

Then we introduce a modification of the above processes labeled by a parameter $p \in [0,1]$:

- the process $\{B^{(p)}_\tau\}_{\tau\in[0,1]}$ is the so–called *skew Brownian motion of parameter* $p$, cf. [**60**]. More explicitly, $B^{(p)}$ is a process such that $|B^{(p)}| = |B|$ in distribution, but in which the sign of each excursion is chosen to be $+1$ (resp. $-1$) with probability $p$ (resp. $1-p$) instead of $1/2$. In the same way, the process $\{\beta^{(p)}_\tau\}_{\tau\in[0,1]}$ is the skew Brownian bridge of parameter $p$. Notice that for $p=1$ we have $B^{(1)} = |B|$ and $\beta^{(1)} = |\beta|$ in distribution.
- the process $\{m^{(p)}_\tau\}_{\tau\in[0,1]}$ is defined by

$$\mathbb{P}(m^{(p)} \in dw) \,:=\, p\,\mathbb{P}(m \in dw) \,+\, (1-p)\,\mathbb{P}(-m \in dw),$$

  i.e. $m^{(p)} = \sigma m$, where $\mathbb{P}(\sigma = 1) = 1 - \mathbb{P}(\sigma = -1) = p$ and $(m,\sigma)$ are independent. The process $\{e^{(p)}_\tau\}_{\tau\in[0,1]}$ is defined in exactly the same manner. For $p=1$ we have $m^{(1)} = m$ and $e^{(1)} = e$.

Finally, we introduce a last process, labeled by two parameters $p, q \in [0,1]$:

- consider a r.v. $U \mapsto [0,1]$ with the arcsin law: $\mathbb{P}(U \le t) = \frac{2}{\pi}\arcsin\sqrt{t}$, and processes $\beta^{(p)}$, $m^{(q)}$ as defined above, with $(U, \beta^{(p)}, m^{(q)})$ independent triple. Then we denote by $\{B^{(p,q)}_\tau\}_{\tau\in[0,1]}$ the process defined by:

$$B^{(p,q)}_\tau \,:=\, \begin{cases} \sqrt{U}\,\beta^{(p)}_{\frac{\tau}{U}} & \text{if } \tau \le U \\ \sqrt{1-U}\,m^{(q)}_{\frac{\tau-U}{1-U}} & \text{if } \tau > U \end{cases}.$$

  Notice that the process $B^{(p,q)}$ differs from the $p$–skew Brownian motion $B^{(p)}$ only for the last excursion in $[0,1]$, whose sign is $+1$ with probability $q$ instead of $p$.

We are going to show that the sequence $\{Q^a_{N,\omega}\}$ has a weak limit as $N \to \infty$ (with a weaker statement if $\omega \in \mathcal{P}$). Again the properties of the limit process differ considerably in the three regimes $\delta^\omega > 1$, $\delta^\omega < 1$ and $\delta^\omega = 1$. However for the precise

description of the limit processes, for the regimes $\delta^\omega = 1$ and $\delta^\omega < 1$ we need to distinguish between $a \in \{\mathrm{f}, \mathrm{c}\}$ and to introduce further parameters $\mathtt{p}_\omega, \mathtt{q}_\omega$, defined as follows:

- case $\delta^\omega = 1$ :
  - $\mathtt{p}_\omega := \mathtt{p}^{=}_\omega$, defined in (4.82). We point out two special cases: if $h_\omega > 0$ then $\mathtt{p}^{=}_\omega = \mathtt{1}$, while if $h_\omega = 0$ and $\Sigma \equiv 0$ then $\mathtt{p}^{=}_\omega = \mathtt{1/2}$;
  - for each $\eta \in \mathbb{S}$, $\mathtt{q}_\omega := \mathtt{q}^{=}_{\omega,\eta}$, defined by (4.84).
- case $\delta^\omega < 1$ :
  - $\omega \notin \mathcal{P}^{<}$: if $h_\omega > 0$ we set $\mathtt{p}_\omega := \mathtt{p}^{<}_\omega := \mathtt{1}$ while if $h_\omega = 0$ we set $\mathtt{p}_\omega := \mathtt{p}^{<}_\omega := \mathtt{1/2}$;
  - $\omega \in \mathcal{P}^{<}$: for each $\eta \in \mathbb{S}$ and $a = \mathrm{f}, \mathrm{c}$, $\mathtt{p}_\omega := \mathtt{p}^{<,a}_{\omega,\eta}$ is defined in (4.71) and (4.73).

THEOREM 4.8 (**Scaling limits**). *If $\omega \notin \mathcal{P}$, then the sequence of measures $\{Q^a_{N,\omega}\}$ on $C([0,1])$ converges weakly as $N \to \infty$. More precisely:*

(1) *for $\delta^\omega > 1$ (localized regime) $Q^a_{N,\omega}$ converges to the measure concentrated on the constant function taking the value zero ;*

(2) *for $\delta^\omega < 1$ (strictly delocalized regime):*
   - *$Q^{\mathrm{f}}_{N,\omega}$ converges to the law of $m^{(\mathrm{p}^{<}_\omega)}$ ;*
   - *$Q^{\mathrm{c}}_{N,\omega}$ converges to the law of $e^{(\mathrm{p}^{<}_\omega)}$ ;*

(3) *for $\delta^\omega = 1$ (critical regime):*
   - *$Q^{\mathrm{f}}_{N,\omega}$ converges to the law of $B^{(\mathrm{p}^{=}_\omega)}$ ;*
   - *$Q^{\mathrm{c}}_{N,\omega}$ converges to the law of $\beta^{(\mathrm{p}^{=}_\omega)}$ .*

*If $\omega \in \mathcal{P}$, then for all $\eta \in \mathbb{S}$ the measures $Q^{\mathrm{c}}_{N,\omega}$ and $Q^{\mathrm{f}}_{N,\omega}$ converge as $N \to \infty$, $[N] = \eta$ to, respectively:*

(1) *for $\delta^\omega < 1$, the law of $e^{(\mathrm{p}^{<,\mathrm{c}}_{\omega,\eta})}$ and $m^{(\mathrm{p}^{<,\mathrm{f}}_{\omega,\eta})}$.*

(2) *for $\delta^\omega = 1$, the law of $\beta^{(\mathrm{p}^{=}_\omega)}$ and $B^{(\mathrm{p}^{=}_\omega, \mathrm{q}^{=}_{\omega,\eta})}$.*

Results on thermodynamic limits in the direction of Theorem 4.7 have been obtained in the physical literature by exact computations either for homogeneous polymers or for $T = 2$ pinning models and copolymers, see e.g. [**51**], while Brownian scaling limits have been heuristically derived at several instances, see e.g. [**70**]. Rigorous results corresponding to our three main theorems have been obtained for

homogeneous pinning/wetting models in [**22, 40**]. We would like to stress the very much richer variety of limit processes that we have obtained in our general context.

**1.6. About the regime $\mathcal{P}$.** We have seen, cf. Theorem 4.7, that if $\omega \in \mathcal{P}^<$ the infinite volume limit (in particular the probability that the walk escapes either to $+\infty$ or to $-\infty$) depends on $a = \mathrm{c}$ or f and on the subsequence $[N] = \eta \in \mathbb{S}$. This reflects directly into Theorem 4.8 and in this case also the $\mathcal{P}^=$ regime is affected, but only for $a = \mathrm{f}$ and the change is restricted to the sign of the very last excursion of the process. It is helpful to keep in mind that $\omega \in \mathcal{P}$ if and only if there is a non trivial unbiased copolymer part, that is $h_\omega = 0$ but the matrix $\Sigma$ is non trivial, and at the same time the polymer is delocalized. It is known (§ 5.4 and [**11**]) that in absence of pinning terms, that is $\omega_n^{(0)} = \widetilde{\omega}_n^{(0)} = 0$ for every $n$, the polymer is localized. However if the pinning rewards are sufficiently large and negative, one easily sees that (de)pinning takes over and the polymer delocalizes. This is the phenomenon that characterizes the regime $\mathcal{P}$ and its lack of uniqueness of limit measures.

Lack of uniqueness of infinite volume measures and dependence on boundary conditions do not come as a surprise if one takes a statistical mechanics viewpoint and if one notices that the system undergoes a *first order* phase transition exactly at $\mathcal{P}$. In order to be more precise let us consider the particular case of

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) \propto \exp\left(\sum_{n=1}^{N}(\omega_n + h)\,\mathrm{sign}\,(S_n) - \beta\sum_{n=1}^{N}\mathbf{1}_{\{S_n=0\}}\right), \tag{4.18}$$

with $h$ and $\beta$ two real parameters and $\omega$ a fixed non trivial centered ($\sum_{n=1}^{T}\omega_n = 0$) periodic configuration of charges. The phase diagram of such a model is sketched in Figure 4.1. In particular it is easy to show that for $h = 0$ and for $\beta$ large and positive the polymer is delocalized and, recalling that for $\beta = 0$ the polymer is localized, by monotonicity of the free energy in $\beta$ one immediately infers that there exists $\beta_c > 0$ such that localization prevails for $\beta < \beta_c$, while the polymer is delocalized (both above and below the interface) if $\beta \geq \beta_c$. However the two regimes of delocalization above or below the interface, appearing for example as soon as $h$ is either positive or negative and $\beta \geq \beta_c$, are characterized by opposite values ($\pm 1$) of $\varrho = \varrho(h,\beta) := \lim_{N\to\infty}\mathbf{E}_{N,\omega}\left[N^{-1}\sum_{n=1}^{N}\mathrm{sign}\,(S_n)\right]$ and of course $\varrho$ is the derivative of the free energy with respect to $h$. Therefore the free energy is not differentiable at

$h = 0$ and we say that there is a *first order phase transition*. First order phase transitions are usually associated to multiple infinite volume limits (*phase coexistence*). A detailed analysis of this interesting phenomenon will be given elsewhere.

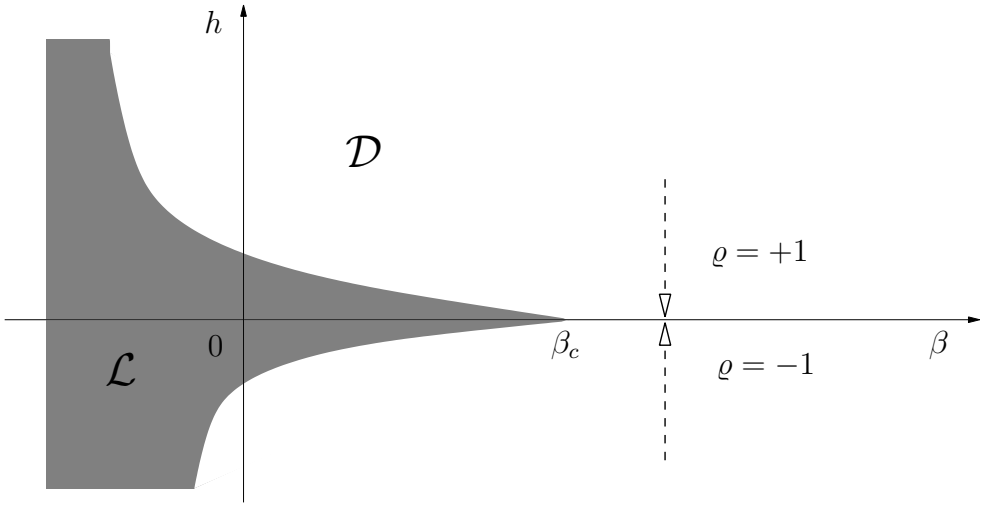


FIGURE 4.1. A sketch of the phase diagram for the model (4.18). In this case, with abuse of notation, $\mathcal{P} = \{(h, \beta) : h = 0, \beta \geq \beta_c\}$. Approaching $\mathcal{P}$ in the sense of the dashed arrowed lines one observes the two sharply different behaviors of paths completely delocalized above ($\varrho = +1$) or below ($\varrho = -1$) the interface.

**1.7. Outline of the exposition.** In Section 2 we study the asymptotic behavior of $Z^{\mathrm{c}}_{N,\omega}$, proving Theorem 4.5. In Section 3 we compute the thermodynamic limits of $\mathbf{P}^{a}_{N,\omega}$, proving Theorem 4.7. In Section 4 we compute the scaling limits of $\mathbf{P}^{a}_{N,\omega}$, proving Theorem 4.8. Finally, in Section 5 we give the proof of some technical results and some additional material.

## 2. Sharp asymptotic behavior of the partition function

In this section we are going to derive the precise asymptotic behavior of $Z^{\mathrm{c}}_{N,\omega}$, in particular proving Theorem 4.5. The key observation is that the study of the partition function for the models we are considering can be set into the framework of the theory of *Markov renewal processes*, see [**5**, Ch. VII.4]. We start recalling the basic notions of this theory and setting the relative notation.

**2.1. Markov Renewal Theory.** Given a finite set $\mathbb{S}$ (for us it will always be $\mathbb{Z}/(T\mathbb{Z})$), by a *kernel* we mean a family of nonnegative $\mathbb{S}\times\mathbb{S}$ matrices $F_{\alpha,\beta}(x)$ depending on a parameter $x \in \mathbb{N}$. We say that the kernel $F_{\alpha,\beta}(x)$ is *semi–Markov* if $F_{\alpha,\cdot}(\cdot)$ is a probability mass function on $\mathbb{S} \times \mathbb{N}$ for every $\alpha \in \mathbb{S}$, that is if $\sum_{\beta,x} F_{\alpha,\beta}(x) = 1$.

A semi–Markov kernel $F_{\alpha,\beta}(x)$ has a simple probabilistic interpretation: it defines a Markov chain $\{(J_k, T_k)\}$ on $\mathbb{S} \times \mathbb{N}$ through the transition kernel given by

$$\mathbb{P}\big[(J_{k+1}, T_{k+1}) = (\beta, x) \,\big|\, (J_k, T_k) = (\alpha, y)\big] = F_{\alpha,\beta}(x)\,. \tag{4.19}$$

In this case we say that the process $\{J_k, T_k\}$ is a (discrete) *Markov–renewal process*, the $\{T_k\}$ being thought of as interarrival times. This provides a generalization of classical renewal processes, since the $\{T_k\}$ are no longer IID but their laws are rather *modulated* by the process $\{J_k\}$. Since the r.h.s. of (4.19) does not depend on $y$, it follows that $\{J_k\}$ is a Markov chain, and it is called the *modulating chain* of the Markov renewal process (observe that in general the process $\{T_k\}$ is *not* a Markov chain). The transition kernel of $\{J_k\}$ is given by $\sum_{x\in\mathbb{N}} F_{\alpha,\beta}(x)$. We will assume that this chain is irreducible (therefore positive recurrent, since $\mathbb{S}$ is finite) and we denote by $\{\nu_\alpha\}_{\alpha\in\mathbb{S}}$ its invariant measure.

Given two kernels $F$ and $G$, their convolution $F * G$ is the kernel defined by

$$(F * G)_{\alpha,\beta}(x) := \sum_{y\in\mathbb{N}} \sum_{\gamma\in\mathbb{S}} F_{\alpha,\gamma}(y) G_{\gamma,\beta}(x-y) = \sum_{y\in\mathbb{N}} \big[F(y) \cdot G(x-y)\big]_{\alpha,\beta}\,, \tag{4.20}$$

where $\cdot$ denotes matrix product. Observe that if $F$ and $G$ are semi–Markov kernels, then $F * G$ is semi–Markov too. With standard notation, the $n$–fold convolution of a kernel $F$ with itself will be denoted by $F^{*n}$, the $n = 0$ case being by definition the identity kernel $[F^{*0}]_{\alpha,\beta}(x) := \mathbf{1}_{(\beta=\alpha)}\mathbf{1}_{(x=0)}$.

A fundamental object associated to a semi–Markov kernel $F$ the so–called *Markov–Green function* (or Markov–renewal kernel), which is the kernel $\boldsymbol{U}$ defined by

$$\boldsymbol{U}_{\alpha,\beta}(x) := \sum_{k=0}^{\infty} \big[F^{*k}\big]_{\alpha,\beta}(x)\,. \tag{4.21}$$

Of course the kernel $\boldsymbol{U}$ is the analog of the Green function of a classical renewal process, and it has a similar probabilistic interpretation in terms of the associated Markov renewal process $\{(J_k, T_k)\}$:

$$\boldsymbol{U}_{\alpha,\beta}(x) = \mathbb{P}_\alpha\big[\exists k \geq 0 :\ T_0 + \ldots + T_k = x\ ,\ J_k = \beta\big]\,, \tag{4.22}$$

where $\mathbb{P}_\alpha$ is the law of $\{(T_k, J_k)\}$ conditioned on $\{J_0 = \alpha, T_0 = 0\}$.

We need some notation to treat our periodic setting: we say that a kernel $F_{\alpha,\beta}(x)$ has period $T \in \mathbb{N}$ if the set $\{x : \boldsymbol{U}_{\alpha,\alpha}(x) \neq 0\}$ is contained in $T\mathbb{Z}$, for the least such $T$ (this definition does not depend on $\alpha$ because the chain $\{J_k\}$ is supposed to be irreducible, see the discussion at p. 208 of [**5**]). It follows that the set $\{x : \boldsymbol{U}_{\alpha,\beta}(x) \neq 0\}$ is contained in the translated lattice $\gamma(\alpha,\beta)+T\mathbb{N}$, where $\gamma(\alpha,\beta) \in \{0,\dots,T-1\}$ (for us it will be $\gamma(\alpha,\beta) = [\beta-\alpha]$).

In analogy to the classical case, the asymptotic behavior of $\boldsymbol{U}_{\alpha,\beta}(x)$ as $x \to \infty$ is of particular interest. Let us define the (possibly infinite) *mean* $\mu$ of a semi–Markov kernel $F_{\alpha,\beta}(x)$ as

$$\mu := \sum_{\alpha,\beta\in\mathbb{S}} \sum_{x\in\mathbb{N}} x\, \nu_\alpha\, F_{\alpha,\beta}(x)\,. \tag{4.23}$$

Then we have an analog of Blackwell's Renewal Theorem, that in our periodic setting reads as

$$\exists \lim_{\substack{x\to\infty \\ [x]=\beta-\alpha}} \boldsymbol{U}_{\alpha,\beta}(x) \;=\; T\frac{\nu_\beta}{\mu}\,, \tag{4.24}$$

cf. Corollary 2.3 p. 10 of [**5**] for the classical case.

We will see that determining the asymptotic behavior of $\boldsymbol{U}_{\alpha,\beta}(x)$ when the kernel $F_{\alpha,\beta}(x)$ is no more semi–Markov is the key to get the asymptotic behavior of the partition function $Z^{\mathrm{c}}_{N,\omega}$.

**2.2. A random walk excursion viewpoint.** Now we are ready to make explicit the link between the partition function for our model and the Theory of Markov Renewal Processes. Let us look back to our Hamiltonian (4.3): its specificity comes from the fact that it can be decomposed in an efficient way by considering the return times to the origin of $S$. More precisely we set for $j \in \mathbb{N}$

$$\tau_0 = 0 \qquad \tau_{j+1} = \inf\{n > \tau_j :\, S_n = 0\}\,,$$

and for $\mathbf{P}$–typical trajectories of $S$ one has an infinite sequence $\tau := \{\tau_j\}_j$ of stopping times. We set $T_j = \tau_j - \tau_{j-1}$ and of course $\{T_j\}_{j=1,2,\dots}$ is, under $\mathbf{P}$, an IID sequence. By conditioning on $\tau$ and integrating on the up–down symmetry of the random walk excursions one easily obtains the following expression for the constrained partition function:

$$Z^{\mathrm{c}}_{N,\omega} \;=\; \mathbf{E}\left[\prod_{j=1}^{\iota_N} \exp\left(\Psi^\omega(\tau_{j-1},\tau_j)\right);\, \tau_{\iota_N} = N\right], \tag{4.25}$$

where $\iota_N = \sup\{k : \tau_k \le N\}$ and we have introduced the *integrated Hamiltonian* $\Psi^\omega(n_1, n_2)$, which gives the energetic contribution of an excursion from $n_1$ to $n_2$:

$$\Psi^\omega(n_1,n_2) = \begin{cases} \omega^{(0)}_{n_2} \,+\, \big(\widetilde{\omega}^{(0)}_{n_2} - \omega^{(+1)}_{n_2}\big) & \text{if } n_2 = n_1+1 \\ \omega^{(0)}_{n_2} \,+\, \log\Big[\frac{1}{2}\Big(1+\exp\sum_{n=n_1+1}^{n_2}\big(\omega^{(-1)}_n - \omega^{(+1)}_n\big)\Big)\Big] & \text{if } n_2 > n_1+1 \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.26}$$

Now we are going to use in an essential way the fact that our charges are $T$–periodic. In fact a look at (4.26) shows that the energy $\Psi^\omega(n_1, n_2)$ of an excursion from $n_1$ to $n_2$ is a function only of $(n_2 - n_1)$, $[n_1]$ and $[n_2]$, where by $[\,\cdot\,]$ we mean the equivalence class modulo $T$, see § 1.4. More precisely for $n_1 \in \alpha$, $n_2 \in \beta$ and $\ell = n_2 - n_1$ we have $\Psi^\omega(n_1, n_2) = \Phi^\omega_{\alpha,\beta}(\ell)$, where $\Phi^\omega$ was defined in (4.13). Then recalling the law $K(n)$ of the first return, introduced in (4.11), we can rewrite (4.25) as

$$Z^{\mathrm{c}}_{N,\omega} \,=\, \sum_{k=1}^{N} \sum_{\substack{t_0,\dots,t_k\in\mathbb{N} \\ 0=:t_0<t_1<\ldots<t_k:=N}} \prod_{j=1}^{k} K\left(t_j - t_{j-1}\right)\, \exp\left(\Phi^\omega_{[t_{j-1}],[t_j]}(t_j - t_{j-1})\right). \tag{4.27}$$

This decomposition of $Z^{\mathrm{c}}_{N,\omega}$ according to the random walk excursions makes explicit the link with Markov Renewal Theory. In fact using the kernel $M_{\alpha,\beta}(x)$ introduced in (4.14) we can rewrite it as

$$\begin{aligned} Z^{\mathrm{c}}_{N,\omega} &= \sum_{k=1}^{N} \sum_{\substack{t_0,\dots,t_k\in\mathbb{N} \\ 0=:t_0<t_1<\ldots<t_k:=N}} \prod_{j=1}^{k} M_{[t_{j-1}],[t_j]}(t_j - t_{j-1}) \\ &= \sum_{k=1}^{N} \sum_{\substack{t_0,\dots,t_k\in\mathbb{N} \\ 0=:t_0<t_1<\ldots<t_k:=N}} \big[M(t_1)\cdot M(t_2-t_1)\cdot\ldots\cdot M(N-t_{k-1})\big]_{0,[N]} \\ &= \sum_{k=0}^{\infty} \big[M^{*k}\big]_{[0],[N]}(N)\,. \end{aligned} \tag{4.28}$$

Therefore it is natural to introduce the kernel $\mathcal{Z}_{\alpha,\beta}(x)$ defined by

$$\mathcal{Z}_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \big[M^{*k}\big]_{\alpha,\beta}(x)\,, \tag{4.29}$$

so that $Z^{\mathrm{c}}_{N,\omega} = \mathcal{Z}_{[0],[N]}(N)$. More generally $\mathcal{Z}_{\alpha,\beta}(x)$ for $[x] = \beta - \alpha$ can be interpreted as the partition function of a directed polymer of size $x$ that starts at a site $(M, 0)$, with $[M] = \alpha$, and which is pinned at the site $(M + x, 0)$.

Our purpose is to get the precise asymptotic behavior of $\mathcal{Z}_{\alpha,\beta}(x)$ as $x \to \infty$, from which we will obtain the asymptotic behavior of $Z^{\mathrm{c}}_{N,\omega}$ and hence the proof of Theorem 4.5. It is clear that equation (4.29) is the same as equation (4.21), except for the fact that in general the kernel $M$ has no reason to be semi–Markov. Nevertheless we will see that with some transformations one can reduce the problem to a semi–Markov setting.

It turns out that for the derivation of the asymptotic behavior of $\mathcal{Z}_{\alpha,\beta}(x)$ it is not necessary to use the specific form (4.14) of the kernel $M_{\alpha,\beta}(x)$, the computations being more transparent if carried out in a general setting. For these reasons, in the following we will assume that $M_{\alpha,\beta}(x)$ is a generic $T$–periodic kernel such that the matrix $B_{\alpha,\beta}$ defined by (4.15) is finite. While these assumption are sufficient to yield the asymptotic behavior of $\mathcal{Z}_{\alpha,\beta}(x)$ when $\delta^\omega > 1$, for the cases $\delta^\omega < 1$ and $\delta^\omega = 1$ it is necessary to know the asymptotic behavior as $x \to \infty$ of $M_{\alpha,\beta}(x)$ itself. Notice that our setting is an *heavy–tailed* one: more precisely we will assume that for every $\alpha, \beta \in \mathbb{S}$:

$$\exists \lim_{\substack{x\to\infty \\ [x]=\beta-\alpha}} x^{3/2}\, M_{\alpha,\beta}(x) \;=:\; L_{\alpha,\beta} \in (0,\infty)\,. \tag{4.30}$$

From equation (4.13) it is easy to check that the kernel $M_{\alpha,\beta}(x)$ defined by (4.14) does satisfy (4.30) (see Section 3 for more details on this issue).

For ease of exposition, we will treat separately the three cases $\delta^\omega > 1$, $\delta^\omega < 1$ and $\delta^\omega = 1$.

**2.3. The localized regime** ($\delta^\omega > 1$)**.** The key idea is to introduce the following exponential perturbation of the kernel $M$ (cf. [**5**, Theorem 4.6]), depending on the positive real parameter $b$:

$$A^b_{\alpha,\beta}(x) := M_{\alpha,\beta}(x)\, e^{-bx}\,.$$

Let us denote by $\Delta(b)$ the Perron–Frobenius eigenvalue of the matrix $\sum_x A^b_{\alpha,\beta}(x)$. As the entries of this matrix are analytic and nonincreasing functions of $b$, $\Delta(b)$ is analytic and nonincreasing too, hence strictly decreasing because $\Delta(0) = \delta^\omega > 1$ and $\Delta(\infty) = 0$. Therefore there exists a single value $\mathrm{F}_\omega > 0$ such that $\Delta\big(\mathrm{F}_\omega\big) = 1$, and we denote by $\{\zeta_\alpha\}_\alpha$, $\{\xi_\alpha\}_\alpha$ the Perron–Frobenius left and right eigenvectors

of $\sum_x A^{\mathrm{F}_\omega}_{\alpha,\beta}(x)$, chosen to have (strictly) positive components and normalized in such a way that $\sum_\alpha \zeta_\alpha \xi_\alpha = 1$ (of course there is still a degree of freedom in the normalization, which however is immaterial).

Now we set

$$\Gamma^{>}_{\alpha,\beta}(x) := A^{\mathrm{F}_\omega}_{\alpha,\beta}(x)\,\frac{\xi_\beta}{\xi_\alpha} = M_{\alpha,\beta}(x)\,e^{-\mathrm{F}_\omega x}\,\frac{\xi_\beta}{\xi_\alpha}\,, \tag{4.31}$$

and it is immediate to check that $\Gamma^{>}$ is a semi–Markov kernel. Furthermore, we can rewrite (4.29) as

$$\mathcal{Z}_{\alpha,\beta}(x) := e^{\mathrm{F}_\omega x}\,\frac{\xi_\alpha}{\xi_\beta}\,\sum_{k=0}^{\infty}\left[(\Gamma^{>})^{*k}\right]_{\alpha,\beta}(x) = e^{\mathrm{F}_\omega x}\,\frac{\xi_\alpha}{\xi_\beta}\,\mathcal{U}_{\alpha,\beta}(x)\,, \tag{4.32}$$

where $\mathcal{U}_{\alpha,\beta}(x)$ is nothing but the Markov–Green function associated to the semi–Markov kernel $\Gamma^{>}_{\alpha,\beta}(x)$. Therefore the asymptotic behavior of $\mathcal{Z}_{\alpha,\beta}(x)$ is easily obtained applying Blackwell's Renewal Theorem (4.24). To this end, let us compute the mean $\mu$ of the semi–Markov kernel $\Gamma^{>}$: it is easily seen that the invariant measure of the associated modulating chain is given by $\{\zeta_\alpha\xi_\alpha\}_\alpha$, therefore

$$\begin{aligned}\mu &= \sum_{\alpha,\beta\in\mathbb{S}}\sum_{x\in\mathbb{N}} x\,\zeta_\alpha\,\xi_\alpha\,\Gamma^{>}_{\alpha,\beta}(x) = \sum_{\alpha,\beta\in\mathbb{S}}\sum_{x\in\mathbb{N}} x e^{-\mathrm{F}_\omega x}\,\zeta_\alpha\,M_{\alpha,\beta}(x)\,\xi_\beta \\ &= -\left(\frac{\partial}{\partial b}\Delta(b)\right)\Big|_{b=\mathrm{F}_\omega} \;\in\; (0,\infty)\,,\end{aligned}$$

(for the last equality see for example [**11**, Lemma 2.1]). Coming back to (4.32), we can now apply Blackwell's Renewal Theorem (4.24) obtaining the desired asymptotic behavior:

$$\mathcal{Z}_{\alpha,\beta}(x) \;\sim\; \xi_\alpha\,\zeta_\beta\,\frac{T}{\mu}\,\exp\left(\mathrm{F}_\omega\,x\right) \qquad\qquad x\to\infty,\quad [x]=\beta-\alpha\,. \tag{4.33}$$

In particular, for $\alpha = [0]$ and $\beta = \eta$ we have part (1) of Theorem 4.5, where $C^{>}_{\omega,\eta} = \xi_0\zeta_\eta T/\mu$.

**2.4. The strictly delocalized case** ($\delta^\omega < 1$)**.** We prove that the asymptotic behavior of $\mathcal{Z}_{\alpha,\beta}(x)$ when $\delta^\omega < 1$ is given by

$$\mathcal{Z}_{\alpha,\beta}(x) \;\sim\; \left(\left[(1-B)^{-1}L\,(1-B)^{-1}\right]_{\alpha,\beta}\right)\frac{1}{x^{3/2}} \qquad\qquad x\to\infty,\quad [x]=\beta-\alpha\,, \tag{4.34}$$

where the matrixes $L$ and $B$ have been defined in (4.30) and (4.15). In particular, taking $\alpha = [0]$ and $\beta = \eta$, (4.34) proves part (2) of Theorem 4.5 with

$$C^{<}_{\omega,\eta} := \left[(1-B)^{-1}L\,(1-B)^{-1}\right]_{0,\eta}\,.$$

To start with, we prove by induction that for every $n \in \mathbb{N}$

$$\sum_{x\in\mathbb{N}} [M^{*n}]_{\alpha,\beta}(x) = [B^n]_{\alpha,\beta}\,. \tag{4.35}$$

The $n = 1$ case is the definition of $B$, while for $n \geq 1$

$$\begin{aligned}\sum_{x\in\mathbb{N}} M^{*(n+1)}(x) \;&=\; \sum_{x\in\mathbb{N}}\sum_{z\leq x} M^{*n}(z)\cdot M(x-z) \;=\; \sum_{z\in\mathbb{N}} M^{*n}(z)\cdot\sum_{x\geq z} M(x-z)\\ &=\; \sum_{z\in\mathbb{N}} M^{*n}(z)\cdot B \;=\; B^n\cdot B \;=\; B^{n+1}\,.\end{aligned}$$

Next we claim that, if (4.30) holds, then for every $\alpha, \beta \in \mathbb{S}$

$$\exists \lim_{\substack{x\to\infty\\ [x]=\beta-\alpha}} x^{3/2}\big[M^{*k}\big]_{\alpha,\beta}(x) \;=\; \sum_{i=0}^{k-1}\big[B^i\cdot L\cdot B^{(k-1)-i}\big]_{\alpha,\beta}\,. \tag{4.36}$$

We proceed by induction on $k$. The $k = 1$ case is given by (4.30), and we have that

$$M^{*(n+1)}(x) = \sum_{y=1}^{x/2}\Big(M(y)\cdot M^{*n}(x-y) \;+\; M(x-y)\cdot M^{*n}(y)\Big)$$

(strictly speaking this formula is true only when $x$ is even, however the odd $x$ case is analogous). By the inductive hypothesis equation (4.36) holds for every $k \leq n$, and in particular this implies that $\{x^{3/2}[M^{*k}]_{\alpha,\beta}(x)\}_{x\in\mathbb{N}}$ is a bounded sequence. Therefore we can apply Dominated Convergence and (4.35), getting

$$\begin{aligned}&\exists \lim_{\substack{x\to\infty\\ [x]=\beta-\alpha}} x^{3/2}\big[M^{*(n+1)}\big]_{\alpha,\beta}(x)\\ &\qquad=\; \sum_{\gamma}\sum_{y=1}^{\infty}\Big(M_{\alpha,\gamma}(y)\sum_{i=0}^{n-1}\big[B^i\cdot L\cdot B^{(n-1)-i}\big]_{\gamma,\beta} \;+\; L_{\alpha,\gamma}\big[M^{*n}\big]_{\gamma,\beta}(y)\Big)\\ &\qquad=\; \sum_{\gamma}\Big(B_{\alpha,\gamma}\sum_{i=0}^{n-1}\big[B^i\cdot L\cdot B^{(n-1)-i}\big]_{\gamma,\beta} \;+\; L_{\alpha,\gamma}\big[B^{*n}\big]_{\gamma,\beta}\Big)\\ &\qquad=\; \sum_{i=0}^{n}\big[B^i\cdot L\cdot B^{n-i}\big]_{\alpha,\beta}\,.\end{aligned}$$

Our purpose is to apply the asymptotic result (4.36) to the terms of (4.29), hence we need a bound to apply Dominated Convergence. What we are going to show is that

$$x^{3/2}\,\big[M^{*k}\big]_{\alpha,\beta}(x) \;\leq\; C\,k^3\,\big[B^k\big]_{\alpha,\beta} \tag{4.37}$$

for some positive constant $C$ and for all $\alpha, \beta \in \mathbb{S}$ and $x, k \in \mathbb{N}$. Observe that the r.h.s. above, as a function of $k$, is a summable sequence because the matrix $B$ has spectral radius $\delta^\omega < 1$. We proceed again by induction: for the $k = 1$ case, thanks to (4.30), it is possible to find $C$ such that (4.37) holds true (this fixes $C$ once for all). Now assuming that (4.37) holds for all $k < n$ we show that it does also for $k = n$ (we suppose for simplicity that $n = 2m$ is even, the odd $n$ case being analogous). Then we have (assuming that also $x$ is even for simplicity)

$$\begin{aligned} x^{3/2} \left[M^{*2m}\right]_{\alpha,\beta}(x) \;&=\; 2 \sum_{y=1}^{x/2} \sum_{\gamma \in \mathbb{S}} \left[M^{*m}\right]_{\alpha,\gamma}(y)\; x^{3/2} \left[M^{*m}\right]_{\gamma,\beta}(x-y) \\ &\le\; 2 \cdot 2^{3/2}\, C\, m^3 \sum_{y=1}^{x/2} \sum_{\gamma \in \mathbb{S}} \left[M^{*m}\right]_{\alpha,\gamma}(y)\; \left[B^m\right]_{\gamma,\beta} \\ &\le\; C\, (2m)^3 \left[B^{2m}\right]_{\alpha,\beta}, \end{aligned}$$

where we have applied (4.35), and (4.37) is proven.

We can finally obtain the asymptotic behavior of $\mathcal{Z}_{\alpha,\beta}(x)$ applying the bound (4.36) to (4.29), using Dominated Convergence thanks to (4.37). In this way we get

$$\begin{aligned} \exists \lim_{\substack{x \to \infty \\ [x] = \beta - \alpha}} x^{3/2} \mathcal{Z}_{\alpha,\beta}(x) \;&=\; \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \left[B^i \cdot L \cdot B^{(k-1)-i}\right]_{\alpha,\beta} \\ &=\; \sum_{i=0}^{\infty} \sum_{k=i+1}^{\infty} \left[B^i \cdot L \cdot B^{(k-1)-i}\right]_{\alpha,\beta} \;=\; \sum_{i=0}^{\infty} \left[B^i \cdot L \cdot (1-B)^{-1}\right]_{\alpha,\beta} \\ &=\; \left[(1-B)^{-1} \cdot L \cdot (1-B)^{-1}\right]_{\alpha,\beta}, \end{aligned}$$

and equation (4.34) is proven.

**2.5. The critical case** ($\delta^\omega = 1$)**.** In the critical case the matrix $B$ defined in (4.15) has Perron–Frobenius eigenvalue equal to 1. Let $\{\zeta_\alpha\}_\alpha$, $\{\xi_\alpha\}_\alpha$ denote its corresponding left and right eigenvectors, always chosen to have positive components and normalized so that $\sum_\alpha \zeta_\alpha\, \xi_\alpha = 1$. Then it is immediate to check that the kernel

$$\Gamma^{=}_{\alpha,\beta}(x) \;:=\; M_{\alpha,\beta}(x)\, \frac{\xi_\beta}{\xi_\alpha} \tag{4.38}$$

is semi–Markov, and the corresponding Markov–Green function $U_{\alpha,\beta}(x)$ is given by

$$U_{\alpha,\beta}(x) \;:=\; \sum_{k=0}^{\infty} \left[(\Gamma^{=})^{*k}\right]_{\alpha,\beta}(x) \;=\; \frac{\xi_\beta}{\xi_\alpha}\, \mathcal{Z}_{\alpha,\beta}(x), \tag{4.39}$$

where the last equality follows easily from (4.29). We are going to derive the asymptotic behavior of $U_{\alpha,\beta}(x)$, and from the above relation we will get the analogous result for $\mathcal{Z}_{\alpha,\beta}(x)$.

Denoting by $\{(T_k, J_k)\}$ under $\mathbb{P}$ the Markov–renewal process generated by the semi–Markov kernel $\Gamma^{=}_{\alpha,\beta}(x)$, for $U_{\alpha,\beta}(x)$ we have the probabilistic interpretation (4.22), that we rewrite for convenience

$$U_{\alpha,\beta}(x) = \mathbb{P}_\alpha\big[\exists k \geq 0 :\ T_0 + \ldots + T_k = x\ ,\ J_k = \beta\big]\,. \tag{4.40}$$

For $\beta \in \mathbb{S}$ we introduce the sequence of stopping times $\{\kappa_n^{(\beta)}\}_{n\geq 0}$ corresponding to the visit of the chain $\{J_k\}$ to the state $\beta$:

$$\kappa_0^{(\beta)} := \inf\{k \geq 0 :\ J_k = \beta\} \qquad \kappa_{n+1}^{(\beta)} := \inf\{k > \kappa_n^{(\beta)} :\ J_k = \beta\}\,, \tag{4.41}$$

and we define the process $\{T_n^{(\beta)}\}_{n\geq 0}$ by setting

$$T_0^{(\beta)} := T_0 + \ldots + T_{\kappa_0^{(\beta)}} \qquad T_n^{(\beta)} : T_{\kappa_{n-1}^{(\beta)}+1} + \ldots + T_{\kappa_n^{(\beta)}}\,. \tag{4.42}$$

The key point is that under $\mathbb{P}_\alpha$ the random variables $\{T_n^{(\beta)}\}$ are the interarrival times of a (possibly delayed) *classical renewal process*, equivalently the sequence $\{T_n^{(\beta)}\}_{n\geq 1}$ is IID and independent of $T_0^{(\beta)}$. We denote for $x \in \mathbb{N}$ by $q^{(\beta)}(x)$ the (mass function of the) law of $T_n^{(\beta)}$ for $n \geq 1$, while the law of $T_0^{(\beta)}$ under $\mathbb{P}_\alpha$ is denoted by $q^{(\alpha;\beta)}(x)$. Since clearly

$$\big\{\exists k \geq 0 :\ T_0 + \ldots + T_k = x\ ,\ J_k = \beta\big\} \quad \Longleftrightarrow \quad \big\{\exists n \geq 0 :\ T_0^{(\beta)} + \ldots + T_n^{(\beta)} = x\big\}\,,$$

from (4.40) we get

$$U_{\alpha,\beta}(x) = \mathbb{P}_\alpha\big[\exists n \geq 0 :\ T_0^{(\beta)} + \ldots + T_n^{(\beta)} = x\big] = \Big(q^{(\alpha;\beta)} * \sum_{n=0}^{\infty} \big(q^{(\beta)}\big)^{*n}\Big)(x)\,, \tag{4.43}$$

which shows that $U_{\alpha,\beta}(x)$ is indeed the Green function of the classical renewal process whose interarrival times are the $\{T_n^{(\beta)}\}_{n\geq 0}$.

Now we claim that the asymptotic behavior of $q^{(\beta)}(x)$ as $x \to \infty$, $x \in \beta$, is given by

$$q^{(\beta)}(x) \ \sim\ \frac{c_\beta}{x^{3/2}} \qquad c_\beta \ :=\ \frac{1}{\zeta_\beta\,\xi_\beta} \sum_{\alpha,\gamma} \zeta_\alpha\, L_{\alpha,\gamma}\, \xi_\gamma \ >\ 0\,, \tag{4.44}$$

see § 5.1 for a proof of this relation. Then the asymptotic behavior of (4.43) is given by

$$U_{\alpha,\beta}(x) \ \sim\ \frac{T^2}{2\pi\, c_\beta}\, \frac{1}{\sqrt{x}} \qquad x \to \infty, \quad [x] = \beta - \alpha\,, \tag{4.45}$$

as it follows by [**27**, Th. B] (the factor $T^2$ is due to our periodic setting). Combining equations (4.39), (4.44) and (4.45) we finally get the asymptotic behavior of $\mathcal{Z}_{\alpha,\beta}(x)$:

$$\mathcal{Z}_{\alpha,\beta}(x) \;\sim\; \frac{T^2}{2\pi}\,\frac{\xi_\alpha\,\zeta_\beta}{\sum_{\gamma,\gamma'}\zeta_\gamma\,L_{\gamma,\gamma'}\,\xi_{\gamma'}}\,\frac{1}{\sqrt{x}} \qquad x\to\infty,\quad [x]=\beta-\alpha\,. \tag{4.46}$$

Taking $\alpha=[0]$ and $\beta=\eta$, we have the proof of part (3) of Theorem 4.5.

## 3. Thermodynamic limits

In this section we study the limit as $N\to\infty$ of the polymer measure $\mathbf{P}^a_{N,\omega}$, using the sharp asymptotics for the partition function obtained in the previous section. We recall that $\mathbf{P}^{\mathrm{c}}_{N,\omega}$ is a probability measure on $\mathbb{Z}^{\mathbb{N}}$, which we endow with the product topology. In particular, weak convergence on $\mathbb{Z}^{\mathbb{N}}$ means convergence of all finite dimensional marginals.

We start giving a very useful decomposition of $\mathbf{P}^a_{N,\omega}$. The intuitive idea is that a path $(S_n)_{n\le N}$ can be split into two main ingredients:

- the family $(\tau_k)_{k=0,1,\dots}$ of *returns to zero* of $S$ (defined in § 2.2);
- the family of *excursions from zero* $(S_{i+\tau_{k-1}} : 0\le i\le \tau_k-\tau_{k-1})_{k=1,2,\dots}$

Moreover, since each excursion can be either positive or negative, it is also useful to consider separately the signs of the excursions $\sigma_k := \mathrm{sign}(S_{\tau_{k-1}+1})$ and the absolute values $(e_k(i) := |S_{i+\tau_{k-1}}| : i=1,\dots,\tau_k-\tau_{k-1})$. Observe that these are trivial for an excursion with length 1: in fact if $\tau_k=\tau_{k-1}+1$ then $\sigma_k=0$ and $e_k(0)=e_k(1)=0$.

Let us first consider the returns $(\tau_k)_{k\le\iota_N}$ under $\mathbf{P}^a_{N,\omega}$, where $\iota_N=\sup\{k:\,\tau_k\le N\}$. The law of this process can be viewed as a probability measure $p^a_{N,\omega}$ on the class $\mathcal{A}_N$ of subsets of $\{1,\dots,N\}$: indeed for $A\in\mathcal{A}_N$, writing

$$A=\{t_1,\dots,t_{|A|}\},\qquad 0=:\,t_0<t_1<\cdots<t_{|A|}\,\le\,N, \tag{4.47}$$

we can set

$$p^a_{N,\omega}(A)\;:=\;\mathbf{P}^a_{N,\omega}(\tau_i=t_i,\;i\le\iota_N). \tag{4.48}$$

The measure $p^a_{N,\omega}$ describes the zero set of the polymer of size $N$, and it is analyzed in detail below. From the inclusion of $\mathcal{A}_N$ into $\{0,1\}^{\mathbb{N}}$, the family of all subsets of $\mathbb{N}$, $p^a_{N,\omega}$ can be viewed as a measure on $\{0,1\}^{\mathbb{N}}$ (this observation will be useful in the following).

Now we pass to the signs: we can see that, given $(\tau_j)_{j\le\iota_N}$, under $\mathbf{P}^a_{N,\omega}$ the signs $(\sigma_k)_{k\le\iota_N}$ form an independent family. Conditionally on $(\tau_j)_{j\le\iota_N}$, the law of $\sigma_k$ is specified by:

- if $\tau_k = 1+\tau_{k-1}$, then $\sigma_k = 0$;
- if $\tau_k > 1+\tau_{k-1}$, then $\sigma_k$ can take the two values $\pm 1$ with

$$\mathbf{P}^a_{N,\omega}\Big(\sigma_k = +1 \,\Big|\, (\tau_j)_{j\le\iota_N}\Big) \;=\; \frac{1}{1+\exp\big\{-(\tau_k-\tau_{k-1})\,h_\omega + \Sigma_{[\tau_{k-1}],[\tau_k]}\big\}}\,. \tag{4.49}$$

Observe that when $\tau_{\iota_N} < N$ (which can happen only for $a = \mathrm{f}$) there is a last (incomplete) excursion in the interval $\{0,\dots,N\}$, and the sign of this excursion is also expressed by (4.49) for $k = \iota_{N+1}$, provided we set $\tau_{\iota_N+1} := N$.

Finally we have the moduli: again, once $(\tau_{k-1},\sigma_k)_{1\le k\le\iota_N+1}$ are given, the excursions $(e_k)_{k=1,\dots,\iota_N+1}$ form an independent family. The conditional law of $e_k(\cdot)$ on the event $\{\tau_{k-1}=\ell_0,\ \tau_k=\ell_1\}$ and for $f=(f_i)_{i=1,\dots,\ell_1-\ell_0}$ is, for $k\le\iota_N$, given by

$$\begin{aligned}&\mathbf{P}^a_{N,\omega}\Big(e_k(\cdot)=f \,\Big|\, (\tau_{j-1},\sigma_j)_{1\le j\le\iota_N+1}\Big)\\ &=\; \mathbf{P}\Big(S_i=f_i:\ i=1,\dots,\ell_1-\ell_0 \,\Big|\, S_i>0:\ i=1,\dots,\ell_1-\ell_0-1,\ S_{\ell_1-\ell_0}=0\Big)\,.\end{aligned} \tag{4.50}$$

In the case $\tau_{\iota_N} < N$ we have a last excursion $e_{\iota_N+1}(\cdot)$: its conditional law, on the event $\{\tau_{\iota_N}=\ell<N\}$ and for $f=(f_i)_{i=1,\dots,N-\ell}$, is given by

$$\begin{aligned}&\mathbf{P}^a_{N,\omega}\Big(e_{\iota_N+1}(\cdot)=f \,\Big|\, (\tau_{j-1},\sigma_j)_{1\le j\le\iota_N+1}\Big)\\ &=\; \mathbf{P}\Big(S_i=f_i:\ i=1,\dots,N-\ell \,\Big|\, S_i>0:\ i=1,\dots,N-\ell\Big),\end{aligned} \tag{4.51}$$

We would like to stress that the above relations fully characterize the polymer measure $\mathbf{P}^a_{N,\omega}$. A remarkable fact is that, conditionally on $(\tau_k)_{k\in\mathbb{N}}$, the joint distribution of $(\sigma_j,e_j)_{j\le\iota_N}$ *does not depend on* $N$: in this sense, all the $N$–dependence is contained in the measure $p^a_{N,\omega}$.

For this reason, this section is mainly devoted to the study of the asymptotic behavior of the zero set measures $p^a_{N,\omega}$ as $N\to\infty$. The main result is that $p^{\mathrm{c}}_{N,\omega}$ and $p^{\mathrm{f}}_{N,\omega}$ have the same weak limit $p_\omega$ on $\{0,1\}^{\mathbb{N}}$ as $N\to\infty$ (with some restrictions when $\omega\in\mathcal{P}^<$). Once this is proven, it follows easily that also the polymer measure $\mathbf{P}^a_{N,\omega}$ converges to a limit measure $\mathbf{P}_\omega$ on $\mathbb{Z}^{\mathbb{N}}$, constructed by pasting the excursion over the limit zero set. More precisely, $\mathbf{P}_\omega$ is the measure under which the processes $(\tau_j)$, $(\sigma_j)$ and $(e_j)$ have the following laws:

- the law of the $(\tau_j)_{j\in\mathbb{N}}$ is determined in an obvious way by the limiting zero set measure $p_\omega$;
- then, conditionally on the $(\tau_j)_{j\in\mathbb{N}}$, the process $(\sigma_j)_{j\in\mathbb{N}}$ is an independent one with marginal laws given by (4.49);
- finally, conditionally on $(\tau_j,\sigma_j)_{j\in\mathbb{N}}$, on the event $\{\tau_{k-1}=\ell_0,\tau_k=\ell_1\}$ with $\ell_0<\ell_1<\infty$ the law of $e_k$ is given by the r.h.s. of (4.50). We have to consider also the case $\ell_0<\infty$, $\ell_1=\infty$, because in the regime $\delta^\omega<1$ it turns out that $\mathbf{P}_\omega(\tau_k=\infty)>0$ (see below and § 5.2): in this case the law of $e_k$ is given for any $n\in\mathbb{N}$ and for $f=(f_i)_{i=1,\dots,n}$ by:

$$\begin{aligned}\mathbf{P}_\omega\Big(e_k(i)=f_i\,:\;i=1,\dots,n\;\Big|\;(\tau_j,\sigma_j)_{j\in\mathbb{N}}\Big)\;&=\;\mathbf{P}^+\Big(S_i=f_i\,:\;i=1,\dots,n\Big)\\ &:=\;\lim_{N\to\infty}\mathbf{P}\Big(S_i=f_i\,:\;i=1,\dots,n\;\Big|\;S_i>0:\;i=1,\dots,N\Big),\end{aligned}\tag{4.52}$$

where the existence of such limit is well known: see e.g. [**35**].

**3.1. Law of the zero level set in the free and constrained cases.** Let us describe more explicitly $p^a_{N,\omega}(A)$, using the (strong) Markov property of $\mathbf{P}^a_{N,\omega}$. We use throughout the chapter the notation (4.47). Recalling the definition (4.14) of $M_{\alpha,\beta}(t)$, we have:

- for $a=\mathrm{c}$ and $A\in\mathcal{A}_N$: $p^{\mathrm{c}}_{N,\omega}(A)\neq 0$ if and only if $t_{|A|}=N$, and in this case:
$$p^{\mathrm{c}}_{N,\omega}(A)\;=\;\frac{1}{Z^{\mathrm{c}}_{N,\omega}}\prod_{i=1}^{|A|}M_{[t_{i-1}],[t_i]}(t_i-t_{i-1})$$
- for $a=\mathrm{f}$ and $A\in\mathcal{A}_N$:

$$p^{\mathrm{f}}_{N,\omega}(A)\;=\;\frac{1}{Z^{\mathrm{f}}_{N,\omega}}\left[\prod_{i=1}^{|A|}M_{[t_{i-1}],[t_i]}(t_i-t_{i-1})\right]P(N-t_{|A|})\,\exp\Big(\widetilde{\Phi}_{[t_{|A|}],[N]}(N-t_{|A|})\Big).$$

where $P(n):=\sum_{k=n+1}^\infty K(k)=\sum_{k=n+1}^\infty\mathbf{P}(\tau_1=k)$ and we have introduced

$$\widetilde{\Phi}_{\alpha,\beta}(\ell)\;:=\;\log\left[\frac{1}{2}\Big(1+\exp\big(-\ell h_\omega+\Sigma_{\alpha,\beta}\big)\Big)\right]\mathbf{1}_{(\ell>1)}\,\mathbf{1}_{(\ell\in\beta-\alpha)}\,,\tag{4.53}$$

which differs from $\Phi$ in not having the terms of interaction with the interface, cf. (4.13).

We are going to show that, for any value of $\delta^\omega$, the measure $p^a_{N,\omega}$ on $\{0,1\}^{\mathbb{N}}$ converges as $N\to\infty$ (with some restrictions if $\omega\in\mathcal{P}^<$) to a limit measure under which the process $([\tau_k],\tau_k-\tau_{k-1})_{k\in\mathbb{N}}$ is a *Markov renewal process*. Moreover, we will

compute explicitly the corresponding semi–Markov kernel, showing that the returns to zero are

(1) integrable if $\delta^\omega > 1$ (localized regime);
(2) defective if $\delta^\omega < 1$ (strictly delocalized regime);
(3) non integrable if $\delta^\omega = 1$ (critical regime).

Thanks to the preceding observations, this will complete the proof of Theorem 4.7. We stress that the key result in our derivation is given by the sharp asymptotics of the partition function $Z^{\mathrm{c}}_{N,\omega}$ obtained in the previous section.

Before going into the proof, we give some preliminary material which is useful for all values of $\delta^\omega$. For $k \in \mathbb{N}$ we define the shift operator:

$$\theta_k : \mathbb{R}^{\mathbb{N}} \mapsto \mathbb{R}^{\mathbb{N}}, \qquad \theta_k \zeta := \zeta_{[k+\cdot]},$$

and it is easy to check that the following relations hold true:

$$Z^{\mathrm{c}}_{N-k,\theta_k\omega} = \mathcal{Z}_{[k],[N]}(N-k), \qquad k \le N. \tag{4.54}$$

$$Z^{\mathrm{f}}_{N,\omega} \;=\; \sum_{t=0}^{N} Z^{\mathrm{c}}_{t,\omega}\, P(N-t)\, \exp\left(\widetilde{\Phi}_{[t],[N]}(N-t)\right), \tag{4.55}$$

$$\mathbf{P}^a_{N,\omega}\left(\tau_1 = k\right) \;=\; M_{0,[k]}(k)\, \frac{Z^a_{N-k,\theta_k\omega}}{Z^a_{N,\omega}}, \qquad 1 \le k \le N, \quad a = \mathrm{c}, \mathrm{f}. \tag{4.56}$$

Finally, using (4.12), (4.14) and (4.13) it is easy to see that (4.30) holds true, namely

$$\exists \lim_{\substack{x\to\infty \\ [x]=\beta-\alpha}} x^{3/2}\, M_{\alpha,\beta}(x) \;=\; L_{\alpha,\beta}, \tag{4.57}$$

where:

$$L_{\alpha,\beta} \;=\; \begin{cases} c_K\, \dfrac{1}{2}\left(1 + \exp\left(\Sigma_{\alpha,\beta}\right)\right) \exp(\omega^{(0)}_\beta) & \text{if } \; h_\omega = 0 \\ c_K\, \dfrac{1}{2}\, \exp(\omega^{(0)}_\beta) & \text{if } \; h_\omega > 0 \end{cases}. \tag{4.58}$$

Since also the asymptotic behavior of $P(\ell)\exp(\widetilde{\Phi}_{\alpha,\beta}(\ell))$ will be needed, we set

$$\widetilde{L}_{\alpha,\beta} \;:=\; \lim_{\ell\to\infty,\; \ell\in\beta-\alpha} \sqrt{\ell}\, P(\ell)\, e^{\widetilde{\Phi}_{\alpha,\beta}(\ell)} \;=\; \begin{cases} c_K\big(1 + \exp(\Sigma_{\alpha,\beta})\big) & \text{if } \; h_\omega = 0 \\ c_K & \text{if } \; h_\omega > 0 \end{cases}, \tag{4.59}$$

as it follows easily from (4.53) and from the fact that $P(\ell) \sim 2\, c_K/\sqrt{\ell}$ as $\ell \to \infty$.

**3.2. The localized regime** ($\delta^\omega > 1$)**.** We prove point (1) of Theorem 4.7. More precisely, we prove the following:

PROPOSITION 4.9. *If $\delta^\omega > 1$ then the polymer measures $\mathbf{P}^{\mathrm{f}}_{N,\omega}$ and $\mathbf{P}^{\mathrm{c}}_{N,\omega}$ converge as $N \to \infty$ to the same limit $\mathbf{P}_\omega$, under which $([\tau_k], \tau_k - \tau_{k-1})_{k\in\mathbb{N}}$ is a Markov renewal process with semi-Markov kernel $(\Gamma^{>}_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{S}, x \in \mathbb{N})$.*

For the definition of $\Gamma^{>}$ see (4.31).

3.2.1. *Proof of Proposition 4.9.* We prove first the case $a = \mathrm{c}$. By (4.54), (4.56) and by the asymptotics of $\mathcal{Z}$ in (4.33) above, we have for all $\alpha, \beta, \gamma \in \mathbb{S}$ and $\ell \in \alpha$, $m \in \beta$

$$\exists \lim_{\substack{N\to\infty \\ N\in\gamma}} \frac{Z^{\mathrm{c}}_{N-m,\theta_m\omega}}{Z^{\mathrm{c}}_{N-\ell,\theta_\ell\omega}} = \lim_{\substack{N\to\infty \\ N\in\gamma}} \frac{\mathcal{Z}_{\beta,\gamma}(N-m)}{\mathcal{Z}_{\alpha,\gamma}(N-\ell)} = e^{-\mathrm{F}_\omega k}\,\frac{\xi_\beta}{\xi_\alpha},$$

and since the right hand side does not depend on $\gamma$, then the limit exists as $N \to \infty$. It follows that for $\ell \in \alpha$, $k + \ell \in \beta$:

$$\lim_{N\to\infty} \mathbf{P}^{\mathrm{c}}_{N-\ell,\theta_\ell\omega}(\tau_1 = k) = M_{\alpha,\beta}(k)\, e^{-\mathrm{F}_\omega k}\,\frac{\xi_\beta}{\xi_\alpha} = \Gamma^{>}_{\alpha,\beta}(k).$$

By the Markov property of $\mathbf{P}^{\mathrm{c}}_{N,\omega}$ this yields

$$\lim_{N\to\infty} \mathbf{P}^{\mathrm{c}}_{N,\omega}(\tau_1 = k_1, \dots, \tau_j = k_j) = \prod_{i=1}^{j} \Gamma^{>}_{[k_{i-1}],[k_i]}(k_i - k_{i-1}), \qquad k_0 := 0.$$

The argument for $\mathbf{P}^{\mathrm{f}}_{N,\omega}$ goes along the very same line: by (4.55),

$$\begin{aligned} e^{-\mathrm{F}_\omega N}\, Z^{\mathrm{f}}_{N-k,\theta_k\omega} &= e^{-\mathrm{F}_\omega N} \sum_{t=0}^{N-k} \mathcal{Z}_{[k],[N-t]}(N-k-t)\, P(t)\, \exp\left(\widetilde{\Phi}_{[N-t],[N]}(t)\right) \\ &= e^{-\mathrm{F}_\omega k} \sum_{\eta\in\mathbb{S}} \sum_{t=0}^{N-k} e^{-\mathrm{F}_\omega t}\, P(t) \left[\exp\left(\widetilde{\Phi}_{\eta,[N]}(t)\right)\, e^{-\mathrm{F}_\omega (N-k-t)}\, \mathcal{Z}_{[k],\eta}(N-k-t)\right]. \end{aligned}$$

Since by (4.33) the expression in brackets converges as $N \to \infty$ and $N \in [t] + \eta$, we obtain

$$\exists \lim_{\substack{N\to\infty \\ N\in\gamma}} e^{-\mathrm{F}_\omega N}\, Z^{\mathrm{f}}_{N-k,\theta_k\omega} = \xi_{[k]}\, e^{-\mathrm{F}_\omega k} \left( \frac{T}{\mu} \sum_{\eta\in\mathbb{S}} \sum_{\substack{t\in\mathbb{N} \\ [t]=\gamma-\eta}} e^{-\mathrm{F}_\omega t}\, P(t)\, \exp\left(\widetilde{\Phi}_{\eta,\gamma}(t)\right) \zeta_\eta \right).$$

Observe that the term in parenthesis is just a function of $\gamma$. Having found the precise asymptotics of $Z^{\mathrm{f}}_{N,\omega}$, we can argue as for $\mathbf{P}^{\mathrm{c}}_{N,\omega}$ to conclude the proof. □

**3.3. The critical regime** ($\delta^\omega = 1$)**.** We prove point (3) of Theorem 4.7. More precisely, we prove the following:

PROPOSITION 4.10. *If $\delta^\omega = 1$ then the polymer measures $\mathbf{P}^{\mathrm{f}}_{N,\omega}$ and $\mathbf{P}^{\mathrm{c}}_{N,\omega}$ converge as $N \to \infty$ to the same limit $\mathbf{P}_\omega$, under which $([\tau_k], \tau_k - \tau_{k-1})_{k\in\mathbb{N}}$ is a Markov renewal process with semi-Markov kernel $(\Gamma^{=}_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{S}, x \in \mathbb{N})$.*

For the definition of $\Gamma^{=}$ see (4.38).

3.3.1. *Proof of Proposition 4.10.* We prove first the case $a = \mathrm{c}$. By (4.57) and and by the asymptotics of $\mathcal{Z}$ in (4.46) above, we obtain for all $k \in \alpha$:

$$\exists \lim_{\substack{N\to\infty \\ N\in\beta}} N^{1/2}\, \mathcal{Z}_{\alpha,\beta}(N-k) \,=\, \frac{T^2}{2\pi}\, \frac{\xi_\alpha\, \zeta_\beta}{\sum_{\gamma,\gamma'} \zeta_\gamma\, L_{\gamma,\gamma'}\, \xi_{\gamma'}}.$$

It follows for all $\alpha, \beta, \gamma \in \mathbb{S}$ and $\ell \in \alpha$, $m \in \beta$

$$\exists \lim_{\substack{N\to\infty \\ N\in\gamma}} \frac{Z^{\mathrm{c}}_{N-m,\theta_m\omega}}{Z^{\mathrm{c}}_{N-\ell,\theta_\ell\omega}} \,=\, \lim_{\substack{N\to\infty \\ N\in\gamma}} \frac{\mathcal{Z}_{\beta,\gamma}(N-m)}{\mathcal{Z}_{\alpha,\gamma}(N)} \,=\, \frac{\xi_\beta}{\xi_\alpha},$$

and since the right hand side does not depend on $\gamma$, then the limit exists as $N \to \infty$. It follows for $\ell \in \alpha$, $k + \ell \in \beta$:

$$\lim_{N\to\infty} \mathbf{P}^{\mathrm{c}}_{N-\ell,\theta_\ell\omega}\,(\tau_1 = k) \,=\, M_{\alpha,\beta}(k)\, \frac{\xi_\beta}{\xi_\alpha} \,=\, \Gamma^{=}_{\alpha,\beta}(k).$$

By the Markov property of $\mathbf{P}^{\mathrm{c}}_{N,\omega}$ this yields

$$\lim_{N\to\infty} \mathbf{P}^{\mathrm{c}}_{N,\omega}\,(\tau_1 = k_1, \ldots, \tau_j = k_j) \,=\, \prod_{i=1}^{j} \Gamma^{=}_{[k_{i-1}],[k_i]}(k_i - k_{i-1}), \qquad k_0 := 0.$$

For $\mathbf{P}^{\mathrm{f}}_{N,\omega}$, by (4.55) we have for $N \in \beta$ and $k \le N$:

$$Z^{\mathrm{f}}_{N-k,\theta_k\omega} \,=\, \sum_{\gamma} \sum_{t=0}^{N-k} \mathcal{Z}_{[k],\gamma}(t)\, P(N-k-t)\, \exp\Big(\widetilde{\Phi}_{\gamma,\,\beta}(N-k-t)\Big).$$

By the previous results and using (4.59) we obtain that for every $k \in \mathbb{N}$

$$\begin{aligned} \exists \lim_{N\to\infty,\, N\in\beta} Z^{\mathrm{f}}_{N-k,\theta_k\omega} \;&=\; \xi_{[k]}\, \frac{T}{2\pi}\, \frac{\sum_\eta \zeta_\eta\, \widetilde{L}_{\eta,\beta}}{\sum_{\eta,\eta'} \zeta_\eta\, L_{\eta,\eta'}\, \xi_{\eta'}} \int_0^1 \frac{dt}{t^{\frac12}(1-t)^{\frac12}} \\ &=\; \xi_{[k]} \left( \frac{T}{2}\, \frac{\sum_\eta \zeta_\eta\, \widetilde{L}_{\eta,\beta}}{\sum_{\eta,\eta'} \zeta_\eta\, L_{\eta,\eta'}\, \xi_{\eta'}} \right). \end{aligned} \tag{4.60}$$

To conclude it suffices to argue as in the constrained case. □

**3.4. The strictly delocalized regime** ($\delta^\omega < 1$)**.** We prove point (2) and the last assertion of Theorem 4.7. In this case the result is different according to whether $\omega \in \mathcal{P}^<$ or $\omega \notin \mathcal{P}^<$ (recall the definition (4.17)). To be more precise, there is first a weak formulation for *all* $\omega$ which gives a thermodynamic limit of $\mathbf{P}^a_{N,\omega}$ depending on the sequence $\{N : [N] = \eta\}$ and on $a = \mathrm{f}, \mathrm{c}$; secondly, there is a stronger formulation only for $\omega \notin \mathcal{P}^<$, which says that such limits coincide for all $\eta \in \mathbb{S}$ and $a = \mathrm{f}, \mathrm{c}$.

It will turn out that in the strictly delocalized regime there exists a.s. a last return to zero, i.e. the process $(\tau_k)_{k\in\mathbb{N}}$ is defective. In order to express this with the language of Markov renewal processes, we introduce the sets $\overline{\mathbb{S}} := \mathbb{S} \cup \{\infty\}$ and $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, extending the equivalence relation to $\overline{\mathbb{N}}$ by $[\infty] = \infty$. Finally we set for all $\alpha, \eta \in \mathbb{S}$:

$$\Lambda^{\mathrm{c}}_{\alpha,\eta} := \left[(1-B)^{-1} L\,(1-B)^{-1}\right]_{\alpha,\eta}, \qquad \mu^{\mathrm{c}}_{\alpha,\eta} := \left[L\,(1-B)^{-1}\right]_{\alpha,\eta},$$

$$\Lambda^{\mathrm{f}}_{\alpha,\eta} := \left[(1-B)^{-1}\widetilde{L}\right]_{\alpha,\eta}, \qquad \mu^{\mathrm{f}}_{\alpha,\eta} := \widetilde{L}_{\alpha,\eta},$$

and for all $\eta \in \mathbb{S}$ and $a = \mathrm{f}, \mathrm{c}$ we introduce the semi-Markov kernel on $\overline{\mathbb{S}} \times \overline{\mathbb{N}}$:

$$\Gamma^{\eta,a}_{\alpha,\beta}(x) := \begin{cases} M_{\alpha,\beta}(k)\,\Lambda^a_{\beta,\eta}/\Lambda^a_{\alpha,\eta} & \alpha \in \mathbb{S},\ x \in \mathbb{N},\ \beta = [x] \in \mathbb{S} \\ \mu^a_{\alpha,\eta}/\Lambda^a_{\alpha,\eta} & \alpha \in \mathbb{S},\ x = \infty,\ \beta = [\infty] \\ 1 & \alpha = \beta = [\infty],\ x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\Gamma^{\eta,a}$ is really a semi-Markov kernel, since for $\alpha \in \mathbb{S}$:

$$\begin{aligned} \sum_{\beta\in\overline{\mathbb{S}}}\sum_{x\in\overline{\mathbb{N}}} \Gamma^{\eta,a}_{\alpha,\beta}(x) &= \frac{\mu^a_{\alpha,\eta}}{\Lambda^a_{\alpha,\eta}} + \sum_{\beta\in\mathbb{S}}\sum_{x\in\mathbb{N}} \frac{M_{\alpha,\beta}(x)\,\Lambda^a_{\beta,\eta}}{\Lambda^a_{\alpha,\eta}} = \frac{\mu^a_{\alpha,\eta}}{\Lambda^a_{\alpha,\eta}} + \frac{1}{\Lambda^a_{\alpha,\eta}}[B\cdot\Lambda^a]_{\alpha,\eta} \\ &= \frac{\mu^a_{\alpha,\eta}}{\Lambda^a_{\alpha,\eta}} + \frac{1}{\Lambda^a_{\alpha,\eta}}(\Lambda^a_{\alpha,\eta} - \mu^a_{\alpha,\eta}) = 1. \end{aligned}$$

We are going to prove the following:

Proposition 4.11. *Let $\delta^\omega < 1$. Then:*

(1) *for $a = \mathrm{f}, \mathrm{c}$, $\mathbf{P}^a_{N,\omega}$ converges as $N \to \infty$, $[N] = \eta$ to a measure $\mathbf{P}^{a,\eta}_\omega$, under which $([\tau_k], \tau_k - \tau_{k-1})_{k\in\mathbb{N}}$ is a Markov renewal process with semi-Markov kernel $(\Gamma^{\eta,a}_{\alpha,\beta}(x) : \alpha, \beta \in \overline{\mathbb{S}}, x \in \overline{\mathbb{N}})$.*

(2) *if* $\omega \notin \mathcal{P}^<$, *then* $\mathbf{P}_\omega^{a,\eta} =: \mathbf{P}_\omega$ *and* $\Gamma^{\eta,a} =: \Gamma^<$ *depend neither on* $\eta$ *nor on* $a$, *and both* $\mathbf{P}_{N,\omega}^{\mathrm{f}}$ *and* $\mathbf{P}_{N,\omega}^{\mathrm{c}}$ *converge as* $N \to \infty$ *to* $\mathbf{P}_\omega$, *under which* $([\tau_k], \tau_k - \tau_{k-1})_{k\in\mathbb{N}}$ *is a Markov renewal process with semi-Markov kernel* $\Gamma^<$.

Remark 4.12. Part (2) of Proposition 4.11 is an easy consequence of part (1). In fact from equations (4.58) and (4.59) it follows immediately that when $\omega \notin \mathcal{P}^<$ then both matrices $(L_{\alpha,\beta})$ and $(\widetilde{L}_{\alpha,\beta})$ are constant in $\alpha$, and therefore $\Lambda^a$ factorizes into a tensor product, i.e.

$$\Lambda^a_{\alpha,\eta} \,=\, \lambda^a_\alpha\, \nu^a_\eta, \qquad \alpha, \eta \in \mathbb{S},$$

where $(\lambda^a_\alpha)_{\alpha\in\mathbb{S}}$ and $(\nu^a_\alpha)_{\alpha\in\mathbb{S}}$ are easily computed. But then it is immediate to check that the semi–Markov kernel $\Gamma^{\eta,a} =: \Gamma^<$ depends neither on $\eta$ nor on $a$.

3.4.1. *Proof of Proposition 4.11.* By the preceding Remark it suffices to prove part (1). For all $k \in \alpha$, by (4.34) we have

$$\exists \lim_{\substack{N\to\infty \\ [N]=\beta}} N^{3/2}\, \mathcal{Z}_{\alpha,\beta}(N-k) \,=\, \big[(1-B)^{-1} L\, (1-B)^{-1}\big]_{\alpha,\beta} \,=\, \Lambda^{\mathrm{c}}_{\alpha,\beta}. \tag{4.61}$$

In particular, we have for all $\alpha, \beta, \eta \in \mathbb{S}$ and $\ell \in \alpha$, $m \in \beta$:

$$\exists \lim_{\substack{N\to\infty \\ N\in\eta}} \frac{Z^{\mathrm{c}}_{N-m,\theta_m\omega}}{Z^{\mathrm{c}}_{N-\ell,\theta_\ell\omega}} \,=\, \lim_{\substack{N\to\infty \\ N\in\eta}} \frac{\mathcal{Z}_{\beta,\eta}(N-m)}{\mathcal{Z}_{\alpha,\eta}(N)} \,=\, \frac{\Lambda^{\mathrm{c}}_{\beta,\eta}}{\Lambda^{\mathrm{c}}_{\alpha,\eta}},$$

Then by (4.56) we get

$$\lim_{\substack{N\to\infty \\ N\in\eta}} \mathbf{P}^{\mathrm{c}}_{N,\omega}(\tau_1 = k) \,=\, \frac{M_{0,[k]}(k)\, \Lambda^{\mathrm{c}}_{[k],\eta}}{\Lambda^{\mathrm{c}}_{0,\eta}} = \Gamma^{\eta,\mathrm{c}}_{0,[k]}(k).$$

By the Markov property of $\mathbf{P}^{\mathrm{c}}_{N,\omega}$ this generalizes to

$$\lim_{\substack{N\to\infty \\ N\in\eta}} \mathbf{P}^{\mathrm{c}}_{N,\omega}\,(\tau_1 = k_1, \ldots, \tau_j = k_j) \,=\, \prod_{i=1}^{j} \Gamma^{\eta,\mathrm{c}}_{[k_{i-1}],[k_i]}(k_i - k_{i-1}), \qquad k_0 := 0.$$

We prove now the case $a = \mathrm{f}$. Recalling (4.55) above, we see here that

$$N^{1/2}\, Z^{\mathrm{f}}_{N-k,\theta_k\omega} \,=\, \sum_{t=0}^{N-k} \mathcal{Z}_{[k],[t+k]}(t)\, N^{1/2}\, P(N-k-t)\, \exp\Big(\widetilde{\Phi}_{[t+k],[N]}(N-k-t)\Big).$$

Then by (4.59) we obtain

$$\exists \lim_{\substack{N\to\infty \\ N\in\eta}} N^{1/2}\, Z^{\mathrm{f}}_{N-k,\theta_k\omega} \,=\, \sum_{t=0}^{\infty} \mathcal{Z}_{[k],[t+k]}(t)\, \widetilde{L}_{[t+k],\eta} \,=\, \big[(1-B)^{-1}\widetilde{L}\big]_{[k],\eta} \,=\, \Lambda^{\mathrm{f}}_{[k],\eta}\,, \tag{4.62}$$

since

$$\sum_{t=0}^{\infty} \mathcal{Z}_{\alpha,\gamma}(t) \;=\; \sum_{t=0}^{\infty}\sum_{k=0}^{\infty} M^{*k}_{\alpha,\gamma}(t) \;=\; \sum_{k=0}^{\infty} B^{*k}_{\alpha,\gamma} \;=\; \left[(I-B)^{-1}\right]_{\alpha,\gamma}. \tag{4.63}$$

Arguing as for $\mathbf{P}^{\mathrm{c}}_{N,\omega}$, we conclude the proof. □

## 4. Scaling limits

In this section we prove that the measures $\mathbf{P}^{a}_{N,\omega}$ converge under Brownian rescaling. The results and proof follow very closely those of [**22**] and we shall refer to this paper for several technical lemmas.

The first step is tightness of $(Q^{a}_{N,\omega})_{N\in\mathbb{N}}$ in $C([0,1])$.

LEMMA 4.13. *For any $\omega$ and $a = \mathrm{c},\mathrm{f}$ the sequence $(Q^{a}_{N,\omega})_{N\in\mathbb{N}}$ is tight in $C([0,1])$.*

For the standard proof we refer to Lemma 4 in [**22**].

In the rest of the section we prove Theorem 4.8.

**4.1. The localized regime** ($\delta^{\omega} > 1$)**.** We prove point (1) of Theorem 4.8. By Lemma 4.13 it is enough to prove that $\mathbf{P}^{a}_{N,\omega}(|X^{N}_{t}| > \varepsilon) \to 0$ for all $\varepsilon > 0$ and $t \in [0,1]$ and one can obtain this estimate explicitly. We point out however that in this regime one can avoid using the compactness lemma and one can obtain a stronger result by elementary means: observe that for any $k, n \in \mathbb{N}$ such that $n > 1$ and $k + n \le N$, we have

$$\begin{aligned}\mathbf{P}^{a}_{N,\omega}\big(S_k = S_{k+n} = 0,\, S_{k+i} \neq 0 \text{ for } i = 1,\dots,n-1\big)\\ \le\; \frac{\frac{1}{2}\left(1+\exp\left(\sum_{i=1}^{n}\left(\omega^{(-1)}_{k+i} - \omega^{(+1)}_{k+i}\right)\right)\right)}{Z^{\mathrm{c}}_{n,\theta_k\omega}} \;=:\; \widehat{K}_k(n),\end{aligned} \tag{4.64}$$

and this holds both for $a = \mathrm{c}$ and $a = \mathrm{f}$. Inequality (4.64) is obtained by using the Markov property of $S$ both in the numerator and the denominator of the expression (4.9) defining $\mathbf{P}^{a}_{N,\omega}(\cdot)$ after having bounded $Z^{a}_{N,\omega}$ from below by inserting the event $S_k = S_{k+n} = 0$. Of course $\lim_{n\to\infty}(1/n)\log \widehat{K}_k(n) = -\mathrm{F}_{\omega}$ uniformly in $k$ (notice that $\widehat{K}_{k+T}(n) = \widehat{K}_k(n)$). Therefore if we fix $\varepsilon > 0$ by the union bound we obtain (we

recall that $\{\tau_j\}_j$ and $\iota_N$ were defined in Section 3)

$$\begin{aligned}\mathbf{P}^a_{N,\omega}\Big(\max_{j=1,2,\dots,\iota_N}\tau_j-\tau_{j-1}>(1+\varepsilon)\log N/\mathrm{F}_\omega\Big)&\\ \le\ \sum_{k\le N-(1+\varepsilon)\log N/\mathrm{F}_\omega}\ \sum_{n>(1+\varepsilon)\log N/\mathrm{F}_\omega}&\widehat{K}_k(n)\\ \le\ N\sum_{n>(1+\varepsilon)\log N/\mathrm{F}_\omega}\ \max_{k=0,\dots,T-1}\widehat{K}_k(n)\ \le\ &\frac{c}{N^\varepsilon},\end{aligned}$$

for some $c>0$.

Let us start with the constrained case: notice that $\mathbf{P}^{\mathrm{c}}_{N,\omega}(\mathrm{d}S)$–a.s. we have $\tau_{\iota_N}=N$ and hence $\max_{j\le\iota_N}\tau_j-\tau_{j-1}\ge\max_{n=1,\dots,N}|S_n|$, since $|S_{n+1}-S_n|\le 1$. Then we immediately obtain that for any $C>1/\mathrm{F}_\omega$

$$\lim_{N\to\infty}\mathbf{P}^{\mathrm{c}}_{N,\omega}\left(\max_{n=1,\dots,N}|S_n|>C\log N\right)\ =\ 0, \tag{4.65}$$

which is of course a much stronger statement than the scaling limit of point (1) of Theorem 4.8. If we consider instead the measure $\mathbf{P}^{\mathrm{f}}_{N,\omega}$, the length of the last excursion has to be taken into account too: however, an argument very close to the one used in (4.64) yields also that the last excursion is exponentially bounded (with the same exponent) and the proof of point (1) of Theorem 4.8 is complete.

**4.2. The strictly delocalized regime** ($\delta^\omega<1$)**.** We prove point (2) of Theorem 4.8. We set for $t\in\{1,\dots,N\}$:

$$D_t\ :=\ \inf\{k=1,\dots,N:\ k>t,\ S_k=0\},\qquad G_t\ :=\ \sup\{k=1,\dots,N:\ k\le t,\ S_k=0\}.$$

The following result shows that in the strictly delocalized regime, as $N\to\infty$, the visits to zero under $\mathbf{P}^a_{N,\omega}$ tend to be very few and concentrated at a finite distance from the origin if $a=\mathrm{f}$ and from 0 or $N$ if $a=\mathrm{c}$.

Lemma 4.14. *If $\delta^\omega<1$ there exists a constant $C>0$ such that for all $L>0$:*

$$\limsup_{N\to\infty}\mathbf{P}^{\mathrm{f}}_{N,\omega}\left(G_N\ge L\right)\ \le\ C\,L^{-1/2}, \tag{4.66}$$

$$\limsup_{N\to\infty}\mathbf{P}^{\mathrm{c}}_{N,\omega}\left(G_{N/2}\ge L\right)\ \le\ C\,L^{-1/2}, \tag{4.67}$$

$$\limsup_{N\to\infty}\mathbf{P}^{\mathrm{c}}_{N,\omega}\left(D_{N/2}\le N-L\right)\ \le\ C\,L^{-1/2}. \tag{4.68}$$

Lemma 4.14 is a quantitative version of point (2) of Theorem 4.7 and it is a rather straightforward complement: the proof is sketched in § 5.2, in particular (4.94).

4.2.1. *The signs.* In order to prove point (2) of Theorem 4.8, it is now enough to argue as in the proof of Theorem 9 in [**22**], with the difference that now the excursions are not necessarily in the upper half plane, i.e. the signs are not necessarily positive. So the proof is complete if we can show that there exists the limit (as $N \to \infty$ along $[N] = \eta$) of the probability that the process (away from $\{0, 1\}$) lives in the upper half plane. In analogy with Section 3.4, in the general case we have different limits depending on the sequence $[N] = \eta$ and on $a = \mathrm{f}, \mathrm{c}$, while if $\omega \notin \mathcal{P}^<$ all such limits coincide.

We start with the constrained case: given Lemma 4.14, it is sufficient to show that

$$\exists \lim_{\substack{N\to\infty \\ N\in\eta}} \mathbf{P}^{\mathrm{c}}_{N,\omega}(S_{N/2} > 0) =: \mathtt{p}^{<,\mathrm{c}}_{\omega,\eta}. \tag{4.69}$$

Formula (4.69) follows from the fact that

$$\mathbf{P}^{\mathrm{c}}_{N,\omega}(S_{N/2} > 0) = \sum_{\alpha,\beta} \sum_{x<N/2} \sum_{y>N/2} \frac{\mathcal{Z}_{0,\alpha}(x)\, \rho^+_{\alpha,\beta}(y-x)\, M_{\alpha,\beta}(y-x)\, \mathcal{Z}_{\beta,[N]}(N-y)}{\mathcal{Z}_{0,[N]}(N)},$$

where for all $z \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{S}$:

$$\rho^+_{\alpha,\beta}(z) := \frac{1}{1 + \exp\left(-z\, h_\omega + \Sigma_{\alpha,\beta}\right)}, \tag{4.70}$$

cf. (4.49). By Dominated Convergence and by (4.58) and (4.63):

$$\begin{aligned}\exists \lim_{\substack{N\to\infty \\ N\in\eta}} N^{3/2} \sum_{x<N/2} \sum_{y>N/2} &\mathcal{Z}_{0,\alpha}(x)\, \rho^+_{\alpha,\beta}(y-x)\, M_{\alpha,\beta}(y-x)\, \mathcal{Z}_{\beta,\eta}(N-y) \\ &= \left[(1-B)^{-1}\right]_{0,\alpha} c_K \frac{1}{2} \exp(\omega^{(0)}_\beta) \left[(1-B)^{-1}\right]_{\beta,\eta}.\end{aligned}$$

By (4.34) we obtain (4.69) with

$$\mathtt{p}^{<,\mathrm{c}}_{\omega,\eta} := \frac{\sum_{\alpha,\beta} \left[(1-B)^{-1}\right]_{0,\alpha} c_K \, \frac{1}{2} \, \exp(\omega^{(0)}_\beta) \left[(1-B)^{-1}\right]_{\beta,\eta}}{\left[(1-B)^{-1} L\, (1-B)^{-1}\right]_{0,\eta}}. \tag{4.71}$$

Observe that by (4.58):

- if $h_\omega > 0$ then in (4.71) the denominator is equal to the numerator, so that $\mathtt{p}^{<,\mathrm{c}}_{\omega,\eta} = 1$ for all $\eta$.
- if $h_\omega = 0$ and $\Sigma \equiv 0$ then in (4.71) the denominator is equal to *twice* the numerator, so that $\mathtt{p}^{<,\mathrm{c}}_{\omega,\eta} = 1/2$ for all $\eta$.
- in the remaining case, i.e. if $\omega \in \mathcal{P}^<$, in general $\mathtt{p}^{<,\mathrm{c}}_{\omega,\eta}$ depends on $\eta$.

Now let us consider the free case. This time it is sufficient to show that

$$\exists \lim_{\substack{N\to\infty \\ N\in\eta}} \mathbf{P}^{\mathrm{f}}_{N,\omega}(S_N > 0) =: \mathtt{p}^{<,\mathrm{f}}_{\omega,\eta}. \tag{4.72}$$

Formula (4.72) follows from the fact that

$$\mathbf{P}^{\mathrm{f}}_{N,\omega}(S_N > 0) = \sum_{\alpha}\sum_{x<N} \frac{\mathcal{Z}_{0,\alpha}(x)\,\cdot \frac{1}{2}P(N-k)}{Z^{\mathrm{f}}_{N,\omega}},$$

and using (4.55), (4.63) and (4.59) we obtain that (4.72) holds with

$$\mathtt{p}^{<,\mathrm{f}}_{\omega,\eta} \;=\; \frac{\sum_\alpha \left[(1-B)^{-1}\right]_{0,\alpha} c_K}{\left[(1-B)^{-1}\widetilde{L}\right]_{0,\eta}}. \tag{4.73}$$

Again, observe that by (4.59):

- if $h_\omega > 0$ then in (4.73) the denominator is equal to the numerator and $\mathtt{p}^{<,\mathrm{f}}_{\omega,\eta} = 1$ for all $\eta$.
- if $h_\omega = 0$ and $\Sigma \equiv 0$ then in (4.73) the denominator is equal to *twice* the numerator, so that $\mathtt{p}^{<,\mathrm{f}}_{\omega,\eta} = 1/2$ for all $\eta$.
- in the remaining case, i.e. if $\omega \in \mathcal{P}^<$, in general $\mathtt{p}^{<,\mathrm{f}}_{\omega,\eta}$ depends on $\eta$ and is different from $\mathtt{p}^{<,\mathrm{c}}_{\omega,\eta}$.

**4.3. The critical regime** ($\delta^\omega = 1$)**.** In this section we prove point (3) of Theorem 4.8. As in the previous section, we first determine the the asymptotic behavior of the zero level set of the copolymer and then we pass to the study of the signs of the excursions.

We introduce the random closed subset $\mathcal{A}^a_N$ of $[0,1]$, describing the zero set of the polymer of size $N$ rescaled by a factor $1/N$:

$$\mathbb{P}(\mathcal{A}^a_N = A/N) \;=\; p^a_{N,\omega}(A), \qquad A \subseteq \{0,\dots,N\},$$

where we recall that $p^a_{N,\omega}(\cdot)$ has been defined in § 3.1. Let us denote by $\mathcal{F}$ the class of *all closed subsets* of $\mathbb{R}^+ := [0,+\infty)$. We are going to put on $\mathcal{F}$ a topological and measurable structure, so that we can view the law of $\mathcal{A}^a_N$ as a probability measure on (a suitable $\sigma$–field of) $\mathcal{F}$ and we can study the weak convergence of $\mathcal{A}^a_N$.

We endow $\mathcal{F}$ with the topology of Matheron, cf. [**47**] and [**32**, § 3], which is a metrizable topology. To define it, to a closed subset $F \subseteq \mathbb{R}^+$ we associate the closed

*nonempty* subset $\widetilde{F}$ of the compact interval $[0, \pi/2]$ defined by $\widetilde{F} := \arctan\big(F \cup \{+\infty\}\big)$. Then the metric $\rho(\cdot, \cdot)$ we take on $\mathcal{F}$ is

$$\rho(F, F') := \max\left\{ \sup_{t\in\widetilde{F}} d(t, \widetilde{F}'), \sup_{t'\in\widetilde{F}'} d(t', \widetilde{F}) \right\} \qquad F,\, F' \in \mathcal{F}\,, \tag{4.74}$$

where $d(s, A) := \inf\{|t - s|, t \in A\}$ is the standard distance between a point and a set. We point out that the r.h.s. of (4.74) is the so–called Hausdorff metric between the compact sets $\widetilde{F}$, $\widetilde{F}'$. Thus given a sequence $\{F_n\}_n \subset \mathcal{F}$ and $F \in \mathcal{F}$, we say that $F_n \to F$ in $\mathcal{F}$ if and only if $\rho(F_n, F) \to 0$. We observe that this is equivalent to requiring that for each open set $G$ and each compact $K$

$$\begin{array}{lll} F \cap G \neq \emptyset & \Longrightarrow & F_n \cap G \neq \emptyset \ \text{ eventually} \\ F \cap K = \emptyset & \Longrightarrow & F_n \cap K = \emptyset \ \text{ eventually} \end{array}\,. \tag{4.75}$$

Another necessary and sufficient condition for $F_n \to F$ is that $d(t, F_n) \to d(t, F)$ for every $t \in \mathbb{R}^+$.

This topology makes $\mathcal{F}$ a separable and compact metric space [**47**, Th. 1-2-1], in particular a Polish space. We endow $\mathcal{F}$ with the Borel $\sigma$–field, and by standard theorems on weak convergence we have that also the space $\mathcal{M}_1(\mathcal{F})$ of probability measures on $\mathcal{F}$ is compact.

The main result of this section is to show that the law of the random set $\mathcal{A}_N^a \in \mathcal{M}_1(\mathcal{F})$ converges as $N \to \infty$ to the law of the zero set of a Brownian motion $\{B(t)\}_{t\in[0,1]}$ for $a = \mathrm{f}$ or of a Brownian bridge $\{\beta(t)\}_{t\in[0,1]}$ for $a = \mathrm{c}$.

PROPOSITION 4.15. *If $\delta^\omega = 1$ then as $N \to \infty$*

$$\mathcal{A}_N^{\mathrm{f}} \implies \{t \in [0, 1] : B(t) = 0\}\,, \tag{4.76}$$

$$\mathcal{A}_N^{\mathrm{c}} \implies \{t \in [0, 1] : \beta(t) = 0\}\,. \tag{4.77}$$

The proof of Proposition 4.15 is achieved comparing the law of $\mathcal{A}_N^{\mathrm{f}}$ and $\mathcal{A}_N^{\mathrm{c}}$ with the law of a random set $\mathcal{R}_N$ defined as follows: recalling that $\{\tau_k\}_{k\in\mathbb{N}}$ denotes the sequence of return times of $S$ to zero, we set

$$\mathcal{R}_N := \text{range}\,\{\tau_i/N,\ i \geq 0\}$$

and we look at the law $\mathcal{R}_N$ under the critical infinite volume measure $\mathbf{P}_\omega$ of Proposition 4.10. Observe that under $\mathbf{P}_\omega$ the process $([\tau_k], \tau_k - \tau_{k-1})_{k\in\mathbb{N}}$ is a Markov renewal

process, whose semi–Markov kernel is given by $\Gamma^{=}$. The key point of the proof is given by the following result:

LEMMA 4.16. *The law of* $\{\mathcal{R}_N\}_N$ *under* $\mathbf{P}_\omega$ *converges weakly to the law of the random set* $\{t \geq 0 : B(t) = 0\}$.

The core of the proof (see Step 1 below) uses the theory of *regenerative sets* and their connection with the concept of *subordinator*, see [**32**]. However we point out that it is also possible to give a more standard proof, using tightness and checking "convergence of the finite dimensional distributions": this approach is outlined in § 5.3.

*Proof of Lemma 4.16* We introduce the random set

$$\mathcal{R}_N^{(\beta)} := \text{range}\{\tau_k/N : k \geq 0,\, [\tau_k] = \beta\} \qquad \beta \in \mathbb{S}\,.$$

Notice that $\mathcal{R}_N = \cup_\beta \mathcal{R}_N^{(\beta)}$. Let us also recall the definitions (4.41) and (4.42):

$$\kappa_0^{(\beta)} := \inf\{k \geq 0 : [\tau_k] = \beta\}, \qquad \kappa_{i+1}^{(\beta)} := \inf\{k > \kappa_i^{(\beta)} : [\tau_k] = \beta\},$$

$$T_0^{(\beta)} := \tau_{\kappa_0^{(\beta)}}, \qquad T_i^{(\beta)} := \tau_{\kappa_i^{(\beta)}} - \tau_{\kappa_{i-1}^{(\beta)}}, \qquad i \geq 1.$$

Then $(T_i^{(\beta)})_{i\geq 1}$ is under $\mathbf{P}_\omega$ an IID sequence, independent of $T_0^{(\beta)}$: see the discussion before (4.43). We divide the rest of the proof in two steps.

**Step 1.** This is the main step: we prove that the law of $\mathcal{R}_N^{(\beta)}$ under $\mathbf{P}_\omega$ converges to the law of $\{t \geq 0 : B(t) = 0\}$. For this we follow the proof of Lemma 5 in [**22**].

Let $\{P(t)\}_{t\geq 0}$ be a Poisson process with rate $\gamma > 0$, independent of $(T_i^{(\beta)})_{i\geq 0}$. Then $\sigma_t = [T_1^{(\beta)} + \cdots + T_{P(t)}^{(\beta)}]/N$ forms a non decreasing CAD process with independent stationary increments and $\sigma_0 = 0$: in other words $\sigma = (\sigma_t)_{t\geq 0}$ is a subordinator. Notice that

$$\mathcal{R}_N^{(\beta)} := T_0^{(\beta)}/N + \widehat{\mathcal{R}}_N^{(\beta)}, \qquad \widehat{\mathcal{R}}_N^{(\beta)} := \text{range}\,\{\sigma_t : t \geq 0\}.$$

Thus $\widehat{\mathcal{R}}_N^{(\beta)}$ is the (closed) range of the subordinator $\sigma$, i.e. by [**32**] a regenerative set. As for any Levy process, the law of $\sigma$ is characterized by the Laplace transform of the one-time distributions:

$$\mathbb{E}\left[\exp\left(-\lambda\sigma_t\right)\right] = \exp\left(-t\phi_N(\lambda)\right), \qquad \lambda \geq 0,\ t \geq 0,$$

for a suitable function $\phi_N : [0,\infty) \mapsto [0,\infty)$, called Lévy exponent, which has a canonical representation, the Lévy–Khintchin formula (see e.g. (1.15) in [**32**]):

$$\begin{aligned}\phi_N(\lambda) &= \int_{(0,\infty)} \left(1 - e^{-\lambda s}\right)\, \gamma\, \mathbb{P}(T_1^{(\beta)}/N \in ds) \\ &= \gamma \sum_{n=1}^{\infty} \left(1 - \exp(-\lambda n/N)\right)\, q^{(\beta)}(n)\ .\end{aligned}$$

Notice now that the law of the regenerative set $\widehat{\mathcal{R}}_N^{(\beta)}$ is invariant under the change of time scale $\sigma_t \longrightarrow \sigma_{ct}$, for $c > 0$, and in particular independent of $\gamma > 0$. Since $\phi_N \longrightarrow c\,\phi_N$ under this change of scale, we can fix $\gamma = \gamma_N$ such that $\phi_N(1) = 1$ and this will be implicitly assumed from now on. By Proposition (1.14) of [**32**], the law of $\widehat{\mathcal{R}}_N^{(\beta)}$ is uniquely determined by $\phi_N$.

By the asymptotics of $q^{(\beta)}$ given in (4.44), one directly obtains that $\phi_N(\lambda) \to \lambda^{1/2} =: \Phi_{BM}(\lambda)$ as $N \to \infty$. It is now a matter of applying the result in [**32**, §3] to obtain that $\widehat{\mathcal{R}}_N^{(\beta)}$ converges in law to the regenerative set corresponding to $\Phi_{BM}$. However by direct computation one obtains that the latter is nothing but the zero level set of a Brownian motion, hence $\widehat{\mathcal{R}}_N^{(\beta)} \Rightarrow \{t \in [0,1] :\ B(t) = 0\}$. From the fact that $T_0^{(\beta)}/N$ tends to 0 a.s., the same weak convergence for $\mathcal{R}_N^{(\beta)}$ follows immediately.

**Step 2.** We notice now that $\mathcal{R}_N = \cup_\beta \mathcal{R}_N^{(\beta)}$ is the union of non independent sets. Therefore, although we know that each $\mathcal{R}_N^{(\beta)}$ converges in law to $\{t \geq 0 : B(t) = 0\}$, it is not trivial that $\mathcal{R}_N$ converges to the same limit. We start showing that for every positive $t \geq 0$, the distance between the first point in $\mathcal{R}_N^{(\alpha)}$ after $t$ and the first point in $\mathcal{R}_N^{(\beta)}$ after $t$ converges to zero in probability. More precisely, for any closed set $F \subset [0,\infty)$ we set:

$$d_t(F) := \inf(F \cap (t,\infty)). \tag{4.78}$$

and we claim that for all $\alpha, \beta \in \mathbb{S}$ and $t \geq 0$, $|d_t(\mathcal{R}_N^{(\alpha)}) - d_t(\mathcal{R}_N^{(\beta)})| \to 0$ in probability.

Recalling (4.43) and setting $q^{(\alpha;\beta)}(t) = \mathbf{P}_{\theta_\alpha\omega}(T_0^{(\beta)} = t)$, for all $\epsilon > 0$:

$$\mathbf{P}_\omega\left(d_t(\mathcal{R}_N^{(\alpha)}) \geq d_t(\mathcal{R}_N^{(\beta)}) + \epsilon\right)$$

$$= \sum_k \sum_\gamma \sum_{y=0}^{\lfloor Nt\rfloor} \mathbf{P}_\omega(\tau_k = y, [\tau_k] = \gamma) \sum_{z=\lfloor Nt\rfloor - y+1}^{\infty} \mathbf{P}_{\theta_{\gamma\omega}}(T_0^{(\beta)} = z)\ \mathbf{P}_{\theta_{\beta\omega}}(T_0^{(\alpha)} \geq \lfloor N\epsilon\rfloor)$$

$$= \sum_\gamma \sum_{y=0}^{\lfloor Nt\rfloor} U_{0,\gamma}(y) \sum_{z=\lfloor Nt\rfloor - y+1}^{\infty} q^{(\gamma;\beta)}(z) \sum_{w=\lfloor N\epsilon\rfloor}^{\infty} q^{(\beta;\alpha)}(w).$$

Arguing as in the proof of (4.44), it is easy to obtain the bound: $q^{(\beta;\alpha)}(w) \leq C_1\, w^{-3/2}$, and by (4.45): $U_{0,\gamma}(y) \leq C_2\, y^{-1/2}$, where $C_1$, $C_2$ are positive constants. Then asymptotically

$$\mathbf{P}_\omega\left(d_t(\mathcal{R}_N^{(\alpha)}) \geq d_t(\mathcal{R}_N^{(\beta)}) + \epsilon\right) \ \leq\ \frac{C_3}{N^{1/2}} \left(\int_0^{t/T} dy \int_{(t-y)/T}^{\infty} dz \int_{\epsilon/T}^{\infty} dw\, \frac{1}{y^{1/2}\, z^{3/2}\, w^{3/2}}\right)$$

for some positive constant $C_3$, having used the convergence of the Riemann sums to the corresponding integral. The very same computations can be performed exchanging $\alpha$ with $\beta$, hence the claim is proven.

Now notice that $d_t(\mathcal{R}_N) = \min_{\alpha\in\mathbb{S}} d_t(\mathcal{R}_N^{(\alpha)})$, and since $\mathbb{S}$ is a finite set we have that also $|d_t(\mathcal{R}_N) - d_t(\mathcal{R}_N^{(\beta)})| \to 0$ in probability for any fixed $\beta \in \mathbb{S}$. Since we already know that $\mathcal{R}_N^{(\beta)}$ converges weakly to the law of $\{t \geq 0 : B(t) = 0\}$, the analogous statement for $\mathcal{R}_N$ follows by standard arguments. More precisely, let us look at $(\mathcal{R}_N, \mathcal{R}_N^{(\beta)})$ as a random element of the space $\mathcal{F} \times \mathcal{F}$: by the compactness of $\mathcal{F}$ it suffices to take any convergent subsequence $(\mathcal{R}_{k_n}, \mathcal{R}_{k_n}^{(\beta)}) \Rightarrow (\mathfrak{B}, \mathfrak{C})$ and to show that $\mathbb{P}(\mathfrak{B} \neq \mathfrak{C}) = 0$. By the Portmanteau Theorem it is sufficient to prove that $\lim_{N\to\infty} \mathbf{P}_\omega(\mathcal{R}_N \neq \mathcal{R}_N^{(\beta)}) = 0$, and this is an immediate consequence of the decomposition

$$\left\{\mathcal{R}_N \neq \mathcal{R}_N^{(\beta)}\right\} \ =\ \bigcup_{t\in\mathbb{Q}^+} \bigcup_{n\in\mathbb{N}} \left\{|d_t(\mathcal{R}_N) - d_t(\mathcal{R}_N^{(\beta)})| > 1/n\right\},$$

which holds by the right–continuity of $t \mapsto d_t$. □

*Proof of (4.77).* First, we compute the Radon-Nykodim density of the law of $\mathcal{A}_N^{\mathrm{c}} \cap [0, 1/2]$ with respect to the law of $\mathcal{R}_N^{1/2} := \mathcal{R}_N \cap [0, 1/2]$: for $F = \{t_1/N, \ldots, t_k/N\} \subset [0, 1/2]$ with $0 =: t_0 < t_1 < \cdots < t_k$ integer numbers, the Radon–Nykodim derivative

of the law of $\mathcal{A}^{\rm c}_N \cap [0,1/2]$ with respect to the law of $\mathcal{R}^{1/2}_N$ for $\mathcal{R}^{1/2}_N = F$ is:

$$f^{\rm c}_N(g_{1/2}(F)) \,=\, f^{\rm c}_N(t_k/N) \,=\, \frac{\sum_{n=N/2}^{N} M_{[t_k],[n]}(n-t_k)\, \mathcal{Z}_{[n],[N]}(N-n)}{\mathcal{Z}_{0,[N]}(N)\; Q_{[t_k]}(N/2-t_k)}\; \frac{\xi_0}{\xi_{[t_k]}},$$

where $Q_\alpha(t) := \sum_\beta \sum_{s=t+1}^{\infty} \Gamma^{=}_{\alpha,\beta}(s)$ and for any closed set $F \subset [0,\infty)$ we set:

$$g_t(F) \,:=\, \sup(F \cap [0,t]). \tag{4.79}$$

By (4.46), for all $\varepsilon > 0$ and uniformly in $g \in [0, 1/2-\varepsilon]$:

$$\begin{aligned} f^{\rm c}_N(g) \;&\sim\; \frac{\sum_\gamma L_{[Ng],\gamma}\; \frac{T^2}{2\pi}\, \frac{\xi_\gamma\, \zeta_{[N]}}{\sum_{\gamma,\gamma'} \zeta_\gamma\, L_{\gamma,\gamma'}\, \xi_{\gamma'}}\; T^{-1} \int_0^{1/2} y^{-1/2}\,(1-y-g)^{-3/2}\, dy}{\frac{T^2}{2\pi}\, \frac{\xi_0\, \zeta_{[N]}}{\sum_{\gamma,\gamma'} \zeta_\gamma\, L_{\gamma,\gamma'}\, \xi_{\gamma'}}\, T^{-1}\, \sum_\gamma L_{[Ng],\gamma}\, \xi_\gamma/\xi_{[Ng]}\; 2\,(1/2-g)^{-1/2}}\; \frac{\xi_0}{\xi_{[Ng]}} \\ &=\; \frac{\sqrt{1/2}}{1-g} \,=:\, r(g). \end{aligned}$$

If $\Psi$ is a bounded continuous functional on $\mathcal{F}$ such that $\Psi(F) = \Psi(F \cap [0,1/2])$ for all $F \in \mathcal{F}$, then, setting $Z_B := \{t \in [0,1] : B(t) = 0\}$ and $Z_\beta := \{t \in [0,1] : \beta(t) = 0\}$, we get:

$$\mathbb{E}[\Psi(Z_\beta)] \,=\, \mathbb{E}\left[\Psi(Z_B)\, r(g_{1/2}(Z_B))\right],$$

see formula (49) in [**22**]. By the asymptotics of $f^{\rm c}_N$ we obtain that

$$\mathbb{E}\left[\Psi(\mathcal{A}^{\rm c}_N)\right] \,=\, \mathbb{E}\left[\Psi(\mathcal{R}^{1/2}_N)\, f^{\rm c}_N(g_{1/2}(\mathcal{R}^{1/2}_N))\right] \,\to\, \mathbb{E}\left[\Psi(Z_B)\, r(g_{1/2}(Z_B))\right] \,=\, \mathbb{E}\left[\Psi(Z_\beta)\right]$$

i.e. $\mathcal{A}^{\rm c}_N \cap [0,1/2]$ converges to $Z_\beta \cap [0,1/2]$. Notice now that the distribution of the random set $\{1-t :\, t \in \mathcal{A}^{\rm c}_N \cap [1/2,1]\}$ under $\mathbf{P}^{\rm c}_{N,\omega}$ is the same as the distribution of $\mathcal{A}^{\rm c}_N \cap [0,1/2]$ under $\mathbf{P}^{\rm c}_{N,\overline{\omega}}$, where $\overline{\omega}_{[i]} := \omega_{[N-i]}$. Therefore we obtain that $\mathcal{A}^{\rm c}_N \cap [1/2,1]$ converges to $Z_\beta \cap [0,1/2]$ and the proof is complete.

*Proof of (4.76).* By conditioning on the last zero, we see that if $\Psi$ is a bounded continuous functional on $\mathcal{F}$ then:

$$\mathbb{E}\left[\Psi(\mathcal{A}^{\rm f}_N)\right] \,=\, \sum_{t=0}^{N} \mathbb{E}\left[\Psi(\mathcal{A}^{\rm c}_t)\right]\, \frac{Z^{\rm c}_{t,\omega}}{Z^{\rm f}_{N,\omega}}\, P\,(N-t)\, \exp\left(\widetilde{\Phi}_{[t],[N]}(N-t)\right).$$

We denote by $\beta^t$ a Brownian bridge over the interval $[0,t]$, i.e. a Brownian motion over $[0,t]$ conditioned to be 0 at time $t$, and we set $Z_{\beta^t} := \{s \in [0,t] : \beta^t(s) = 0\}$.

By (4.77), (4.46) and (4.60) we obtain as $N \to \infty$:

$$\mathbb{E}\left[\Psi(\mathcal{A}_N^{\mathrm{f}})\right] \;=\; \sum_{t=0}^{N} \sum_{\gamma} 1_{(t\in\gamma)}\, \mathbb{E}\left[\Psi(\mathcal{A}_t^{\mathrm{c}})\right]\, \frac{Z_{t,\omega}^{\mathrm{c}}}{Z_{N,\omega}^{\mathrm{f}}}\, P\left(N-t\right)\, \exp\left(\widetilde{\Phi}_{\gamma,[N]}(N-t)\right)$$

$$\sim \int_0^1 \mathbb{E}[\Psi(Z_{\beta^t})]\, \frac{1}{t^{\frac12}(1-t)^{\frac12}}\, dt \cdot \sum_{\gamma} \frac{1}{T}\, \frac{T^2}{2\pi}\, \frac{\xi_0\, \zeta_\gamma}{\sum_{\eta,\eta'} \zeta_\eta\, L_{\eta,\eta'}\, \xi_{\eta'}}\, \frac{\widetilde{L}_{\gamma,[N]}}{\xi_0\, \frac{T}{2}\, \frac{\sum_\eta \zeta_\eta\, \widetilde{L}_{\eta,[N]}}{\sum_{\eta,\eta'} \zeta_\eta\, L_{\eta,\eta'}\, \xi_{\eta'}}}$$

$$= \int_0^1 \mathbb{E}[\Psi(Z_{\beta^t})]\, \frac{1}{\pi\, t^{\frac12}(1-t)^{\frac12}}\, dt \;=\; \mathbb{E}[\Psi(Z_B)]. \quad \square$$

4.3.1. *The signs.* To complete the proof of point (3) of Theorem 4.8 in the critical case ($\delta^\omega = 1$) we follow closely the proof given in Section 8 of [**22**]. We have already proven the convergence of the set of zeros and we have to "paste" the excursions. From Section 3 we know that, *conditionally on the zeros*:

- the signs $\{\sigma_k\}_k$ and the absolute values $\{e_k(\cdot)\}_k$ of the excursions are independent;
- the (conditional) law of $e_k(\cdot)$ is the same as under the original random walk measure $\mathbf{P}$.

Furthermore, the weak convergence under diffusive rescaling on $e_k(\cdot)$ towards the Brownian excursion $e(\cdot)$ follows by the arguments described in [**22**]. Then it only remains to concentrate on the signs.

We start with the constrained case: we are going to show that for all $t \in (0,1)$

$$\exists \lim_{N\to\infty} \mathbf{P}_{N,\omega}^{\mathrm{c}}(S_{\lfloor tN\rfloor} > 0) \;=:\; \mathtt{p}_\omega^{=}, \tag{4.80}$$

and the limit is independent of $t$. We point out that actually we should fix the extremities of the excursion embracing $t$, that is we should rather prove that

$$\lim_{N\to\infty} \mathbf{P}_{N,\omega}^{\mathrm{c}}\big(S_{\lfloor tN\rfloor} > 0\,,\; G_{\lfloor tN\rfloor}/N \in (a, a+\varepsilon)\,,\; D_{\lfloor tN\rfloor}/N \in (b, b+\varepsilon)\big) \;=\; \mathtt{p}_\omega^{=}\,, \tag{4.81}$$

for $a < t < b$ and $\varepsilon > 0$ (recall the definition of $G_t$ and $D_t$ in § 4.2). However to lighten the exposition we will stick to (4.80), since proving (4.81) requires only minor changes.

We have, recalling (4.70):

$$\mathbf{P}_{N,\omega}^{\mathrm{c}}(S_{\lfloor tN\rfloor} > 0) = \sum_{\alpha,\beta} \sum_{x<\lfloor tN\rfloor} \sum_{y>\lfloor tN\rfloor} \frac{\mathcal{Z}_{0,\alpha}(x)\, \rho_{\alpha,\beta}^{+}(y-x)\, M_{\alpha,\beta}(y-x)\, \mathcal{Z}_{\beta,[N]}(N-y)}{\mathcal{Z}_{0,[N]}(N)}\,.$$

By Dominated Convergence and by (4.46):

$$\exists \lim_{\substack{N\to\infty \\ N\in\eta}} N^{1/2} \sum_{x<\lfloor tN\rfloor} \sum_{y>\lfloor tN\rfloor} \mathcal{Z}_{0,\alpha}(x)\, \rho^+_{\alpha,\beta}(y-x)\, M_{\alpha,\beta}(y-x)\, \mathcal{Z}_{\beta,\eta}(N-y)$$
$$= \frac{1}{T^2} \int_0^t dx \int_t^1 dy\, [x(y-x)^3(1-y)]^{-\frac{1}{2}} \left(\frac{T^2}{2\pi}\right)^2 \frac{\xi_0\, \zeta_\alpha\, \xi_\beta\, \zeta_\eta}{(\sum_{\gamma,\gamma'} \zeta_\gamma\, L_{\gamma,\gamma'}\, \xi_{\gamma'})^2}\, c_K\, \frac{1}{2}\, \exp(\omega^{(0)}_\beta)$$

see (4.58). We obtain (4.80) with

$$\mathtt{p}^{=}_{\omega} \;:=\; \frac{\sum_{\alpha,\beta} \zeta_\alpha\, c_K\, \frac{1}{2}\, \exp(\omega^{(0)}_\beta)\, \xi_\beta}{\sum_{\alpha,\beta} \zeta_\alpha\, L_{\alpha,\beta}\, \xi_\beta} \tag{4.82}$$

Observe the following: by (4.58),

- if $h_\omega > 0$ then in (4.82) the denominator is equal to the numerator, so that $\mathtt{p}^{=}_{\omega} = 1$.
- if $h_\omega = 0$ and $\Sigma \equiv 0$ then in (4.82) the denominator is equal to *twice* the numerator, so that $\mathtt{p}^{=}_{\omega} = 1/2$.

Now let us consider the free case. We are going to show that for all $t \in (0, 1]$:

$$\exists \lim_{\substack{N\to\infty \\ [N]=\eta}} \mathbf{P}^{\mathrm{f}}_{N,\omega}(S_{\lfloor tN\rfloor} > 0) \;=\; \left(1 - \frac{2\,\arcsin\sqrt{t}}{\pi}\right) \mathtt{p}^{=}_{\omega} \;+\; \frac{2\,\arcsin\sqrt{t}}{\pi}\, \mathtt{q}^{=}_{\omega,\eta} \;=:\; \mathtt{p}^{=,\mathrm{f}}_{\omega,\eta}(t)\,, \tag{4.83}$$

where $\mathtt{p}^{=}_{\omega}$ is the same as above, see (4.82), while $\mathtt{q}^{=}_{\omega,\eta}$ is defined in (4.84) below. We stress again that we should actually fix the values of $G_{\lfloor tN\rfloor}$ and $D_{\lfloor tN\rfloor}$ like in (4.81), proving that the limiting probability is either $\mathtt{p}^{=}_{\omega}$ or $\mathtt{q}^{=}_{\omega,\eta}$ according to whether $D_{\lfloor tN\rfloor} \le N$ or $D_{\lfloor tN\rfloor} > N$, but this will be clear from the steps below. Formula (4.83) follows from the fact that

$$\mathbf{P}^{\mathrm{f}}_{N,\omega}(S_{\lfloor tN\rfloor} > 0) = \sum_{\alpha,\beta} \sum_{x<\lfloor tN\rfloor} \sum_{y>\lfloor tN\rfloor} \frac{\mathcal{Z}_{0,\alpha}(x)\, \rho^+_{\alpha,\beta}(y-x)\, M_{\alpha,\beta}(y-x)\, Z^{\mathrm{f}}_{N-y,\theta_{[y]}\omega}}{Z^{\mathrm{f}}_{N,\omega}}$$
$$+ \sum_{\alpha} \sum_{x<\lfloor tN\rfloor} \frac{\mathcal{Z}_{0,\alpha}(x)\, \rho^+_{\alpha,[N]}(N-x)\, P(N-x)\, \exp\left(\widetilde{\Phi}_{[x],[N]}(N-x)\right)}{Z^{\mathrm{f}}_{N,\omega}}\,.$$

By (4.60), letting $N \to \infty$, the first term in the r.h.s. converges to:

$$\int_0^t \frac{dx}{x^{\frac{1}{2}}} \int_t^1 \frac{dy}{(y-x)^{\frac{3}{2}}} \cdot$$
$$\cdot \sum_{\alpha,\beta} \frac{1}{T^2} \frac{T^2\, \xi_0\, \zeta_\alpha}{2\pi \sum_{\gamma,\gamma'} \zeta_\gamma\, L_{\gamma,\gamma'}\, \xi_{\gamma'}}\, c_K\, \frac{1}{2}\, \exp(\omega_\beta^{(0)})\, \frac{\xi_\beta\, \frac{T}{2}\, \sum_\gamma \zeta_\gamma\, \widetilde{L}_{\gamma,\eta}}{\sum_{\gamma,\gamma'} \zeta_\gamma\, L_{\gamma,\gamma'}\, \xi_{\gamma'}} \cdot \frac{\sum_{\gamma,\gamma'} \zeta_\gamma\, L_{\gamma,\gamma'}\, \xi_{\gamma'}}{\xi_0\, \frac{T}{2}\, \sum_\gamma \zeta_\gamma\, \widetilde{L}_{\gamma,\eta}}$$
$$= \left(1 - \frac{2\, \arcsin \sqrt{t}}{\pi}\right) \cdot \mathtt{p}_\omega^=$$

while letting $N \to \infty$ with $[N] = \eta$, the second term converges to

$$\int_0^t \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}\, \frac{1}{T} \sum_\alpha \frac{T^2\, \xi_0\, \zeta_\alpha}{2\pi \sum_{\gamma,\gamma'} \zeta_\gamma\, L_{\gamma,\gamma'}\, \xi_{\gamma'}}\, c_K \cdot \frac{\sum_{\gamma,\gamma'} \zeta_\gamma\, L_{\gamma,\gamma'}\, \xi_{\gamma'}}{\xi_0\, \frac{T}{2}\, \sum_\gamma \zeta_\gamma\, \widetilde{L}_{\gamma,\eta}}$$
$$= \frac{2\, \arcsin \sqrt{t}}{\pi} \cdot \frac{c_K \sum_\gamma \zeta_\gamma}{\sum_\gamma \zeta_\gamma\, \widetilde{L}_{\gamma,\eta}} \,.$$

Therefore we obtain (4.83) with:

$$\mathtt{q}_{\omega,\eta}^= = \frac{c_K \sum_\gamma \zeta_\gamma}{\sum_\gamma \zeta_\gamma\, \widetilde{L}_{\gamma,\eta}} \,. \tag{4.84}$$

We observe that, by (4.59):

- if $h_\omega > 0$ or if $h_\omega = 0$ and $\Sigma \equiv 0$, then $\mathtt{p}_{\omega,\eta}^{=,\mathrm{f}}(t) = \mathtt{q}_{\omega,\eta}^= = \mathtt{p}_\omega^=$ for all $t$ and $\eta$
- in the remaining case, i.e. if $\omega \in \mathcal{P}^=$, in general $\mathtt{p}_{\omega,\eta}^{=,\mathrm{f}}(t)$ depends on $t$ and $\eta$.

Now that we have proven the convergence of the probabilities of the signs of the excursion, in order to conclude the proof of point (3) of Theorem 4.8 it is enough to argue as in the proof of Theorem 11 in [**22**].

## 5. Appendix

**5.1. An asymptotic result.** We are going to prove that equation (4.44) holds true. Before starting, let us recall an elementary fact about Markov chains. Let $Q_{\alpha,\beta}$ denote the transition matrix of an irreducible, positive recurrent Markov chain, and let us introduce the matrix $Q^{(\gamma)}$ and the (column) vector $|\gamma\rangle$ defined by

$$\left[Q^{(\gamma)}\right]_{\alpha,\beta} := Q_{\alpha,\beta}\, \mathbf{1}_{(\beta \neq \gamma)} \qquad \left[|\gamma\rangle\right]_\alpha := \mathbf{1}_{(\alpha=\gamma)} \,.$$

Since for any $\gamma$ the matrix $Q^{(\gamma)}$ has spectral radius strictly smaller than 1, we can define the geometric series

$$(1 - Q^{(\gamma)})^{-1} \;:=\; \sum_{k=0}^{\infty} \big(Q^{(\gamma)}\big)^k .$$

The interesting point is that, for every $\gamma$, the row vector $\langle\gamma| \cdot (1 - Q^{(\gamma)})^{-1}$ is (a multiple of) the left Perron–Frobenius eigenvector of the matrix $Q$ (by $\langle\gamma|$ we denote the transposed of $|\gamma\rangle$). Similarly the column vector $(1-Q^{(\gamma)})^{-1} \cdot Q \cdot |\gamma\rangle$ is (a multiple of) the right Perron–Frobenius eigenvector of $Q$. More precisely we have

$$\big[\langle\gamma| \cdot (1 - Q^{(\gamma)})^{-1}\big]_\alpha = \frac{\nu_\alpha}{\nu_\gamma} \qquad \big[(1 - Q^{(\gamma)})^{-1} \cdot Q \cdot |\gamma\rangle\big]_\alpha = 1 \,, \tag{4.85}$$

where $\{\nu_\alpha\}_\alpha$ is the invariant measure of the chain, that is $\sum_\alpha \nu_\alpha Q_{\alpha,\beta} = \nu_\beta$ and $\sum_\alpha \nu_\alpha = 1$. Equation (4.85) can be proved by exploiting its probabilistic interpretation in terms of expected number of visits to state $\alpha$ before the first return to site $\gamma$, see [**5**, § I.3].

Next we turn to the asymptotic behavior of $q^{(\beta)}(x)$, giving the law of $T_0^{(\beta)}$ under $\mathbb{P}_\beta$ (recall the notations introduced in § 2.5). With a standard renewal argument, we can express it as

$$q^{(\beta)}(x) \;=\; \sum_{y=0}^{x-1} \sum_{\gamma\in\mathbb{S}} V^{(\beta)}_{\beta,\gamma}(y)\, \Gamma^{=}_{\gamma,\beta}(x-y) \;=\; \big(V^{(\beta)} * \Gamma^{=}\big)_{\beta,\beta}(x) \,, \tag{4.86}$$

where the kernel $V^{(\beta)}$ is defined by

$$V^{(\beta)}_{\alpha,\gamma}(x) \;=\; \sum_{k=0}^{\infty} \big[\big(\Gamma^{(\beta)}\big)^{*k}\big]_{\alpha,\gamma}(x) \,,$$

and we have set $\Gamma^{(\beta)}_{\alpha,\gamma}(x) := \Gamma^{=}_{\alpha,\gamma}(x)\mathbf{1}_{(\gamma\neq\beta)}$. Let us look more closely at both terms in the r.h.s. of (4.86).

- For the semi–Markov kernel $\Gamma^{=}$, recall its definition (4.38), the asymptotic behavior as $x \to \infty$, $[x] = \beta - \gamma$ is given by

$$\Gamma^{=}_{\gamma,\beta}(x) \;\sim\; \frac{\widetilde{L}_{\gamma,\beta}}{x^{3/2}} \qquad \widetilde{L}_{\gamma,\beta} := L_{\gamma,\beta}\frac{\xi_\beta}{\xi_\gamma} \,. \tag{4.87}$$

  Moreover, we have that

$$\sum_{x\in\mathbb{N}} \Gamma^{=}_{\gamma,\beta}(x) \;=\; B_{\gamma,\beta}\frac{\xi_\beta}{\xi_\gamma} \;=:\; \widetilde{B}_{\gamma,\beta} \,. \tag{4.88}$$

- On the other hand, for the kernel $V^{(\beta)}$ we can apply the theory developed in § 2.4 for the case $\delta^\omega < 1$, because the matrix

$$\sum_{x\in\mathbb{N}} \Gamma^{(\beta)}_{\alpha,\gamma}(x) \;=\; \big[\widetilde{B}^{(\beta)}\big]_{\alpha,\gamma}$$

has Perron–Frobenius eigenvalue strictly smaller than 1 (we recall the convention $[Q^{(\beta)}]_{\alpha,\gamma} := Q_{\alpha,\gamma}\mathbf{1}_{(\gamma\neq\beta)}$ for any matrix $Q$). Since

$$\Gamma^{(\beta)}_{\alpha,\gamma}(x) \;\sim\; \frac{\big[\widetilde{L}^{(\beta)}\big]_{\alpha,\gamma}}{x^{3/2}} \qquad x\to\infty\,,\quad [x]=\gamma-\alpha\,,$$

we can apply (4.34) to get that as $x\to\infty$, $[x]=\alpha-\gamma$

$$V^{(\beta)}_{\alpha,\gamma}(x) \;\sim\; \Big(\big[(1-\widetilde{B}^{(\beta)})^{-1}\widetilde{L}^{(\beta)}(1-\widetilde{B}^{(\beta)})^{-1}\big]_{\alpha,\gamma}\Big)\,\frac{1}{x^{3/2}}\,. \tag{4.89}$$

Moreover applying an analog of (4.35) we get that

$$\sum_{y\in\mathbb{N}} V^{(\beta)}_{\alpha,\gamma}(y) = \sum_{k=0}^{\infty}\big[\big(\widetilde{B}^{(\beta)}\big)^k\big]_{\alpha,\gamma} = \big[(1-\widetilde{B}^{(\beta)})^{-1}\big]_{\alpha,\gamma}\,. \tag{4.90}$$

We are finally ready to get the asymptotic behavior of $q^{(\beta)}$. As both $V^{(\beta)}$ and $\Gamma^{=}$ have a $x^{-3/2}$–like tail, it is easy to check from (4.86) that as $x\to\infty$, $x\in T\mathbb{N}$

$$q^{(\beta)}(x) \;\sim\; \sum_{\gamma\in\mathbb{S}}\left\{\Big(\sum_{y\in\mathbb{N}} V^{(\beta)}_{\beta,\gamma}(y)\Big)\Gamma^{=}_{\gamma,\beta}(x) \;+\; V^{(\beta)}_{\beta,\gamma}(x)\Big(\sum_{y\in\mathbb{N}}\Gamma^{=}_{\gamma,\beta}(y)\Big)\right\},$$

and applying (4.90), (4.87), (4.89) and (4.88) we get that $q^{(\beta)}(x)\sim c_\beta/x^{3/2}$ as $x\to\infty$, $x\in T\mathbb{N}$, with

$$\begin{aligned} c_\beta \;&=\; \Big[(1-\widetilde{B}^{(\beta)})^{-1}\cdot\widetilde{L}\Big]_{\beta,\beta} \;+\; \Big[(1-\widetilde{B}^{(\beta)})^{-1}\cdot\widetilde{L}^{(\beta)}\cdot(1-\widetilde{B}^{(\beta)})^{-1}\cdot\widetilde{B}\Big]_{\beta,\beta} \\ &=\; \Big[(1-\widetilde{B}^{(\beta)})^{-1}\cdot\widetilde{L}\cdot(1-\widetilde{B}^{(\beta)})^{-1}\cdot\widetilde{B}\Big]_{\beta,\beta} \\ &=\; \langle\beta|\cdot(1-\widetilde{B}^{(\beta)})^{-1}\cdot\widetilde{L}\cdot(1-\widetilde{B}^{(\beta)})^{-1}\cdot\widetilde{B}\cdot|\beta\rangle\,. \end{aligned}$$

To obtain the second equality we have used the fact that

$$\Big[(1-\widetilde{B}^{(\beta)})^{-1}\cdot\widetilde{B}\Big]_{\beta,\beta} \;=\; \Big[\langle\beta|\cdot(1-\widetilde{B}^{(\beta)})^{-1}\cdot\widetilde{B}\Big]_{\beta} \;=\; 1\,,$$

which follows from (4.85) applied to the matrix $Q=\widetilde{B}$. Again from (4.85) we get

$$c_\beta \;=\; \frac{1}{\nu_\beta}\sum_{\alpha,\gamma\in\mathbb{S}}\nu_\alpha\widetilde{L}_{\alpha,\gamma}\,,$$

where $\{\nu_\alpha\}_\alpha$ is the invariant measure (that is the normalized left Perron–Frobenius eigenvector) of the matrix $\widetilde{B}$. However from the definition (4.88) of $\widetilde{B}$ it is immediate

to see that $\{\nu_\alpha\} = \{\zeta_\alpha\,\xi_\alpha\}$, and recalling the definition (4.87) of $\widetilde{L}$ we obtain the expression for $c_\beta$ we were looking for:

$$c_\beta \;=\; \frac{1}{\zeta_\beta\,\xi_\beta}\sum_{\alpha,\gamma}\zeta_\alpha\,L_{\alpha,\gamma}\,\xi_\gamma. \tag{4.91}$$

**5.2. Some computations on the thermodynamic limit measure.** We want now to give a description of the typical paths under $\mathbf{P}_\omega^{a,\eta}$ in the delocalization regime, i.e. when $\delta^\omega < 1$. We are going to compute the distribution of two interesting random variables under $\mathbf{P}_\omega^{a,\eta}$ in this case: the last return to zero and the total number of returns to zero. Other analogous computations are possible using the same procedure.

5.2.1. *The last return to zero.* We want to study the law under $\mathbf{P}_\omega^{a,\eta}$ of the last zero $\ell := \sup\{i \in \mathbb{N} : S_i = 0\}$ in the strictly delocalized regime. For simplicity we consider the case $a = \mathrm{c}$, the case $a = \mathrm{f}$ being completely analogous. We compute first the law of $\ell_k := \sup\{i \le k : S_i = 0\}$ with $k \in \mathbb{N}$: for $x \le k < N$ and $N \in \eta$:

$$\mathbf{P}^{\mathrm{c}}_{N,\omega}(\ell_k \ge x) \;=\; \sum_{y=x}^{k}\mathcal{Z}_{0,[y]}(y)\sum_{z=k+1}^{N}\frac{M_{[y],[z]}(z-y)\,\mathcal{Z}_{[z],\eta}(N-z)}{\mathcal{Z}_{0,\eta}(N)} \tag{4.92}$$

By (4.57) and (4.61) we obtain:

$$\lim_{\substack{N\to\infty\\ N\in\eta}}\mathbf{P}^{\mathrm{c}}_{N,\omega}(\ell_k \ge x) = \sum_{y=x}^{k}\mathcal{Z}_{0,[y]}(y)\left[\sum_{z=0}^{\infty}\frac{L_{[y],\eta-[z]}}{\Lambda^{\mathrm{c}}_{0,\eta}}\,\mathcal{Z}_{\eta-[z],\eta}(z) + \sum_{z=k+1}^{\infty}M_{[y],[z]}(z-y)\,\frac{\Lambda^{\mathrm{c}}_{[z],\eta}}{\Lambda^{\mathrm{c}}_{0,\eta}}\right]$$

Notice now that, by (4.63):

$$\sum_{z=0}^{\infty}L_{[y],\eta-[z]}\,\mathcal{Z}_{\eta-[z],\eta}(z) \;=\; \sum_{\gamma}L_{[y],\gamma}\sum_{z=0}^{\infty}\mathcal{Z}_{\gamma,\eta}(z) \;=\; \left[L\cdot(I-B)^{-1}\right]_{[y],\eta} \;=\; \mu^{\mathrm{c}}_{[y],\eta}. \tag{4.93}$$

Therefore, we have proven that:

$$\mathbf{P}^{\mathrm{c},\eta}_\omega(\ell_k \ge x) \;=\; \lim_{\substack{N\to\infty\\ N\in\eta}}\mathbf{P}^{\mathrm{c}}_{N,\omega}(\ell_k \ge x) \;=\sum_{y=x}^{k}\mathcal{Z}_{0,[y]}(y)\left[\frac{\mu^{\mathrm{c}}_{[y],\eta}}{\Lambda^{\mathrm{c}}_{0,\eta}} + \sum_{z=k+1}^{\infty}M_{[y],[z]}(z-y)\,\frac{\Lambda^{\mathrm{c}}_{[z],\eta}}{\Lambda^{\mathrm{c}}_{0,\eta}}\right]$$

and letting $k \to \infty$ we obtain:

$$\mathbf{P}^{\mathrm{c},\eta}_\omega(\ell \ge x) \;=\; \sum_{y=x}^{\infty}\mathcal{Z}_{0,[y]}(y)\,\frac{\mu^{\mathrm{c}}_{[y],\eta}}{\Lambda^{\mathrm{c}}_{0,\eta}}.$$

For the proof of Lemma 4.14 above, notice for instance that by (4.92):

$$\mathbf{P}^{\mathrm{c}}_{N,\omega}(G_{N/2} \geq L) \,=\, \mathbf{P}^{\mathrm{c}}_{N,\omega}(\ell_{N/2} \geq L) \tag{4.94}$$

$$\leq\, C_1\, N^{3/2} \sum_{t=L}^{\lfloor N/2 \rfloor} t^{-3/2} \sum_{k=\lfloor N/2 \rfloor+1}^{N+1} (k-t)^{-3/2}\, (N+2-k)^{-3/2} \,\leq\, C_2\, L^{-1/2},$$

where $C_1, C_2$ are positive constants.

5.2.2. *The number of returns to zero.* Analogously, we want to study the law of the total number of returns to zero $\mathcal{N} := \#\{i \in \mathbb{N} : S_i = 0\}$ under $\mathbf{P}^{\mathrm{c},\eta}_{\omega}$. We study first Let $\mathcal{N}_K := \#\{i : 1 \leq i \leq K : S_i = 0\}$ for $k \in \mathbb{N}$. For $k \leq K$ and $N \in \eta$:

$$\mathbf{P}^{\mathrm{c}}_{N,\omega}(\mathcal{N}_K = k) \,=\, \sum_{x=1}^{K} M^{*k}_{0,[x]}(x) \sum_{y=K+1}^{N} \frac{M_{[x],[y]}(y-x)\, \mathcal{Z}_{[y],\eta}(N-y)}{\mathcal{Z}_{0,\eta}(N)}$$

Then by (4.57) and (4.61):

$$\lim_{\substack{N\to\infty \\ N\in\eta}} \mathbf{P}^{\mathrm{c}}_{N,\omega}(\mathcal{N}_K = k)$$

$$= \sum_{x=0}^{K} M^{*k}_{0,[x]}(x) \left[ \sum_{y=0}^{\infty} \frac{L_{[x],\eta-[y]}}{\Lambda^{\mathrm{c}}_{0,\eta}} \mathcal{Z}_{\eta-[y],\eta}(y) + \sum_{z=K+1}^{\infty} M_{[x],[y]}(y-x) \frac{\Lambda^{\mathrm{c}}_{[y],\eta}}{\Lambda^{\mathrm{c}}_{0,\eta}} \right].$$

By (4.93), letting $K \to \infty$ we obtain:

$$\mathbf{P}^{\mathrm{c},\eta}_{\omega}(\mathcal{N} = k) \,=\, \frac{1}{\Lambda^{\mathrm{c}}_{0,\eta}} \left[B^k \cdot \mu^{\mathrm{c}}\right]_{0,\eta}.$$

**5.3. On the weak convergence of the critical zero set.** We are going to outline an alternative proof of Lemma 4.16, that is we are going to show that when $\delta^{\omega} = 1$ as $N \to \infty$

$$\mathcal{R}_N \text{ under } \mathbf{P}_{\omega} \implies \{t \geq 0 : B(t) = 0\}. \tag{4.95}$$

To keep the notation transparent, it is convenient to denote by $\mathcal{G}_N \in \mathcal{M}_1(\mathcal{F})$ the image law of $\mathcal{R}_N$ under $\mathbf{P}_{\omega}$. That is $\mathcal{G}_N$ is a probability law on $\mathcal{F}$ (the class of all closed subsets of $\mathbb{R}^+$) defined for a measurable subset $A \subseteq \mathcal{F}$ by

$$\mathcal{G}_N(A) \,:=\, \mathbf{P}_{\omega}\big(\mathcal{R}_N \in A\big).$$

In the same way the law of $\{t \geq 0 : B(t) = 0\}$ will be denoted by $\mathcal{G}^{(BM)}$. Then we can reexpress our goal (4.95) as

$$\mathcal{G}_N \implies \mathcal{G}^{(BM)} . \tag{4.96}$$

Remember the definition (4.78) of the mapping $d_t : \mathcal{F} \mapsto \mathbb{R}^+ \cup \{+\infty\}$. We claim that to prove (4.96) it suffices to show that, for every $n \in \mathbb{N}$ and for all $t_1, \ldots, t_n \in \mathbb{R}$, the law of the vector $(d_{t_1}, \ldots, d_{t_n})$ under $\mathcal{G}_N$ converges to the law of the same vector under $\mathcal{G}^{(BM)}$:

$$(d_{t_1}, \ldots, d_{t_n}) \circ \big(\mathcal{G}_N\big)^{-1} \implies (d_{t_1}, \ldots, d_{t_n}) \circ \big(\mathcal{G}^{(BM)}\big)^{-1} . \tag{4.97}$$

The intuitive explanation of why (4.97) should imply (4.96) is that an element $\xi \in \mathcal{F}$ can be *identified* with the process $\{d_t(\xi)\}_{t\in\mathbb{R}^+}$, since $\xi = \{t \in \mathbb{R}^+ : \ d_{t-}(\xi) = t\}$. Hence the convergence in $\mathcal{M}_1(\mathcal{F})$ can be read in terms of the random process $\{d_t(\cdot)\}_{t\in\mathbb{R}^+}$, and using the compactness of $\mathcal{M}_1(\mathcal{F})$ it turns out that (4.97) is indeed sufficient to ensure (4.96). Let us sketch more in detail these arguments.

(1) The Borel $\sigma$–field of $\mathcal{F}$ coincides with $\sigma(\{d_t\}_{t\in\mathbb{R}^+})$, i.e. with the $\sigma$-field generated by $\{d_t\}_{t\in\mathbb{R}^+}$, and also with $\sigma(\{d_t\}_{t\in I})$ where $I$ is any dense subset of $\mathbb{R}^+$.

(2) Suppose that we are given $\{\nu_k\}$, $\nu \in \mathcal{M}_1(\mathcal{F})$ such that $\nu_k \Rightarrow \nu$: this fact does not entail the convergence of all the finite dimensional marginals of $\{d_t\}$, that is *it is not true* that the law of the vector $(d_{t_1}, \ldots, d_{t_n})$ under $\nu_k$ converges to the law of the same vector under $\nu$, because the mappings $d_t(\cdot)$ are not continuous on $\mathcal{F}$. Nevertheless one can show that this convergence does hold for *almost all* choices of the indexes $t_1, \ldots, t_n$. More precisely, given any measure $\nu \in \mathcal{M}_1(\mathcal{F})$ there exists a subset $I_\nu \subseteq \mathbb{R}^+$ with $Leb({I_\nu}^c) = 0$ with the following property: for any sequence $\{\nu_k\}$ with $\nu_k \Rightarrow \nu$, for any $n \in \mathbb{N}$ and for all $t_1, \ldots, t_n \in I_\nu$, the law of the vector $(d_{t_1}, \ldots, d_{t_n})$ under $\nu_k$ converges as $k \to \infty$ to the law of the same vector under $\nu$. This is a well-known feature of processes whose discontinuity points form a negligible set, in particular CADLAG processes: in fact the set $I_\nu$ can be chosen as the set of $t \in \mathbb{R}^+$ such that $\nu\big\{\xi : \ d_{t-}(\xi) = d_t(\xi)\big\} = 1$, because $d_{t-}(\xi) = d_t(\xi)$ implies that $d_t(\cdot)$ is continuous at $\xi$.

(3) Since $\mathcal{M}_1(\mathcal{F})$ is compact, to prove (4.96) it suffices to show that any convergent subsequence of $\{\mathcal{G}_N\}_N$ converges to $\mathcal{G}^{(BM)}$. Thus we take a convergent subsequence $\mathcal{G}_{k_n} \Rightarrow \nu$ for some $\nu \in \mathcal{M}_1(\mathcal{F})$ and we want to prove that $\nu = \mathcal{G}^{(BM)}$. By point (2) there exists a dense subset $I_\nu \subseteq \mathbb{R}^+$ such that for $t_1, \dots, t_n \in I_\nu$ the law of the vector $(d_{t_1}, \dots, d_{t_n})$ under $\mathcal{G}_{k_n}$ converges to the law of the same vector under $\nu$, and since we are assuming that (4.97) holds this means that the vector $(d_{t_1}, \dots, d_{t_n})$ has the same law under $\nu$ and under $\mathcal{G}^{(BM)}$. This is equivalent to say that $\nu$ and $\mathcal{G}^{(BM)}$ coincide on the $\sigma$–field $\sigma(\{d_t\}_{t\in I_\nu})$, and by point (1) it follows that indeed $\nu = \mathcal{G}^{(BM)}$.

Thus it only remains to show that (4.97) holds, and this can be done by direct computation. For simplicity we consider only the case $n = 1$ of the one–time marginals, but everything can be extended to the case $n > 1$.

For any $t > 0$ the law of $d_t$ under $\mathcal{G}^{(BM)}$ is given by

$$\mathcal{G}^{(BM)}\big(d_t \in \mathrm{d}y\big) \;=\; \frac{t^{1/2}}{\pi\, y(y-t)^{1/2}}\, \mathbf{1}_{(y>t)}\, dy \;=:\; \rho_t(y)\, dy\,,$$

see [**60**]. Hence we have to show that for every $x \in \mathbb{R}^+$

$$\lim_{N\to\infty} \mathbf{P}_\omega\big(d_t(\mathcal{R}_N) > x\big) \;=\; \int_x^\infty \rho_t(y)\, dy\,.$$

We recall that $\mathcal{R}_N = \mathrm{range}\{\tau_n/N :\ n \geq 0\}$ is the range of the process $\{\tau_n\}_{n\in\mathbb{N}}$ rescaled by a factor $1/N$, and that under $\mathbf{P}_\omega$ the process $\{\tau_n\}_{n\in\mathbb{N}}$ is a Markov–renewal process with semi–Markov kernel $\Gamma^{=}_{\alpha,\beta}(x)$ defined by (4.38). We also use the notation $U_{\alpha,\beta}(x)$ for the corresponding Markov–Green function, defined by (4.39). Then using the Markov property we get

$$\begin{aligned}
\mathbf{P}_\omega\big(d_t(\mathcal{R}_N) > x\big) &= \sum_{k\in\mathbb{N}} \mathbf{P}_\omega\big(\tau_k \leq Nt\,,\ \tau_{k+1} > Nx\big) \\
&= \sum_{\alpha,\beta\in\mathbb{S}} \sum_{y=1}^{Nt} \sum_{w=Nx}^{\infty} \sum_{k\in\mathbb{N}} \mathbf{P}_\omega\big(\tau_k = y\,,\ [\tau_k] = \alpha\big)\, \mathbf{P}_{\theta_y\omega}\big(\tau_1 = w - y\,,\ [\tau_1] = \beta - \alpha\big) \\
&= \sum_{\alpha,\beta\in\mathbb{S}} \sum_{y=1}^{Nt} U_{0,\alpha}(y) \sum_{w=Nx}^{\infty} \Gamma^{=}_{\alpha,\beta}(w-y)
\end{aligned}$$

The asymptotic behavior of the terms appearing in the expression can be extracted from (4.45) and (4.30): the net result is that as $z \to \infty$

$$\sqrt{z}\, U_{0,\alpha}(z) \;\xrightarrow{[z]=\alpha}\; \frac{T^2}{2\pi}\, \frac{\zeta_\alpha \xi_\alpha}{\sum_{\gamma,\gamma'} \zeta_\gamma L_{\gamma,\gamma'} \xi_{\gamma'}} \;=:\; c^U_{0,\alpha}$$

$$z^{3/2}\, \Gamma^{=}_{\alpha,\beta}(z) \;\xrightarrow{[z]=\beta-\alpha}\; \frac{\xi_\beta}{\xi_\alpha}\, L_{\alpha,\beta} \;=:\; c^\Gamma_{\alpha,\beta}\,.$$

Therefore we have as $N \to \infty$

$$\mathbf{P}_\omega\big(d_t(\mathcal{R}_N) > x\big) \;\sim\; \sum_{\alpha,\beta\in\mathbb{S}} c^U_{0,\alpha}\, c^\Gamma_{\alpha,\beta} \sum_{y=1}^{Nt} \frac{1}{\sqrt{y}}\, \mathbf{1}_{([y]=\alpha)} \sum_{w=Nx}^{\infty} \frac{1}{(w-y)^{3/2}}\, \mathbf{1}_{([w]=\beta)}$$

$$\sim\; \frac{1}{T^2} \Bigg( \sum_{\alpha,\beta\in\mathbb{S}} c^U_{0,\alpha}\, c^\Gamma_{\alpha,\beta} \Bigg) \frac{1}{N^2} \sum_{s\in(0,\frac{t}{T})\cap\frac{\mathbb{Z}}{N}} \frac{1}{\sqrt{s}} \sum_{u\in(\frac{x}{T},\infty)\cap\frac{\mathbb{Z}}{N}} \frac{1}{(u-s)^{3/2}}\,,$$

and from the explicit expressions for $c^U_{0,\alpha}$, $c^\Gamma_{\alpha,\beta}$ together with the convergence of the Riemann sums to the corresponding integral we get

$$\exists \lim_{N\to\infty} \mathbf{P}_\omega\big(d_t(\mathcal{R}_N) > x\big) \;=\; \frac{1}{2\pi} \int_0^{t/T} \mathrm{d}s\, \frac{1}{\sqrt{s}} \int_{x/T}^{\infty} \mathrm{d}u\, \frac{1}{(u-s)^{3/2}}$$

$$=\; \frac{1}{\pi} \int_0^{t/T} \mathrm{d}s\, \frac{1}{\sqrt{s}}\, \frac{1}{\sqrt{x/T - s}} \;=\; \frac{1}{\pi} \int_0^{t} \mathrm{d}y\, \frac{1}{\sqrt{y}}\, \frac{1}{\sqrt{x-y}} \;=\; \int_x^\infty \mathrm{d}z\, \rho_t(z)\,,$$

that is what was to be proven.

**5.4. A localization argument.** Let us give a proof that for the *copolymer near a selective interface* model, described in § 1.1, the charge $\omega$ never belongs to $\mathcal{P}$ (see (4.17) for the definition of $\mathcal{P}$). More precisely, we are going to show that if $h_\omega = 0$ and $\Sigma \not\equiv 0$ then $\delta^\omega > 1$, that is the periodic copolymer with zero–mean, nontrivial charges is always localized. As a matter of fact this is an immediate consequence of the estimates on the critical line obtained in [**11**]. However we want to give here an explicit proof, both because it is more direct and because the model studied in [**11**] is built over the simple random walk measure, corresponding to $p = 1/2$ with the language of § 1, while we consider the case $p < 1/2$.

We give some preliminary notation: given an *irreducible* $T \times T$ matrix $Q_{\alpha,\beta}$ with *nonnegative entries*, its Perron–Frobenius eigenvalue (= spectral radius) will be denoted by $\mathtt{Z} = \mathtt{Z}(Q)$ and the corresponding left and right eigenvectors (with any normalization) will be denoted by $\{\zeta_\alpha\}, \{\xi_\alpha\}$. Being a simple root of the characteristic polynomial, $\mathtt{Z}(Q)$ is an *analytic* function of the entries of $Q$, and one can check

that

$$\frac{\partial \mathtt{Z}}{\partial Q_{\alpha,\beta}} \;=\; \frac{\zeta_\alpha\,\xi_\beta}{\big(\sum_\gamma \zeta_\gamma \xi_\gamma\big)}\,. \tag{4.98}$$

Hence $\mathtt{Z}(Q)$ is a strictly increasing function of each of the entries of $Q$. We also point out a result proved by Kingman [**42**]: if the matrix is a function of a real parameter $Q = Q(t)$ such that all the entries $Q_{\alpha,\beta}(t)$ are *log–convex* functions of $t$ (that is $t \mapsto \log Q_{\alpha,\beta}(t)$ is convex for all $\alpha, \beta$), then also $t \mapsto \mathtt{Z}(Q(t))$ is a log–convex function of $t$.

Next we come to the *copolymer near a selective interface* model: with reference to the general Hamiltonian (4.3), we are assuming that $\omega_n^{(0)} = \widetilde{\omega}_n^{(0)} = 0$ and $h_\omega = 0$ (where $h_\omega$ was defined in (4.8)). In this case the integrated Hamiltonian $\Phi_{\alpha,\beta}(\ell)$, see (4.13), is given by

$$\Phi_{\alpha,\beta}(\ell) = \begin{cases} 0 & \text{if } \ \ell = 1 \ \text{ or } \ \ell \notin \beta - \alpha \\ \log\Big[\frac{1}{2}\Big(1 + \exp\big(\Sigma_{\alpha,\beta}\big)\Big)\Big] & \text{if } \ \ell > 1 \ \text{ and } \ \ell \in \beta - \alpha \end{cases}.$$

We recall that the law of the first return to zero of the original walk is denoted by $K(\cdot)$, see (4.11), and we introduce the function $q : \mathbb{S} \to \mathbb{R}^+$ defined by

$$q(\gamma) := \sum_{x \in \mathbb{N},\ [x] = \gamma} K(x)$$

(notice that $\sum_\gamma q(\gamma) = 1$). Then the matrix $B_{\alpha,\beta}$ defined by (4.15) becomes

$$B_{\alpha,\beta} \;=\; \begin{cases} \frac{1}{2}\Big(1 + \exp\big(\Sigma_{\alpha,\beta}\big)\Big)\, q(\beta - \alpha) & \text{if } \ \beta - \alpha \neq [1] \\ K(1) \;+\; \frac{1}{2}\Big(1 + \exp\big(\Sigma_{\alpha,\alpha+[1]}\big)\Big) \cdot \big(q([1]) - K(1)\big) & \text{if } \ \beta - \alpha = [1] \end{cases} \tag{4.99}$$

By (4.16), to prove localization we have to show that the Perron–Frobenius eigenvalue of the matrix $(B_{\alpha,\beta})$ is strictly greater than 1, that is $\mathtt{Z}(B) > 1$.

Applying the elementary convexity inequality $(1 + \exp(x))/2 \geq \exp(x/2)$ to (4.99) we get

$$B_{\alpha,\beta} \;\geq\; \widetilde{B}_{\alpha,\beta} \;:=\; \begin{cases} \exp\big(\Sigma_{\alpha,\beta}/2\big)\, q(\beta - \alpha) & \text{if } \ \beta - \alpha \neq [1] \\ K(1) \;+\; \exp\big(\Sigma_{\alpha,\alpha+[1]}/2\big) \cdot \big(q([1]) - K(1)\big) & \text{if } \ \beta - \alpha = [1] \end{cases}. \tag{4.100}$$

By hypothesis $\Sigma_{\alpha_0,\beta_0} \neq 0$ for some $\alpha_0, \beta_0$, therefore the inequality above is strict for $\alpha = \alpha_0$, $\beta = \beta_0$. We have already observed that the P–F eigenvalue is a strictly

increasing function of the entries of the matrix, hence $\mathtt{Z}(B) > \mathtt{Z}(\widetilde{B})$. Therefore it only remains to show that $\mathtt{Z}(\widetilde{B}) \geq 1$, and the proof will be completed.

Again an elementary convexity inequality applied to the second line of (4.100) yields

$$\widetilde{B}_{\alpha,\beta} \;\geq\; \widehat{B}_{\alpha,\beta} \;:=\; \exp\big(\,c(\beta-\alpha)\,\Sigma_{\alpha,\beta}/2\,\big)\cdot q(\beta-\alpha) \tag{4.101}$$

where

$$c(\gamma) \;:=\; \begin{cases} 1 & \text{if } \;\gamma \neq [1] \\ \frac{q([1])-K(1)}{q([1])} & \text{if } \;\gamma = [1] \end{cases} .$$

We are going to prove that $\mathtt{Z}(\widehat{B}) \geq 1$. Observe that setting $v_\alpha := \Sigma_{[0],\alpha}$ we can write

$$\Sigma_{\alpha,\beta} = \Sigma_{[0],\beta} - \Sigma_{[0],\alpha} = v_\beta - v_\alpha\,.$$

Then we make a similarity transformation via the matrix $L_{\alpha,\beta} := \exp(v_\beta/2)\,\mathbf{1}_{(\beta=\alpha)}$, getting

$$\begin{aligned} C_{\alpha,\beta} \;&:=\; \big[L\cdot\widehat{B}\cdot L^{-1}\big]_{\alpha,\beta} \;=\; \exp\Big(\big(c(\beta-\alpha)-1\big)\Sigma_{\alpha,\beta}/2\Big)\cdot q(\beta-\alpha) \\ &=\; \exp\Big(d\,\Sigma_{\alpha,\alpha+[1]}\,\mathbf{1}_{(\beta-\alpha=1)}\Big)\cdot q(\beta-\alpha)\,, \end{aligned}$$

where we have introduced the constant $d := -K(1)/(\,2\,q([1])\,)$. Of course $\mathtt{Z}(\widehat{B}) = \mathtt{Z}(C)$. Also notice that by the very definition of $\Sigma_{\alpha,\beta}$ we have $\Sigma_{\alpha,\alpha+[1]} = \omega^{(-1)}_{\alpha+[1]} - \omega^{(+1)}_{\alpha+[1]}$, hence the hypothesis $h_\omega = 0$ yields $\sum_{\alpha\in\mathbb{S}}(\Sigma_{\alpha,\alpha+[1]}) = 0$.

Thus we are finally left with showing that $\mathtt{Z}(C) \geq 1$ where $C_{\alpha,\beta}$ is an $\mathbb{S}\times\mathbb{S}$ matrix of the form

$$C_{\alpha,\beta} \;=\; \exp\big(\,w_\alpha\,\mathbf{1}_{(\beta-\alpha=1)}\,\big)\cdot q(\beta-\alpha) \qquad \text{where} \qquad \sum_\alpha w_\alpha = 0 \qquad \sum_\gamma q(\gamma) = 1\,.$$

To this end, we introduce an interpolation matrix

$$C(t)_{\alpha,\beta} \;:=\; \exp\big(\,t\cdot w_\alpha\,\mathbf{1}_{(\beta-\alpha=1)}\,\big)\cdot q(\beta-\alpha)\,,$$

defined for $t\in\mathbb{R}$, and notice that $C(1) = C$. Let us denote by $\eta(t) := \mathtt{Z}\big(C(t)\big)$ the Perron–Frobenius eigenvalue of $C(t)$: as the entries of $C(t)$ are log–convex functions of $t$, it follows that also $\eta(t)$ is log–convex, therefore in particular convex. Moreover $\eta(0) = 1$ (the matrix $C(0)$ is bistochastic) and using (4.98) one easily checks that $\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)|_{t=0} = 0$. Since clearly $\eta(t)\geq 0$ for all $t\in\mathbb{R}$, by convexity it follows that indeed $\eta(t)\geq 1$ for all $t\in\mathbb{R}$, and the proof is complete.

CHAPTER 5

# A general copolymer model with continuous increments

In this chapter we introduce and study a modification of the copolymer near a selective interface model defined in Chapter 1a. The difference is that we change the reference measure $\mathbf{P}$ on which the model is built: instead of the law of the simple symmetric random walk on $\mathbb{Z}$, we allow $\mathbf{P}$ to be the law of a more general real random walk (see § 1.1 for some motivations for this choice). More precisely we will consider the case when the typical increment of the walk is bounded, centered and has an absolutely continuous law (plus a standard regularity hypothesis on the density in order to apply the Central Local Limit Theorem). About the charges $\{\omega_n\}_n$, we focus on the *random case*.

Besides giving a proof of the existence of the free energy (which in this setting is not trivial) we analyze the phase diagram of the model, pointing out the close analogies with the simple random walk case described in Chapter 1a. We also consider briefly the issue of extending to this model the coarse graining of the free energy expressed by Theorem 1a.5 of Chapter 1a (work in progress), giving some partial results in this direction and discussing what is still missing.

## 1. The model

**1.1. Motivations.** Up to now all the polymer models we have worked on were built as modifications of the law of a $(1+1)$–dimensional directed walk, the latter being of the form $\{(n, S_n)\}_n$ where $\{S_n\}$ is a symmetric nontrivial random walk on $\mathbb{Z}$ with increments in $\{0, \pm 1\}$. One could object that from the viewpoint of modeling a real polymer chain these restrictions are too severe, that is we are working with oversimplified models. A possible answer to this objection is that the phenomena that we want to understand, like localization/delocalization, *should not* depend too much on the microscopic details of the model, at least at a qualitative level. Even more, one could maintain that the essential reasons of the phenomenon we are investigating may be even more visible in an extremely simplified model. The paradigm in this

direction is given by the *Ising model*, which despite its extreme simplicity is able to explain the origin of the ferromagnetic behavior.

Nevertheless, it would be certainly very interesting to be able to study more refined models, at least for the purpose of understanding to what extent the results one obtains are indeed independent of the microscopic details of the models. In our situation, possibly the more direct refinement that one could think of considering is to work with a $(1+1)$–dimensional directed walk $\{(n, S_n)\}_n$ in which $\{S_n\}_n$ is allowed to be a generic real random walk.

It may not be a priori evident why this should be a more realistic model: after all it is always a directed walk model in which the first component is deterministic. However we claim that, for the purpose of modeling a copolymer in the proximity of a flat interface, any $d$–dimensional random walk $\{Y_n\}_n$ is essentially equivalent to a $(1+1)$–dimensional directed walk $\{(n, S_n)\}_n$ for a suitable choice of the real random walk $\{S_n\}_n$. In fact, assuming that the interface is the hyperplane $\{x_d = 0\}$ and denoting by $\boldsymbol{Q}$ the law of the $d$–dimensional random walk $\{Y_n\}_n$, the analogue of the polymer measure introduced in Chapter 1a, see equation (1a.1), can be written as

$$\frac{\mathrm{d}\boldsymbol{Q}^{\lambda,h}_{N,\omega}}{\mathrm{d}\boldsymbol{Q}}(Y) \;\propto\; \lambda \sum_{n=1}^{N} (\omega_n + h)\, \mathrm{sign}\left((Y_n)_d\right),$$

where by $(Y_n)_d$ we mean the $d$–th component of the vector $Y_n \in \mathbb{R}^d$. Now observe that for the purpose of investigating the localization/delocalization phenomenon it is sufficient to look at the $d$–th coordinate $\{(Y_n)_d\}_n$ under the polymer measure, which simply amounts to defining the copolymer model over the $(1+1)$–dimensional directed walk $(n, (Y_n)_d)$ (observe that $\{(Y_n)_d\}_n$ is a real random walk). A graphical representation of this correspondence is given in Fig. 5.1 for the case of a two–dimensional random walk in which the step law is concentrated on the surface of a sphere (which means that the distance between monomers is fixed).

We take the above considerations as sufficient motivation and we proceed to the definition and analysis of the model.

**1.2. Definition of the model.** We take a real random walk $\{S_n\}_{n\geq 0}$, that is $S_0 = 0$ and $S_n - S_{n-1} =: X_n$ with $\{X_n\}_n$ an IID sequence. The law of the walk will be denoted by $\mathbf{P}$. Our assumptions are that:

- the typical step of the walk is bounded (to be definite we take $|X_1| \leq 1$) and centered: $\mathbf{E}[X_1] = 0$;

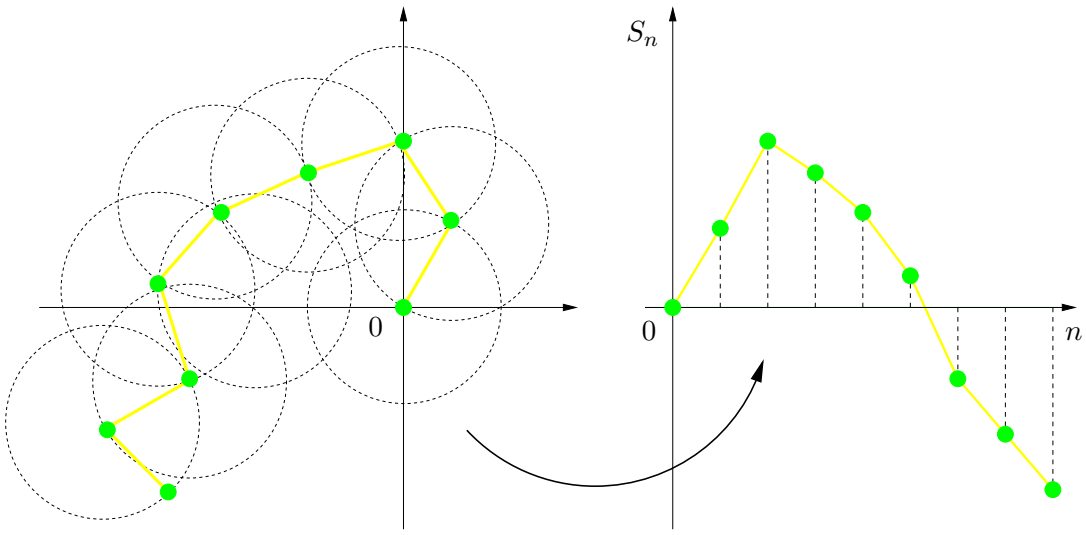


FIGURE 5.1. The correspondence between a two–dimensional random walk and a $(1+1)$–dimensional directed walk, for the purpose of modeling a polymer chain in the proximity of an interface (the $x$–axis in this case).

- the law of $X_1$ is absolutely continuous w.r.t. Lebesgue measure, with density $f$: $\mathbf{P}(X_1 \in \mathrm{d}x) = f(x)\,\mathrm{d}x$;
- for some $n_0 \in \mathbb{N}$ the density $f_{n_0}(x) := f^{*n_0}(x)$ of $S_{n_0}$ is essentially bounded: $f_{n_0}(x) \in L^\infty(\mathbb{R}, \mathrm{d}x)$.

We point out that the last hypothesis is made in order to apply the so–called Local Central Limit Theorem, see § 1.3 below. We denote by $\sigma^2 := \mathbf{E}\big[{X_1}^2\big] < \infty$ the variance of the typical step of the walk.

For the *charges* we place ourself in the *random setting*: we take the sequence $\omega = \{\omega_n\}_{n\geq 1}$ to be a typical realization of a sequence of IID random variables, whose global law is denoted by $\mathbb{P}$. The assumptions we make on the law of $\omega_1$ are exactly the same as in Chapter 1a, namely that it has finite exponential moments: $\mathsf{M}(\alpha) := \mathbf{E}[\exp(\alpha\omega_1)] < \infty$ for every $\alpha \in \mathbb{R}$ and that it is centered: $\mathbb{E}[\omega_1] = 0$. We also fix $\mathbb{E}[{\omega_1}^2] = 1$.

For technical reasons it will be convenient to assume that $\omega$ is a double–sided sequence, that is $\omega = \{\omega_n\}_{n\in\mathbb{Z}}$, though for the definition of the copolymer model we will only need the $\omega_n$ with $n \geq 1$. The enlarged $\omega$–space will be denoted by $\Omega$, and

of course we look at $\mathbb{P}$ as a probability measure on $\Omega$. We also recall for $k \in \mathbb{Z}$ the notation $\theta^k$ for the translation on $\Omega$, defined by $(\theta^k\omega)_n := \omega_{k+n}$.

Now we are ready to define the copolymer measure in our setting: for $\lambda, h \geq 0$ and $N \in \mathbb{N}$ we define $\mathbf{P}_{N,\omega}^{\lambda,h}$ through its Radon–Nikodym derivative:

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}^{\lambda,h}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{\widetilde{Z}_{N,\omega}} \exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h)\,\mathrm{sign}(S_n)\right) =: \frac{\mathcal{G}_{N,\omega}^{\lambda,h}(S)}{\widetilde{Z}_{N,\omega}}\,. \tag{5.1}$$

For definiteness we put $\mathrm{sign}(0) := 0$, but observe that in this new setting this has no role, because the event that $S_n = 0$ for some $n$ has zero probability. We point out that in this continuous model the charges are assigned to the points rather than to the bonds of the polymeric chain (we recall the discussion in the caption of Fig. 1a.2 of Chapter 1a for the discrete setting).

The normalization factor (*partition function*) $\widetilde{Z}_{N,\omega} = \widetilde{Z}_{N,\omega}^{\lambda,h}$ appearing in (5.1) is of course given by

$$\widetilde{Z}_{N,\omega} = \mathbf{E}\left[\exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h)\,\mathrm{sign}(S_n)\right)\right] = \mathbf{E}\big[\mathcal{G}_{N,\omega}\big]\,, \tag{5.2}$$

and the corresponding *free energy* $f(\lambda, h)$ is defined by

$$f(\lambda, h) := \lim_{n\to\infty} \frac{1}{N} \log \widetilde{Z}_{N,\omega}^{\lambda,h}\,. \tag{5.3}$$

A proof of the existence of such a limit, both $\mathbb{P}(\mathrm{d}\omega)$–a.s. and in $\mathbb{L}_1(\mathbb{P})$, and of the fact that $f(\lambda, h)$ is nonrandom (a phenomenon called *self–averaging*) will be given in full detail in Section 2.

Before proceeding with the analysis of the phase diagram of the model, it is convenient to recall some basic results.

**1.3. Local Limit Theorem and Fluctuation Theory.** Since the random walk we consider has a typical step with finite nonzero variance $\sigma^2$, the Central Limit Theorem (CLT) holds, that is we have the weak convergence as $N \to \infty$ of the law of $S_N/(\sigma\sqrt{N})$ towards the standard Gaussian Law:

$$\forall t \in \mathbb{R} : \quad \mathbf{P}\left[\,\frac{S_N}{\sigma\sqrt{N}} \leq t\,\right] \;\longrightarrow\; \int_{-\infty}^{t} \mathrm{d}x\, \frac{e^{-x^2/2}}{\sqrt{2\pi}} \qquad (N \to \infty)\,.$$

However in the following we will need rather precise estimates on the *density* of $S_N$ for large $N$. This does not follow automatically from the CLT, and some further assumptions are required. It turns out that the (mild) assumption that for some

$n_0 \in \mathbb{N}$ the density $f_{n_0}(x) := f^{*n_0}(x)$ of $S_{n_0}$ is essentially bounded is sufficient to guarantee the *uniform convergence* of the density of $S_N/(\sigma\sqrt{N})$ towards the standard Gaussian density: this is the content of the so–called Local Central Limit Theorem (LLT), cf. [**37**].

THEOREM 5.1 (LLT). *Under the above assumptions, the density $f_N(x)$ of $S_N$ is bounded and continuous for large $N$. Moreover, the (continuous version of the) density of $S_N/(\sigma\sqrt{N})$ converges uniformly to the standard Normal density:*

$$\sup_{x\in\mathbb{R}} \left| \sigma\sqrt{N} f_n\big(\sigma\sqrt{N}x\big) \; - \; \frac{e^{-x^2/2}}{\sqrt{2\pi}} \right| \to 0 \qquad (n\to\infty)\,. \tag{5.4}$$

The usefulness of the LLT is that it allows a precise control of the probability of events like $\{S_N \in I_N\}$ when the area of $I_N$ grows slower than $\sqrt{N}$. A typical example in this direction is provided by the following lemma, which is an immediate consequence of (5.4).

LEMMA 5.2. $\forall\, x \in \mathbb{R}$

$$\sqrt{N}\cdot f_N(x) \;\to\; \frac{1}{\sqrt{2\pi}} \qquad (N\to\infty) \tag{5.5}$$

$$\mathbf{P}\big[\,|S_N| \le x\,\big] \;\sim\; \frac{2x}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{N}} \qquad (N\to\infty)\,, \tag{5.6}$$

*where in both relations the convergence is uniform for $x$ in compact sets.*

We conclude this section by recalling some results from the Fluctuation Theory for random walks about conditioning a random walk to stay positive (for more details see Section 2 of Chapter 6). We start with the asymptotic behavior of the probability of the first entrance of the walk in the negative half–line: it is a classical result [**29**] that, whenever the step of the walk has zero mean and finite nonzero variance $\sigma^2$, as $n\to\infty$

$$\mathbf{P}\big[\,S_1>0,\dots,S_n>0\,\big] \;=\; \sum_{k=n+1}^{\infty} \mathbf{P}\big[\,S_1>0,\dots,S_{k-1}>0, S_k\le 0\,\big] \;\sim\; \frac{2\,C}{\sqrt{n}}\,, \tag{5.7}$$

for some $C\in(0,\infty)$. The local version of this relation holds too, namely

$$\mathbf{P}\big[\,S_1>0,\dots,S_{n-1}>0, S_n\le 0\,\big] \;\sim\; \frac{C}{n^{3/2}} \qquad (n\to\infty)\,, \tag{5.8}$$

and it has been proven in [**3**]. Of course both (5.7) and (5.8) hold also for the first entrance in the positive half–line (with possibly a different constant $C$).

We will also need to control the probabilities of the first entrances in the case when the random walk does not necessarily starts from 0. More precisely, for $x \in \mathbb{R}$ let us denote by $\mathbf{P}_x$ the law of $\{S_n + x\}_n$ under $\mathbf{P}$. Then for $x \geq 0$ we have

$$\mathbf{P}_x\big[\, S_1 > 0, \ldots, S_n > 0 \,\big] \;\sim\; \frac{2\, C_x}{\sqrt{n}} \qquad (n \to \infty) \tag{5.9}$$

$$\mathbf{P}_x\big[\, S_1 > 0, \ldots, S_{n-1} > 0, S_n \leq 0 \,\big] \;\sim\; \frac{C_x}{n^{3/2}} \qquad (n \to \infty)\,, \tag{5.10}$$

where $C_x$ is a positive nondecreasing function of $x \geq 0$. We point out that, up to multiplicative constants, the function $C_x$ coincides with the renewal function associated to the descending ladder heights process of the random walk, see [**6**, § 3] for more on this issue. A direct proof of (5.10) can be also given using the methods of Chapter 6. Again, an analog of (5.10) is valid also for the first entrance in the positive half–line, when the random walk stars from $x \leq 0$.

**1.4. The phase diagram.** Next we turn to the analysis of the phase diagram of the model we have introduced. As in the discrete case, the first step is the identification of the free energy coming from delocalized paths: restricting to trajectories that stay positive up to epoch $N$ we have that for $\mathbb{P}$–a.e. $\omega$

$$\begin{aligned} \frac{1}{N} \log \widetilde{Z}^{\lambda,h}_{N,\omega} \;&\geq\; \frac{1}{N} \log \mathbf{E}\left[ \exp\left( \lambda \sum_{n=1}^{N} (\omega_n + h)\, \mathrm{sign}\,(S_n) \right) ;\, S_1 > 0, \ldots, S_N > 0 \right] \\ &=\; \frac{\lambda}{N} \sum_{n=1}^{N} (\omega_n + h) \;+\; \frac{1}{N} \log \mathbf{P}\big(S_1 > 0, \ldots, S_N > 0\big) \;\stackrel{N\to\infty}{\longrightarrow}\; \lambda h, \end{aligned} \tag{5.11}$$

where in the last line we have used the strong law of large numbers and the asymptotic behavior given by (5.7).

Arguing as in Chapter 1a, we partition the $(\lambda, h)$–space in two regions:

- the localized region: $\mathcal{L} = \{(\lambda, h) :\; f(\lambda, h) > \lambda h\}$;
- the delocalized region: $\mathcal{D} = \{(\lambda, h) :\; f(\lambda, h) = \lambda h\}$.

For the critical line $h_c(\cdot)$ separating the two regions we have the following result, in complete analogy with the discrete case:

Proposition 5.3. *There exists a continuous increasing function $h_c : [0, \infty) \to [0, \infty)$ with $h_c(0) = 0$ such that*

$$\mathcal{D} \;=\; \big\{(\lambda, h) :\; h \geq h_c(\lambda)\big\} \qquad \mathcal{L} \;=\; \big\{(\lambda, h) :\; h < h_c(\lambda)\big\}\,.$$

Thus the picture of the phase diagram in this continuous setting looks quite similar to the discrete case analyzed in Chapter 1a, at least at a qualitative level. Now we are going to make this statement quantitative, showing that for the critical curve $h_c(\cdot)$ of our continuous model we have *exactly the same upper and lower bound* that hold in the discrete case, namely

$$h^{(2/3)}(\cdot) \;=:\; \underline{h}(\cdot) \;\le\; h_c(\cdot) \;\le\; \overline{h}(\cdot) \;:=\; h^{(1)}(\cdot)\,, \tag{5.12}$$

where we recall the definition of $h^{(m)}(\cdot)$ for $m > 0$:

$$h^{(m)}(\lambda) \;:=\; \frac{\log \mathsf{M}(-2m\lambda)}{2m\lambda}\,,$$

and $\mathsf{M}(\alpha) := \mathbb{E}\big[\exp(\alpha\omega_1)\big]$ is the moment generating function of the environment.

Before proceeding, let us spend some words on Proposition 5.3: using convexity arguments as in [**9**, § 1.2] it is not difficult to prove the existence of the critical line, together with some of its properties. However showing that $h_c(\cdot)$ is indeed increasing and not only nondecreasing, that it is continuous also at $\lambda = 0$ and that $h_c(\lambda) < \infty$ for every $\lambda \ge 0$ does not follow immediately. A rather cheap (if not elementary) way of proving these properties is to supply convex analysis with the knowledge of the bounds (5.12) on $h_c(\cdot)$ (whose proof is independent of Proposition 5.3).

1.4.1. *Upper bound.* The proof of the upper bound in (5.12) is completely analogous to the one given in Chapter 1a for the discrete setting, that is it suffices to apply the annealing procedure. However, in order not to end up with a useless bound, we have to suitably modify the partition function, as in § 2.3 of Chapter 1a. More precisely, subtracting to the Hamiltonian the term $\lambda \sum_{n=1}^N (\omega_n + h)$ (that does not depend on $S$ and that once averaged on the environment is simply $\lambda h N$) and using the fact that the limit (5.3) holds also in $\mathbb{L}_1(\mathbb{P})$ we can write

$$f(\lambda, h) - \lambda h \;=\; \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^N (\omega_n + h)\, \mathbf{1}_{\{\mathrm{sign}(S_n)=-1\}}\right)\right].$$

However by Jensen's inequality we can bring the expectation $\mathbb{E}$ inside the log, and performing the integration over the disorder we get

$$f(\lambda, h) - \lambda h \;\le\; \lim_{N\to\infty} \frac{1}{N} \log \mathbf{E}\left[\exp\left(\sum_{n=1}^N \big(\log \mathsf{M}(-2\lambda) - 2\lambda h\big) \mathbf{1}_{\{\mathrm{sign}(S_n)=-1\}}\right)\right]. \tag{5.13}$$

For $h \ge \overline{h}(\lambda)$ the argument of of the exponential is nonpositive: thus $f(\lambda, h) - \lambda h \le 0$ and by (5.11) we have $f(\lambda, h) = \lambda h$, hence we have proven that $h_c(\cdot) \le \overline{h}(\cdot)$.

Observe that for $h < \overline{h}(\lambda)$ the r.h.s. of (5.13) equals $\big(\log \mathsf{M}(-2\lambda) - 2\lambda h\big) > 0$, hence $\overline{h}(\cdot)$ is indeed the best upper bound on $h_c(\cdot)$ that one can extract from (5.13). Also notice that the arguments of Chapter 3 can be applied to our continuous setting with essentially no change: therefore the technique of *constrained annealing* via empirical averages of local functions cannot improve the upper bound we have found.

1.4.2. *Lower bound.* A proof of the lower bound in (5.12) can be obtained by following very closely the proof in the discrete setting given in § 6.2 of Chapter 2. For this reason we simply outline the main steps. Let us introduce a notation for the modified Hamiltonian

$$\mathcal{H}'_{N,\omega} \;:=\; -2\lambda \sum_{n=1}^{N} (\omega_n + h)\, \mathbf{1}_{\{\mathrm{sign}(S_n)=-1\}}\,,$$

so that the reduced free energy $f(\lambda, h) - \lambda h$ can be expressed for $\mathbb{P}$–a.e. $\omega$ as

$$f(\lambda, h) - \lambda h \;=\; \lim_{N\to\infty} \frac{1}{N} \log I^{\lambda,h}_{N,\omega}\,, \tag{5.14}$$

where

$$I^{\lambda,h}_{N,\omega} \;:=\; \inf_{-1\le x\le 1} \mathbf{E}_x\Big[ \exp\big(\mathcal{H}'_{N,\omega}\big)\, \mathbf{1}_{\{|S_N|\le 1\}} \Big]\,.$$

The proof of relation (5.14) is the core of Section 2.

We stress that $I_{N,\omega}$ takes for the continuous setting the role that the pinned partition function $Z_{N,\omega}(0)$ (see (1a.9) of Chapter 1a) has in the discrete setting. In fact, using (5.14) together with the superadditivity of the process $\{I_{N,\omega}\}_N$ (proved in § 2.1), the arguments of the first part of § 6.2 of Chapter 2 can be easily adapted to the continuous setting. In particular, in order to prove that a point $(\lambda, h)$ is localized, it suffices to find a number $C > 1$ and a random variable $T : \Omega \to \mathbb{N}$ with the following two properties:

$$(1)\quad I^{\lambda,h}_{T(\omega),\omega} \ge C \quad \mathbb{P}(\mathrm{d}\omega)\text{–a.s.} \qquad\qquad (2)\quad \mathbb{E}\big[T\big] < \infty\,. \tag{5.15}$$

Thus it only remains to show that for every $(\lambda, h)$ with $h < \underline{h}(\lambda)$ one can build a random time $T$ satisfying (5.15). However, if we define $T = T_{A,\varepsilon,q}$ as in (2.57) of Chapter 2, then using the asymptotic relation (5.10) we can easily get a lower bound on $I_{T(\omega),\omega}$ like (2.58), for a possibly different value of the constant $c'$ (see also (2.55)). Therefore one can tune the parameters $A, \varepsilon, q$ exactly as it is done in the end of § 6.2 of Chapter 2, see page 81, and condition (5.15) will be satisfied.

## 2. Existence of the free energy

In this section we give a proof of the existence of the free energy, that is of the limit (5.3). The standard procedure to get the existence of such a limit is to modify the partition function of the model (without changing the Laplace asymptotic behavior) in order to perform superadditivity arguments. For instance in the discrete case it is sufficient to restrict the sum defining the partition function $\widetilde{Z}_{N,\omega}$ to the trajectories such that $S_N = 0$: in our continuous setting this is no longer possible, because the event $\{S_N = 0\}$ has probability 0. This obstacle is easily bypassed and it is not difficult to find a useful modification of the partition function. The drawback is that showing that the modified partition function yields the same Laplace asymptotic behavior as the original one is no longer trivial.

Remember that by hypothesis the steps of our random walk are bounded by 1: $|S_n - S_{n-1}| \le 1$. We also recall the notation $\mathcal{G}_{N,\omega} := \mathcal{G}^{\lambda,h}_{N,\omega}$ for the Boltzmann factor appearing in the definition of the copolymer measure (5.1), and the expression (5.2) for the partition function $\widetilde{Z}_{N,\omega}$. The modified partition function to which we will apply superadditivity arguments will be

$$I_{N,\omega} \;=\; I^{\lambda,h}_{N,\omega} \;:=\; \inf_{-1\le x\le +1} \mathbf{E}_x\big[\mathcal{G}_{N,\omega}\,\mathbf{1}_{\{|S_N|\le 1\}}\big]\,, \tag{5.16}$$

where $\mathbf{P}_x$ is the law of the random walk starting at $x \in \mathbb{R}$, introduced in the preceding section.

The proof is organized in three steps: in § 2.1 we show that the limit (5.3) exists if we replace $\widetilde{Z}_{N,\omega}$ by $I_{N,\omega}$, and then in § 2.2 and § 2.3 we prove some comparison inequalities showing that $I_{N,\omega}$ and $\widetilde{Z}_{N,\omega}$ are equivalent for the sake of computing the free energy. To this purpose it will be convenient to consider an intermediate partition function $J_{N,\omega}$ defined as follows:

$$J_{N,\omega} \;=\; J^{\lambda,h}_{N,\omega} \;:=\; \mathbf{E}\big[\mathcal{G}_{N,\omega}\,\mathbf{1}_{\{|S_N|\le 1\}}\big]\,. \tag{5.17}$$

**2.1. Step 1.** We start showing that the sequence of random variables $\{\log I_{N,\omega}\}_N$ satisfies the hypothesis of *Kingman's Superadditive Ergodic Theorem* [**43**].

We begin with the upper bound on $\mathbb{E}[\log I_{N,\omega}]$: using Jensen's inequality, the definition (5.16) of $I_{N,\omega}$ and a rough bound on $\mathcal{G}_{N,\omega}$ we get

$$\mathbb{E}\left[\log I_{N,\omega}\right] \;\le\; \log \mathbf{E}\mathbb{E}\left[\exp\left(\lambda\sum_{n=1}^{N}(|\omega_n|+h)\right)\mathbf{1}_{\{|S_N|\le 1\}}\right] \;\le\; \big(\log \mathbb{E}\big[e^{\lambda|\omega_1|}\big] + \lambda h\big)N\,,$$

hence $\sup_N\{\mathbb{E}[\log I_{N,\omega}]/N\} < +\infty$. Also the superadditivity is easily obtained: making explicit the functional dependence of $\mathcal{G}_{N,\omega}$ on the path $(S_1,\dots,S_N)$ when it is convenient and using the Markov property, we obtain that $\forall\ x \in [-1,+1]$

$$\begin{aligned}
\mathbf{E}_x\left[\,\mathcal{G}_{N+M,\omega}\,\mathbf{1}_{\{|S_{N+M}|\le 1\}}\right] \;&\ge\; \mathbf{E}_x\left[\,\mathcal{G}_{N+M,\omega}\,\mathbf{1}_{\{|S_N|\le 1\}}\mathbf{1}_{\{|S_{N+M}|\le 1\}}\right]\\
&=\; \int_{-1}^{1}\mathrm{d}z\; f_N(z-x)\,\mathbf{E}_x\big[\,\mathcal{G}_{N,\omega}(S_1,\dots,S_{N-1},z)\big]\cdot\mathbf{E}_z\left[\,\mathcal{G}_{M,\theta^N\omega}\,\mathbf{1}_{\{|S_M|\le 1\}}\right]\\
&\ge\; \left(\int_{-1}^{1}\mathrm{d}z\; f_N(z-x)\,\mathbf{E}_x\big[\,\mathcal{G}_{N,\omega}(S_1,\dots,z)\big]\right)\cdot\inf_{z\in[-1,1]}\mathbf{E}_z\big[\,\mathcal{G}_{M,\theta^N\omega}\,\mathbf{1}_{\{|S_M|\le 1\}}\big]\\
&=\; \mathbf{E}_x\big[\,\mathcal{G}_{N,\omega}\cdot\mathbf{1}_{\{|S_N|\le 1\}}\big]\cdot I_{M,\theta^N\omega}\;\ge\; I_{N,\omega}\cdot I_{M,\theta^N\omega}
\end{aligned}$$

Reading only the extremities of this chain of inequalities, we have

$$\mathbf{E}_x\left[\,\mathcal{G}_{N+M,\omega}\,\mathbf{1}_{\{|S_{N+M}|\le 1\}}\right]\ge\; I_{N,\omega}\cdot I_{M,\theta^N\omega}\qquad\Rightarrow\qquad I_{N+M,\omega}\ge\; I_{N,\omega}\cdot I_{M,\theta^N\omega}$$

so that the superadditivity of the process $\{\log I_{N,\omega}\}_N$ is proved. We can thus apply Kingman's Theorem, concluding that the sequence $\{\log I^{\lambda,h}_{N,\omega}/N\}_N$ converges $\mathbb{P}(\mathrm{d}\omega)$–a.s. and in $\mathbb{L}_1(\mathbb{P})$ to a limit $f_\omega(\lambda,h)$ which is $\theta$–invariant. By tail triviality, $f$ is $\mathbb{P}(\mathrm{d}\omega)$–a.s. constant and we consequently omit the $\omega$ dependence: $f = f(\lambda,h)$.

**2.2. Step 2.** Now we show that also the sequence $\{\log J^{\lambda,h}_{N,\omega}/N\}_N$ has, $\mathbb{P}(\mathrm{d}\omega)$–a.s. and in $\mathbb{L}_1(\mathbb{P})$, the limit $f(\lambda,h)$ as $N\to\infty$. We start noting that by definition

$$J^{\lambda,h}_{N,\omega}\;\ge\; I^{\lambda,h}_{N,\omega}\qquad\Rightarrow\qquad \liminf_{N\to\infty}\frac{\log J^{\lambda,h}_{N,\omega}}{N}\;\ge\; f(\lambda,h) \tag{5.18}$$

for $\mathbb{P}$–a.e. $\omega$, so it remains to find a bound for the lim sup.

We recall that by hypothesis the density $f$ of $X_1$ is supported in the interval $[-1,+1]$, and that for some $n_0\in\mathbb{N}$ the density $f_{n_0}$ of $S_{n_0}$ is bounded, hence we can find two positive constants $A$, $M$ such that $f_{n_0}(x)\le A\cdot\mathbf{1}_{\{|x|\le M\}}$. Then for $N\in\mathbb{N}$ by the Markov property we get a first upper bound for $J$:

$$\begin{aligned}
J_{n_0+N,\omega}\;&=\;\int_{\mathbb{R}}\mathrm{d}z\; f_{n_0}(z)\,\mathbf{E}\big[\,\mathcal{G}_{n_0,\omega}(S_1,\dots,S_{n_0-1},z)\big]\cdot\mathbf{E}_z\left[\,\mathcal{G}_{N,\theta^{n_0}\omega}\,\mathbf{1}_{\{|S_N|\le 1\}}\right]\\
&\le\; A\,K(\omega)\cdot\int_{-M}^{M}\mathrm{d}z\;\mathbf{E}_z\left[\,\mathcal{G}_{N,\theta^{n_0}\omega}\cdot\mathbf{1}_{\{|S_N|\le 1\}}\right]
\end{aligned}\tag{5.19}$$

where the constant $K(\omega)$ that we have used to bound $\mathbf{E}[\dots]$ is simply

$$K(\omega)=K(\lambda,h,\{\omega_i\}_{1\le i\le n_0}) := \exp\left(\lambda\sum_{n=1}^{N_0}(|\omega_n|+h)\right).$$

Next we want to obtain an analogous *lower bound* for $I$. Observe that by (5.5) we can find $n_1 \in \mathbb{N}$ such that

$$f_{n_1}(y) \geq \frac{1}{2}\,\frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{n_1}} \qquad \forall y \in [-(M+1),(M+1)]\,.$$

Then for $N \in \mathbb{N}$ and for all $x \in [-1,+1]$ we get

$$\begin{aligned}
&\mathbf{E}_x\Big[\mathcal{G}_{n_1+N,\omega}\,\mathbf{1}_{\{|S_{n_1+N}|\leq 1\}}\Big] \\
&\quad = \int_{\mathbb{R}} \mathrm{d}z\, f_{n_1}(z-x)\,\mathbf{E}\big[\,\mathcal{G}_{n_1,\omega}(S_1,\ldots,S_{n_1-1},z)\big]\cdot\mathbf{E}_z\left[\,\mathcal{G}_{N,\,\theta^{n_1}\omega}\,\mathbf{1}_{\{|S_N|\leq 1\}}\right] \\
&\quad \geq \frac{1}{2}\,\frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{n_1}}\,K(\omega)^{-1}\cdot\int_{-M}^{M}\mathrm{d}z\,\mathbf{E}_z\left[\,\mathcal{G}_{N,\,\theta^{n_1}\omega}\cdot\mathbf{1}_{\{|S_N|\leq 1\}}\right]\,,
\end{aligned}$$

hence

$$I_{n_1+N,\omega} \;\geq\; \frac{1}{2}\,\frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{n_1}}\,K(\omega)^{-1}\cdot\int_{-M}^{M}\mathrm{d}z\,\mathbf{E}_z\left[\,\mathcal{G}_{N,\,\theta^{n_1}\omega}\cdot\mathbf{1}_{\{|S_N|\leq 1\}}\right]\,. \tag{5.20}$$

Combining (5.19) with (5.20) we get that for all $N \in \mathbb{N}$

$$J_{N,\omega} \;\leq\; A'\,K(\omega)^2\,I_{N+(n_1-n_0),\theta^{(n_0-n_1)}\omega}\,,$$

for some positive constant $A'$ (we recall that we consider two–sided sequence of charges: $\omega = \{\omega_n\}_{n\in\mathbb{Z}}$, hence the translations $\theta^k$ are meaningful for all $k \in \mathbb{Z}$). It follows that

$$\limsup_{N\to\infty}\frac{\log J^{\lambda,h}_{N,\,\omega}}{N} \;\leq\; \limsup_{N\to\infty}\frac{\log I^{\lambda,h}_{N+(n_1-n_0),\,\theta^{(n_0-n_1)}\omega}}{N} \;=\; f(\lambda,h)\,,$$

$\mathbb{P}(\mathrm{d}\omega)$–a.s., that is what was to be proven. Notice that the bounds we have obtain yield easily also the $\mathbb{L}_1(\mathbb{P})$ convergence of $\{\log J^{\lambda,h}_{N,\,\omega}/N\}_N$ towards $f(\lambda,h)$.

**2.3. Step 3.** Finally we are left with comparing $J_{N,\omega}$ with the original partition function $\widetilde{Z}_{N,\omega}$, which amounts to removing the restriction $\{|S_N| \leq 1\}$. Observe that by definition $\widetilde{Z}_{N,\omega} \geq J_{N,\omega}$, hence we can concentrate on finding a suitable upper bound.

The procedure we follow is very similar to the simple random walk case, cf. [**35**]. The idea is to look at the last point up to epoch $N$ at which the random walk changes its sign. More precisely, we define the random variable $U$ by

$$U \;:=\; \min\big\{k \in \{1,\ldots,N\} :\, \mathrm{sign}(S_k) = \mathrm{sign}(S_{k+1}) = \ldots = \mathrm{sign}(S_N)\big\}\,,$$

and we disintegrate the partition function according to the range of $U$. It is convenient to consider separately the cases $\{S_N > 0\}$ and $\{S_N < 0\}$, that is we split

$$\widetilde{Z}_{N,\omega} \;=\; \widetilde{Z}^{>}_{N,\omega} \;+\; \widetilde{Z}^{<}_{N,\omega} \;:=\; \mathbf{E}\big[\mathcal{G}_{N,\omega}\,,\, S_N > 0\big] \;+\; \mathbf{E}\big[\mathcal{G}_{N,\omega}\,,\, S_N < 0\big]\,.$$

Then we can write

$$\begin{aligned}\widetilde{Z}^{>}_{N,\omega} \;&=\; \sum_{k=1}^{N} \mathbf{E}\big[\mathcal{G}_{N,\omega}\,,\, U = k\,,\, S_N > 0\big] \\ &=\; \sum_{k=1}^{N} \int_0^1 \mathrm{d}z\, f_k(z)\, \mathbf{E}\big[\mathcal{G}_{k,\omega}(S_1,\dots,S_{k-1},z)\big]\, e^{\lambda \sum_{i=k+1}^{N}(\omega_i+h)}\, \mathbf{P}_z\big[S_1 > 0,\dots,S_{N-k} > 0\big]\,.\end{aligned}$$

Now by the asymptotic behavior in (5.9) and (5.10) it follows that one can find a positive constant $D$ such that for all $z \in [0,1]$, for all $N \in \mathbb{N}$ and for all $k \in \{1,\dots,N\}$ one has

$$\mathbf{P}_z\big[S_1 > 0,\dots,S_{N-k} > 0\big] \;\le\; D\,N\,\mathbf{P}_z\big[S_1 > 0,\dots,S_{N-k-1} > 0, S_{N-k} \le 0\big]\,.$$

Performing this substitution we obtain

$$\widetilde{Z}^{>}_{N,\omega} \;\le\; e^{2\lambda|\omega_N|}\, D\,N\,\mathbf{E}\big[\mathcal{G}_{N,\omega}\,,\, |S_N| \le 1\big] \;=\; \big(e^{2\lambda|\omega_N|}\, D\,N\big)\, J_{N,\omega}\,.$$

As the very same arguments can be performed for $\widetilde{Z}^{<}_{N,\omega}$, we have definitively shown that

$$J^{\lambda,h}_{N,\omega} \;\le\; \widetilde{Z}^{\lambda,h}_{N,\omega} \;\le\; \big(e^{2\lambda|\omega_N|}\, D'\,N\big)\, J^{\lambda,h}_{N,\omega}\,,$$

for some positive constant $D'$. From this relation the convergence of $\{\log \widetilde{Z}^{\lambda,h}_{N,\omega}/N\}_n$ towards $f(\lambda,h)$ both $\mathbb{P}(\mathrm{d}\omega)$–a.s. and in $\mathbb{L}_1(\mathbb{P})$ follows immediately.

## 3. Towards the coarse graining of the free energy

The coarse graining of the free energy for the copolymer near a selective interface model is expressed by Theorem 1a.5 of Chapter 1a: it holds when the underlying random walk is the simple symmetric random walk on $\mathbb{Z}$, and the proof of it is the main result of the paper [**12**] by Bolthausen and den Hollander. The purpose of this section is to discuss the issue of extending it to the continuous setting adopted in this chapter.

The idea that lies at the basis of the coarse graining is that when $\lambda \to 0$ the reward to stay close to the interface gets small and consequently the typical excursions of the polymer away from the interface tend to become very long. Therefore it should

be possible to approximate both the polymer and the charges by Brownian motions, and in fact Theorem 1a.5 provides a quantitative version of this approximation.

According to this heuristic point of view, the microscopic details of the random walk and of the charges should not be too relevant, until we work with processes in the domain of attraction of the Brownian motion. This is indeed true for the charges, as we already mentioned in Chapter 1a: in fact the original proof of Theorem 1a.5 in [**12**] was given for the Bernoulli case $\mathbb{P}(\omega_1 = +1) = 1 - \mathbb{P}(\omega_1 = -1) = 1/2$, but it can be easily extended to the general $\omega$ case considered here.

On the other hand, the extension to the more general random walks considered in this chapter appears to be more challenging. In order to outline the reasons of this fact, we have to look more closely at the original proof of Theorem 1a.5. Without going into the details, which are quite long and extremely delicate, we point out that the proof is divided in four main steps, which we can roughly describe as follows:

(1) first it is shown that when $\lambda$ and $h$ are small one can safely throw away the *short excursions* of the walk in the computation of the partition function;

(2) then the $\{\omega_n\}$ are replaced by standard Gaussian variables;

(3) the law of the (long) excursions under the rescaled simple random walk measure is then replaced by the law under the Brownian motion measure, ending up with a Brownian copolymer model without the short excursions;

(4) finally, one reintroduces the short excursion for the Brownian copolymer model.

We observe that in step (2) the random walk plays a minor role and it is not difficult to adapt the proof to our continuous random walk setting, while step (4) is a problem involving only the Brownian copolymer model, hence it requires no change. Therefore the crucial points are step (1) and step (3), that will be analyzed separately.

**3.1. Step (1): throwing away the short excursions.** For the first step the original proof makes use of several peculiar properties enjoyed by the excursions of the simple random walk, namely:

- there is a complete decoupling between the epochs of the returns to zero $\{\tau_k^{\mathrm{SRW}}\}_k$ and the signs $\{\sigma_k^{\mathrm{SRW}}\}_k$ of the excursions: in fact the sequence $\{\tau_k^{\mathrm{SRW}}\}_k$ is independent of the sequence $\{\sigma_k^{\mathrm{SRW}}\}_k$;
- the signs $\{\sigma_k^{\mathrm{SRW}}\}_k$ form an independent sequence of Bernoulli variables with $\mathbf{P}(\sigma_k^{\mathrm{SRW}} = +1) = \mathbf{P}(\sigma_k^{\mathrm{SRW}} = -1) = 1/2$;
- the zeros $\{\tau_k^{\mathrm{SRW}}\}_k$ form a classical renewal process, that is the interarrival times $\{\tau_{k+1}^{\mathrm{SRW}} - \tau_k^{\mathrm{SRW}}\}_k$ are independent positive random variables.

The first observation is that the returns to zero are no longer meaningful for a continuous random walk, since $\mathbf{P}(S_n = 0 \text{ for some } n) = 0$. We point out two possible definitions:

(a) the epochs at which the random walk crosses the interface:

$$\tau_0 := 0 \qquad \tau_{k+1} := \inf\left\{n > \tau_k : \operatorname{sign}(S_n) \neq \operatorname{sign}(S_{n-1})\right\}, \tag{5.21}$$

with the signs of the excursions $\{\sigma_k\}_{k\geq 1}$ defined by $\sigma_k := \operatorname{sign}(S_{\tau_{k-1}})$;

(b) the epochs at which the walk gets close to the interface:

$$\tau_0 := 0 \qquad \tau_{k+1} := \inf\left\{n > \tau_k : S_n \in [-1, +1]\right\}, \tag{5.22}$$

with the signs of the excursions $\{\sigma_k\}_{k\geq 1}$ defined by $\sigma_k := \operatorname{sign}(S_{\tau_k})$.

Notice that with the first definition there is a striking difference with respect to the simple random walk case, because the sequence $\{\sigma_k\}_{k\geq 1}$ is almost deterministic: in fact $\mathbf{P}(\mathrm{d}S)$–a.s. we have that $\{\sigma_k\}_{k\geq 1} = \operatorname{sign}(S_1) \cdot \{(-1)^k\}_{k\geq 1}$.

In any case none of the above mentioned properties of $\{\sigma_k^{\mathrm{SRW}}\}_k$ and $\{\tau_k^{\mathrm{SRW}}\}_k$ holds anymore for $\{\sigma_k\}_k$ and $\{\tau_k\}_k$, with any of the two definitions (a) or (b).

The most serious problem is that the interarrival times $\{T_k\}_k$ where $T_k := \tau_{k+1} - \tau_k$ are no longer independent. Nevertheless they enjoy a useful property. Let us introduce the sequence of random variables $\{J_k\}_{k\geq 0}$ defined by $J_k := S_{\tau_k}$ (notice that $J_k \in [-1, +1]$ because by hypothesis our random walk has steps bounded by 1 in absolute value). Then it is not difficult to check that the joint process $\{(J_k, T_k)\}_k$ is a *Markov renewal process* [**5**], that is a Markov chain on $[-1, +1] \times \mathbb{N}$ such that the transition kernel

$$\mathbf{P}\big(J_{k+1} \in \mathrm{d}y,\, T_{k+1} = n \,\big|\, J_k = x,\, T_k = m\big)$$

does not depend on $m$. This implies that, conditionally on $\{J_k\}_k$, the variables $\{T_k\}_k$ are independent.

Another remarkable fact is that the asymptotic behavior of the probability tail of the variables $\{T_k\}_k$ is similar to the simple random walk case, also conditionally on the $\{J_k\}_k$:

$$\mathbf{P}\big(T_{k+1} = n \,\big|\, J_k = x\big) \;\sim\; \frac{\text{const.}(x)}{n^{3/2}} \qquad (n \to \infty)\,.$$

If one chooses the definition (a) this relation is just a rephrasing of (5.10), and it is not difficult to check that it holds also with definition (b).

Thus the situation is not extremely bad. After all, we have seen that Markov renewal processes have been the fundamental tool in the study of periodic inhomogeneous polymer models performed in Chapter 4. The reason is that a lot of fundamental asymptotic results (renewal theorems) of classical renewal processes can be extended to the Markov case. We stress however that the processes of Chapter 4 enjoy the peculiar property of having a modulating chain $\{J_k\}_k$ with finite state space and this is indeed a great simplification, as it is explained in [**5**, Ch. VII.4]. Dealing with the case when the modulating chain has uncountable state space is much more delicate and the results are more involved (see for instance [**4**]), especially in the case of heavy tails.

Up to now we have not succeeded in extending the proof of step (1) to the continuous random walk setting. Nevertheless we point out that we are able to prove a weaker form of step (1) that, *provided one can extend to the continuous setting step* (3), is sufficient to yield the first part of Theorem 1a.5, namely the scaling limit of the free energy expressed by equation (1a.16). This would be an interesting result, but unfortunately we do not have yet a complete proof of step (3) in the continuous setting.

**3.2. Step (3): from random walk to Brownian motion.** The central point of step (3) is a sharp comparison between the law of the (long) excursions of the rescaled simple random walk and the law of the excursions of the Brownian motion. Without getting too much into the details, we mention that a fundamental estimate of the proof is the following one (cf. equations (4.62) and (4.66) in [**12**]): for $k, l \in \mathbb{N}$ such that $k + l \in 2\mathbb{N}$, as $k, l \to \infty$ jointly we have

$$\mathbf{P}^{(SRW)}\big(S_i \neq 0 \text{ for } k < i < k + l,\, S_{k+l} = 0\big) \;=\; (1 + o(1))\,\frac{2}{\pi}\,\frac{\sqrt{k}}{(k+l)\sqrt{l}}\,. \qquad (5.23)$$

where $\mathbf{P}^{(SRW)}$ is the law of the simple symmetric random walk on $\mathbb{Z}$. The proof of this relation is obtained by conditioning on the position of the walk at epoch $k$ and then using the reflection principle together with a strong approximation of the mass function of the simple random walk by the Gaussian density.

Now let us set $\ell_N := \max\{k = 0, \dots, N : \tau_k \le N\}$, where the $\{\tau_k\}_k$ are defined by (5.21) (but we could also choose definition (5.22)). Then the continuous analogue of (5.23) should be that for $k, l \in \mathbb{N}$ and as $k, l \to \infty$ jointly

$$\mathbf{P}\big(\ell_{k+l-1} \le k,\, \ell_{k+l} = k + l\big) \;=\; (1 + o(1))\, \frac{1}{\pi}\, \frac{\sqrt{k}}{(k+l)\sqrt{l}} \tag{5.24}$$

(the reason for the missing factor 2 with respect to (5.23) lies in the periodicity of the returns of the simple random walk).

In order to prove (5.24), we start conditioning on the position at epoch $k$:

$$\begin{aligned}\mathbf{P}\big(\ell_{k+l-1} \le k,\, \ell_{k+l} = k + l\big) \;=\; &\int_0^{+\infty} \mathrm{d}x\, f_k(x)\, \mathbf{P}_x\big(S_1 > 0, \dots, S_{l-1} > 0, S_l \le 0\big) \\ &+ \int_{-\infty}^{0} \mathrm{d}x\, f_k(x)\, \mathbf{P}_x\big(S_1 < 0, \dots, S_{l-1} < 0, S_l \ge 0\big)\,.\end{aligned}$$

Let us consider the first integral in the r.h.s. above, the second one being analogous: when $k$ is large the asymptotic behavior of $f_k(x)$ is given by the Local Central Limit Theorem (5.4) (actually one should use a stronger version valid in a ratio sense, see [**61**]). On the other hand the asymptotic behavior of the term $\mathbf{P}_x(S_1 > 0, \dots, S_{l-1} > 0, S_l \le 0)$ is not immediate: notice in fact that the relevant values of $x$ are those of order $\sqrt{k}$ and $k \to \infty$, hence one cannot use (5.10).

Conditioning on the position at epoch $l$ we can write

$$\mathbf{P}_x(S_1 > 0, \dots, S_{l-1} > 0, S_l \le 0) \;=\; \int_0^1 \mathrm{d}y\, \phi_l^{(y)}(x + y)\, f_l(-y - x)$$

where $\phi_l^{(y)}(z)$ is the value at $z \in \mathbb{R}$ of the density of the random variable $\widehat{S}_l$ *conditionally on the event* $\{\widehat{S}_1 > y, \dots, \widehat{S}_l > y\}$, where we have introduced the dual random walk $\{\widehat{S}_n\}_n := \{-S_n\}_n$. Notice however that the value of $y \in (0, 1)$ is actually irrelevant for the asymptotic behavior of $\phi_l^{(y)}(z)$, because we are interested in the regime when both $l$ and $z$ are large, and it is sufficient to consider the case $y = 0$.

Therefore an important role is played by the asymptotic behavior of the density of the variable $\widehat{S}_l$ conditionally on the event $\{\widehat{S}_1 > 0, \dots, \widehat{S}_l > 0\}$, where $\{\widehat{S}_n\}_n = \{-S_n\}_n$ is a random walk satisfying the hypothesis stated in § 1.2. It has been known

for a long time [**10**] that the only hypothesis of finite nonzero variance $\sigma$ guarantees the weak convergence

$$\frac{\widehat{S}_l}{\sigma\sqrt{l}} \text{ conditionally on } \{\widehat{S}_1 > 0, \dots, \widehat{S}_l > 0\} \quad \Rightarrow \quad xe^{-x^2/2}\mathbf{1}_{(x\geq 0)}\mathrm{d}x\,.$$

However what we need is rather a local refinement of this weak convergence, exactly as the Local Limit Theorem (5.4) is a local refinement of the Central Limit Theorem.

Such a Local Limit Theorem for random walks conditioned to stay positive does not seem to be known in the literature. We give a proof in Chapter 6 in a very general setting, using the Fluctuation Theory for random walks. Besides being an interesting result in itself, this theorem is a key step to prove the asymptotic behavior (5.24). Unfortunately there is still some technical points to be solved in order to extend the proof of step (3) to the continuous setting, but we think that a complete solution is not too far.

CHAPTER 6

# A local limit theorem for random walks conditioned to stay positive

In this chapter we study the asymptotic behavior of random walks conditioned to stay positive. We consider a real random walk $S_n = X_1 + \ldots + X_n$ attracted (without centering) to the normal law: this means that for a suitable norming sequence $a_n$ we have the weak convergence $S_n/a_n \Rightarrow \varphi(x)\mathrm{d}x$, $\varphi(x)$ being the standard normal density. A local refinement of this convergence is provided by Gnedenko's and Stone's Local Limit Theorems, in the lattice and nonlattice case respectively.

Now let $\mathcal{C}_n$ denote the event $(S_1 > 0, \ldots, S_n > 0)$ and let $S_n^+$ denote the random variable $S_n$ conditioned on $\mathcal{C}_n$: it is known that $S_n^+/a_n \Rightarrow \varphi^+(x)\mathrm{d}x$, where $\varphi^+(x) := x\exp(-x^2/2)\mathbf{1}_{(x\geq 0)}$. What we are going to establish is an equivalent of Gnedenko's and Stone's Local Limit Theorems for this weak convergence. We also consider the particular case when $X_1$ has an absolutely continuous law: in this case the uniform convergence of the density of $S_n^+/a_n$ towards $\varphi^+(x)$ holds under a standard additional hypothesis, in analogy to the classical case. We finally discuss an application of our main results to the asymptotic behavior of the joint renewal measure of the ladder variables process. Unlike the classical proofs of the LLT, we make no use of characteristic functions: our techniques are rather taken from the so–called Fluctuation Theory for random walks.

The article [**15**] has been taken from the content of this chapter.

## 1. Introduction and results

**1.1. The nonlattice case.** Let $S_n = X_1 + \ldots + X_n$ be a real random walk attracted (without centering) to the normal law. This means that $\{X_k\}$ is an IID sequence of real random variables, and for a suitable norming sequence $a_n$ we have the weak convergence

$$S_n/a_n \Rightarrow \varphi(x)\,\mathrm{d}x\,, \qquad \varphi(x) := \frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}\,. \tag{6.1}$$

This is the case for example when $\mathbf{E}(X_1) = 0$ and $\mathbf{E}(X_1^2) =: \sigma^2 \in (0, \infty)$ with $a_n := \sigma\sqrt{n}$, by the Central Limit Theorem.

We recall that, by the standard theory of stability [**29**, §IX.8 & §XVII.5], for equation (6.1) to hold it is necessary and sufficient that $\mathbf{E}(X_1) = 0$, that the truncated variance $V(t) := \mathbf{E}(X_1^2\,\mathbf{1}_{(|X_1|\le t)})$ be *slowly varying* at $\infty$ (that is $V(ct)/V(t) \to 1$ as $t \to \infty$ for every $c > 0$) and that the sequence $a_n$ satisfy the condition $a_n^2 \sim nV(a_n)$ as $n \to \infty$.

For the moment we assume that the law of $X_1$ is *nonlattice*, that is not supported in $(b + c\mathbb{Z})$ for any $b \in \mathbb{R}, c > 0$. Then a local refinement of (6.1) is provided by Stone's Local Limit Theorem [**66, 67**], that in our notations reads as (cf. [**7**, §8.4])

$$a_n\,\mathbf{P}\big(S_n \in [x, x+h)\,\big) = h\,\varphi(x/a_n) + o(1) \qquad (n \to \infty)\,, \tag{6.2}$$

*uniformly* for $x \in \mathbb{R}$ and $h$ in compact sets in $\mathbb{R}^+$.

What we are interested in is the asymptotic behavior of the random walk $\{S_n\}$ conditioned to stay positive. More precisely, let $\mathcal{C}_n := (S_1 > 0, \dots, S_n > 0)$ and let $S_n^+$ denote the random variable $S_n$ under the conditional probability $\mathbf{P}(\,\cdot\,|\,\mathcal{C}_n)$: if (6.1) holds then one has an analogous weak convergence result for $S_n^+/a_n$, namely

$$S_n^+/a_n \Rightarrow \varphi^+(x)\,\mathrm{d}x\,, \qquad \varphi^+(x) := x\,e^{-x^2/2}\,\mathbf{1}_{(x\ge 0)}\,. \tag{6.3}$$

This is an immediate consequence of the fact [**39, 10, 24**] that, whenever (6.1) holds, the whole process $\{S_{\lfloor nt\rfloor}/a_n\}_{t\in[0,1]}$ under $\mathbf{P}(\,\cdot\,|\,\mathcal{C}_n)$ converges weakly as $n \to \infty$ to the standard Brownian meander process $\{B_t^+\}_{t\in[0,1]}$, and $\varphi^+(x)\,\mathrm{d}x$ is the law of $B_1^+$, cf. [**60**].

Our main result is an analogue of Stone's LLT for the weak convergence (6.3).

THEOREM 6.1. *If $X_1$ is nonlattice and* (6.1) *holds, then*

$$a_n\,\mathbf{P}\big(S_n \in [x, x+h)\,\big|\,\mathcal{C}_n\big) = h\,\varphi^+(x/a_n) + o(1) \qquad (n \to \infty)\,, \tag{6.4}$$

*uniformly for $x \in \mathbb{R}$ and $h$ in compact sets in $\mathbb{R}^+$.*

The main difficulty with respect to the classical case is given by the fact that under the conditional probability $\mathbf{P}(\,\cdot\,|\,\mathcal{C}_n)$ the increments of the walk $\{X_k\}$ are no longer independent. This is a major point in that the standard proof of Stone's LLT relies heavily on characteristic functions methods. As a matter of fact, we make no use of characteristic functions: our methods are rather of combinatorial nature, and

we make an essential use of the so–called Fluctuation Theory for random walks. The core of our proof consists in expressing the law of $S_n$ under $\mathbf{P}(\,\cdot\,|\,\mathcal{C}_n)$ as a suitable mixture of the laws of $\{S_k\}_{0\le k\le n}$ *under the unconditioned measure* $\mathbf{P}$, to which Stone's LLT can be applied. Thus our "Positive LLT" is in a sense directly derived from Stone's LLT.

We point out that our methods may in principle be applied to the case when the random walk is attracted to a generic stable law (the analogue of (6.3) in this case is also provided by [**24**]), so that it should be possible to obtain an equivalent of Theorem 6.1 in this general setting.

**1.2. The lattice case.** Let us consider now the lattice case: we assume that $X_1$ is supported in $(b+c\mathbb{Z})$, for the least such $c$. In this case the local version of (6.1) is given by Gnedenko's Local Limit Theorem [**7**, §8.4], which says that

$$\frac{a_n}{c}\,\mathbf{P}\big(S_n = bn+cx\big) = \varphi\big((bn+cx)/a_n\big) + o(1) \qquad (n\to\infty)\,, \tag{6.5}$$

uniformly for $x\in\mathbb{Z}$.

We can derive the local version of (6.3) also in this setting.

THEOREM 6.2. *If $X_1$ is lattice with span 1 and* (6.1) *holds, then*

$$\frac{a_n}{c}\,\mathbf{P}\big(S_n = bn+cx\,\big|\,\mathcal{C}_n\big) = \varphi^+\big((bn+cx)/a_n\big) + o(1) \qquad (n\to\infty)\,,$$

*uniformly for $x\in\mathbb{Z}$.*

The proof is omitted since it can be recovered from the proof of Theorem 1 with only slight modifications (some steps are even simpler).

**1.3. The density case.** When the law of $X_1$ is absolutely continuous with respect to Lebesgue measure and (6.1) holds, one may ask whether the density of $S_n/a_n$ converges to $\varphi(x)$ in some pointwise sense. However, it is easy to build examples [**37**, §46] satisfying (6.1), such that for every $n$ the density of $S_n/a_n$ is unbounded in any neighborhood of 0: therefore without some extra–assumption one cannot hope for convergence to hold at each point. Nevertheless, if one looks for the *uniform convergence* of the density, then there is a simple condition which turns out to be necessary and sufficient.

ASSUMPTION 6.3. *The law of $X_1$ is absolutely continuous, and for some $k\in\mathbb{N}$ the density $f_k(x)$ of $S_k$ is essentially bounded: $f_k(x)\in L^\infty(\mathbb{R},\mathrm{d}x)$.*

It is easy to see that if this assumption holds, then for large $n$ the density $f_n(x)$ admits a bounded and continuous version. A proof that Assumption 6.3 yields the uniform convergence of the (continuous versions of the) density of $S_n/a_n$ towards $\varphi(x)$, namely

$$\sup_{x\in\mathbb{R}} \left| a_n f_n(a_n x) - \varphi(x) \right| \to 0 \qquad (n \to \infty)\,,$$

can be found in [**37**, §46]. On the other side, the necessity of Assumption 6.3 for the above convergence to hold is evident.

We can derive a completely analogous result for $S_n^+$.

THEOREM 6.4. *Assume that $X_1$ satisfies Assumption 6.3, and that* (6.1) *holds. Then:*

(1) *$S_n^+$ has an absolutely continuous law, whose density $f_n^+(x)$ is bounded and continuous (except at $x = 0$) for large $n$;*
(2) *the (continuous version of the) density of $S_n^+/a_n$ converges uniformly to $\varphi^+(x)$:*

$$\sup_{x\in\mathbb{R}} \left| a_n f_n^+(a_n x) - \varphi^+(x) \right| \to 0 \qquad (n \to \infty)\,.$$

This Theorem can be proved following very closely the proof of Theorem 1: in fact equation (6.19) in Section 3 provides an explicit expression for $f_n^+(x)$, that can be shown to converge to $\varphi^+(x)$ with the very same arguments given in Section 4.

**1.4. Asymptotic behavior of the ladder renewal measure.** As a by–product of the Local Limit Theorems described above, we have a result on the asymptotic behavior of the renewal measure of the ladder variables process. For simplicity we take the arithmetic setting, assuming that $X_1$ is supported by $\mathbb{Z}$ and it is aperiodic, but everything works similarly in the general lattice and nonlattice cases. The renewal mass function $u(n,x)$ of the ladder variables process is defined for $n \in \mathbb{N},\ x \in \mathbb{Z}$ by

$$u(n,x) := \sum_{r=0}^{\infty} \mathbf{P}\big(T_r = n, H_r = x\big) = \mathbf{P}\big(n \text{ is a ladder epoch}, S_n = x\big)\,, \qquad (6.6)$$

where $\{(T_k, H_k)\}$ is the (strict, ascending) ladder variables process associated to the random walk (the definitions are given in Section 2). Generalizing some earlier result of [**41**], in [**3**] it has been shown that, for $\{x_n\}$ such that $x_n/a_n \to 0$,

$$u(n,x_n) \sim \frac{1}{\sqrt{2\pi}\, n\, a_n}\, U(x_n - 1) \sim \frac{1}{n}\, \mathbf{P}\big(S_n = x_n\big)\, U(x_n - 1) \qquad (n \to \infty)\,, \quad (6.7)$$

where $U(x) := \sum_{r=0}^{\infty} \mathbf{P}(H_r \leq x)$ is the distribution function of the renewal measure associated to the ladder heights process (as a matter of fact, the proof of (6.7) given in [**3**] is carried out under the assumption that $\mathbf{E}(X_1^2) < \infty$, but it can be easily extended to the general case).

With our methods we are able to show that the same relation is valid for $x = O(a_n)$, with no further restriction on $X_1$ other than the validity of (6.1).

THEOREM 6.5. *Let $X_1$ be arithmetic with span 1 and such that equation* (6.1) *holds. Then for $x \in \mathbb{Z}$*

$$u(n,x) = \frac{1}{n}\,\mathbf{P}\big(S_n = x\big)\,U(x-1)\,\big(1+o(1)\big) \qquad (n \to \infty)\,, \tag{6.8}$$

*uniformly for $x/a_n \in [\varepsilon, 1/\varepsilon]$, for every fixed $\varepsilon > 0$.*

The proof of this theorem is a direct consequence of Theorem 6.2: the details are worked out in Section 2.5.

Notice that in the r.h.s. of (6.8) we could as well write $U(x)$ instead of $U(x-1)$, since $x \to \infty$ as $n \to \infty$. Also observe that putting together equation (6.7) with Theorem 6.5 one has the stronger result that equation (6.8) holds uniformly for $x/a_n \in [0, K]$, for every fixed $K > 0$.

We point out that Theorem 6.5 has been obtained also in [**13**], where the authors study random walks conditioned to stay positive in a different sense.

**1.5. Outline of the exposition.** The exposition is organized as follows: in Section 2 we recall some basic facts on Fluctuation Theory and stable laws, and we set the relative notation; we also give the proof of Theorem 6.5. The rest of the chapter is devoted to the proof of Theorem 6.1, which has been split in two parts. The first one, in Section 3, contains the core of the proof: using Fluctuation Theory we obtain an alternative expression for the law of $S_n^+$, see equation (6.19), and we prove a crucial weak convergence result connected to the renewal measure of the ladder variables process. Then in Section 4 we apply these preliminary results, together with Stone's LLT, to complete the proof. Finally, some minor points have been deferred to the appendix.

## 2. Fluctuation Theory and some applications

In this section we are going to recall some basic facts about Fluctuation Theory for random walks, especially in connection with the theory of stable laws, and to derive some preliminary results. Standard references on the subject are [**29**] and [**7**].

**2.1. Regular variation.** A positive sequence $d_n$ is said to be *regularly varying* of index $\alpha \in \mathbb{R}$ (we denote this by $d_n \in R_\alpha$) if $d_n \sim L_n\, n^\alpha$ as $n \to \infty$, where $L_n$ is *slowly varying* at $\infty$ in that $L_{\lfloor tn \rfloor}/L_n \to 1$ as $n \to \infty$, for every $t > 0$. If $d_n \in R_\alpha$ with $\alpha \neq 0$, up to asymptotic equivalence we can (and will) always assume [**7**, Th.1.5.3] that $d_n = d(n)$, with $d(\cdot)$ a continuous, strictly monotone function, whose inverse will be denoted by $d^{-1}(\cdot)$. Observe that if $d_n \in R_\alpha$ then $d^{-1}(n) \in R_{1/\alpha}$ and $1/d_n \in R_{-\alpha}$.

Let us recall two basic facts on regularly varying sequences that will be used a number of times in the sequel. The first one is a uniform convergence property [**7**, Th.1.2.1]: if $d_n \in R_\alpha$, then

$$d_{\lfloor tn \rfloor} = t^\alpha\, d_n \big(1 + o(1)\big) \qquad (n \to \infty)\,, \tag{6.9}$$

uniformly for $t \in [\varepsilon, 1/\varepsilon]$, for every fixed $\varepsilon > 0$. The second basic fact [**7**, Prop.1.5.8] is that if $d_n \in R_\alpha$ with $\alpha > -1$, then

$$\sum_{k=1}^{n} d_k \sim \frac{n d_n}{\alpha + 1} \qquad (n \to \infty)\,. \tag{6.10}$$

**2.2. Ladder variables and stability.** The first (strict ascending) *ladder epoch* $T_1$ of a random walk $S_n = X_1 + \ldots + X_n$ is the first time the random walk enters the positive half line, and the corresponding *ladder height* $H_1$ is the position of the walk at that time:

$$T_1 := \inf\{n > 0 : S_n > 0\} \qquad H_1 := S_{T_1}\,.$$

Iterating these definitions one gets the following ladder variables: more precisely, for $k > 1$ one defines inductively

$$T_k := \inf\{n > T_{k-1} : S_n > H_{k-1}\} \qquad H_k := S_{T_k}\,,$$

and for convenience we put $(T_0, H_0) := (0, 0)$. The *weak* ascending ladder variables are defined in a similar way, just replacing $>$ by $\geq$ in the relations $(S_n > 0)$ and $(S_n > H_{k-1})$ above. In the following we will rather consider the weak *descending* ladder

variables $(\overline{T}_k, \overline{H}_k)$, which are by definition the weak ascending ladder variables of the walk $\{-S_n\}$. Observe that, by the strong Markov property, both $\{(T_k, H_k)\}_k$ and $\{(\overline{T}_k, \overline{H}_k)\}_k$ are bidimensional renewal processes, that is random walks on $\mathbb{R}^2$ with step law supported in the first quadrant.

It is known that $X_1$ is in the domain of attraction (without centering) of a stable law if and only if $(T_1, H_1)$ lies in a bivariate domain of attraction, cf. [**38, 25, 26**]. This fact will play a fundamental role in our derivation: let us specialize it to our setting. By hypothesis $X_1$ is attracted to the normal law, that is $S_n/a_n \Rightarrow \varphi(x)\,\mathrm{d}x$, so that by the standard theory of stability $a_n \in R_{1/2}$. We define two sequences $b_n,\ c_n$ by

$$\log \frac{n}{\sqrt{2}} = \sum_{m=1}^{\infty} \frac{\rho_m}{m}\, e^{-\frac{m}{b_n}} \qquad c_n := a(b_n)\,, \tag{6.11}$$

where $\rho_m := \mathbf{P}(S_m > 0)$: then $b_n \in R_2$, $c_n \in R_1$ and we have the weak convergence

$$\left(\frac{T_n}{b_n}\,, \frac{H_n}{c_n}\right) \Rightarrow Z\,, \qquad \mathbf{P}\big(Z \in (\mathrm{d}x,\, \mathrm{d}y)\big) = \frac{e^{-1/2x}}{\sqrt{2\pi}\, x^{3/2}} \mathbf{1}_{(x\geq 0)}\, \mathrm{d}x \cdot\, \delta_1(\mathrm{d}y)\,, \tag{6.12}$$

where $\delta_1(\mathrm{d}y)$ denotes the Dirac measure at $y = 1$.

Thus the first ladder epoch $T_1$ is attracted to the positive stable law of index $1/2$, as for the simple random walk case:

$$\frac{T_n}{b_n} \Rightarrow Y, \qquad \mathbf{P}\big(Y \in \mathrm{d}x\big) = \frac{e^{-1/2x}}{\sqrt{2\pi}\, x^{3/2}} \mathbf{1}_{(x\geq 0)}\, \mathrm{d}x\,,$$

while for $\{H_k\}$ one has a generalized law of large numbers, with norming sequence $c_n$: $H_n/c_n \Rightarrow 1$ (that is $H_1$ is *relatively stable*, cf. [**7**, §8.8]).

We stress that we choose the sequence $a_n$ to be increasing, and by (6.11) $b_n$ and $c_n$ are increasing too. We also recall that the norming sequence $b_n$ is sharply linked to the probability tail of the random variable $T_1$, by the relation

$$\mathbf{P}\big(T_1 > b_n\big) \sim \sqrt{\frac{2}{\pi}}\, \frac{1}{n}\,. \tag{6.13}$$

In fact, this condition is necessary and sufficient in order that a sequence $b_n$ be such that $T_n/b_n \Rightarrow Y$, cf. [**29**, §XIII.6].

Remark 6.6. It has already been noticed that when the step $X_1$ has finite (nonzero) variance and zero mean,

$$\mathbf{E}\big(X_1\big) = 0 \qquad \mathbf{E}\big(X_1^2\big) =: \sigma^2 \in\ (0, \infty)\,,$$

by the Central Limit Theorem one can take $a_n = \sigma\sqrt{n}$ in order that equation (6.1) holds. In other words, $X_1$ is in the *normal domain of attraction* of the normal law. In this case the first ladder height $H_1$ is integrable [**23**] and the behavior of the tail of $T_1$ is given by

$$\mathbf{P}\big(T_1 > n\big) \sim \frac{2\,\mathbf{E}(H_1)}{\sigma\sqrt{2\pi}} \frac{1}{\sqrt{n}} \qquad (n \to \infty)\,,$$

cf. [**29**, Th.1 in §XII.7 & Th.1 in §XVIII.5]. This means that also $T_1$ and $H_1$ belong to the normal domain of attraction of their respective limit law, and one can take

$$b_n = \frac{\mathbf{E}(H_1)^2}{\sigma^2}\, n^2 \qquad c_n = \mathbf{E}(H_1)\, n$$

in order that (6.12) holds (we have used the law of large numbers for $H_1$ and relation (6.13) for $T_1$).

**2.3. An asymptotic result.** As an application of the results exposed so far, we derive the asymptotic behavior of $\mathbf{P}(\mathcal{C}_n)$ as $n \to \infty$, which will be needed in the sequel. The connection with Fluctuation Theory is given by the fact that

$$\mathcal{C}_n := \big(S_1 > 0, \ldots, S_n > 0\big) = \big(\overline{T}_1 > n\big)\,.$$

In analogy to what we have seen for $T_1$, the fact that the random walk is attracted to the normal law implies that $\overline{T}_1$ lies in the domain of attraction of the positive stable law of index $1/2$. Therefore $\mathbf{P}(\mathcal{C}_n) \in R_{-1/2}$, and denoting by $\psi(t) := \mathbf{E}(\exp(-t\overline{T}_1))$ the Laplace transform of $\overline{T}_1$, by standard Tauberian theorems [**29**, Ex.(c) in §XIII.5] we have that

$$\mathbf{P}(\mathcal{C}_n) \sim \frac{1}{\sqrt{\pi}} \big(1 - \psi(1/n)\big) \qquad (n \to \infty)\,.$$

Now, for $\psi(t)$ we have the following explicit expression [**29**, Th.1 in §XII.7]:

$$-\log(1-\psi(t)) \;=\; \sum_{m=1}^{\infty} \frac{\overline{\rho}_m}{m} e^{-mt} \;=\; -\log(1-e^{-t}) - \sum_{m=1}^{\infty} \frac{\rho_m}{m} e^{-mt}\,,$$

where $\overline{\rho}_m := \mathbf{P}(S_m \leq 0)$. A look to (6.11) then yields the desired asymptotic behavior:

$$\mathbf{P}(\mathcal{C}_n) \sim \frac{1}{\sqrt{2\pi}} \frac{b^{-1}(n)}{n} \qquad (n \to \infty)\,. \tag{6.14}$$

**2.4. Two combinatorial identities.** The power of Fluctuation Theory for the study of random walks is linked to some fundamental identities. The most famous one is the so-called Duality Lemma [**29**, §XII] which can be expressed as

$$\mathbf{P}\big(n \text{ is a ladder epoch}, S_n \in \mathrm{d}x\big) = \mathbf{P}\big(\mathcal{C}_n, S_n \in \mathrm{d}x\big)\,, \tag{6.15}$$

where by ($n$ is a ladder epoch) we mean of course the disjoint union $\cup_{k\geq 0}(T_k = n)$, and by $\mathbf{P}(A, Z \in \mathrm{d}x)$ we denote the finite measure $B \mapsto \mathbf{P}(A, Z \in B)$. A second important identity, recently discovered by Alili and Doney [**3**], will play a fundamental role for us:

$$\mathbf{P}\big(T_k = n, H_k \in \mathrm{d}x\big) = \frac{k}{n}\,\mathbf{P}\big(H_{k-1} < S_n \leq H_k, S_n \in \mathrm{d}x\big)\,. \tag{6.16}$$

We point out that both the above identities are of purely combinatorial nature, in the sense that they can be proved by relating the events on the two sides with suitable one to one, measure preserving transformations on the sample paths space.

**2.5. Proof of Theorem 6.5.** We recall that by hypothesis $\varepsilon$ is a fixed positive number. We start from the definition (6.6) of $u(n,x)$: applying the Duality Lemma (6.15) we get

$$u(n,x) = \mathbf{P}\big(\,\mathcal{C}_n, S_n = x\big) = \mathbf{P}\big(\,\mathcal{C}_n\big)\,\mathbf{P}\big(S_n = x \,\big|\, \mathcal{C}_n\big)\,. \tag{6.17}$$

Observe that

$$\inf_{z\in[\varepsilon,1/\varepsilon]} \varphi^+(z) > 0 \qquad \inf_{z\in[\varepsilon,1/\varepsilon]} \varphi(z) > 0\,,$$

which implies that both Theorem 6.2 and Gnedenko's LLT (6.5) hold also in a ratio sense, namely

$$\mathbf{P}\big(S_n = x \,\big|\, \mathcal{C}_n\big) = \frac{1}{a_n}\,\varphi^+(x/a_n)\,\big(1+o(1)\big) \qquad (n\to\infty)$$
$$\mathbf{P}\big(S_n = x\,\big) = \frac{1}{a_n}\,\varphi\,(x/a_n)\,\big(1+o(1)\big) \qquad (n\to\infty)\,,$$

uniformly for $x/a_n \in [\varepsilon, 1/\varepsilon]$. Since $\varphi^+(z) = \sqrt{2\pi}\,z\,\varphi(z)$ for $z>0$, it follows that

$$\mathbf{P}\big(S_n = x \,\big|\, \mathcal{C}_n\big) = \sqrt{2\pi}\,\frac{x}{a_n}\,\mathbf{P}\big(S_n = x\,\big)\,\big(1+o(1)\big) \qquad (n\to\infty)\,, \tag{6.18}$$

uniformly for $x/a_n \in [\varepsilon, 1/\varepsilon]$.

The asymptotic behavior of $\mathbf{P}(\mathcal{C}_n)$ is given by (6.14), and comparing equation (6.8) with (6.18) and (6.17) we are left with proving that

$$U(x) = x\,\frac{b^{-1}(n)}{a(n)}\,\big(1+o(1)\big) \qquad (n\to\infty)\,,$$

uniformly for $x/a_n \in [\varepsilon, 1/\varepsilon]$. We recall that $U(x)$ is the distribution function of the renewal measure associated to the ladder height process $\{H_k\}$, which is relatively stable, since $H_n/c_n \Rightarrow 1$ as $n \to \infty$. Then Theorem 8.8.1 in [**7**] gives that $U(x) \sim c^{-1}(x)$ as $x \to \infty$, hence it rests to show that

$$\frac{x}{c^{-1}(x)} \frac{b^{-1}(n)}{a(n)} \to 1 \qquad (n \to \infty)\,,$$

uniformly for $x/a_n \in [\varepsilon, 1/\varepsilon]$, or equivalently, setting $x = z\, a_n$, that

$$\frac{z\, b^{-1}(n)}{c^{-1}(z\, a(n))} \to 1 \qquad (n \to \infty)\,,$$

uniformly for $z \in [\varepsilon, 1/\varepsilon]$. However, as $c^{-1}(\cdot) \in R_1$, by (6.9) we have that

$$c^{-1}(z\, a(n)) \sim z\, c^{-1}(a(n)) \qquad (n \to \infty)\,,$$

uniformly for $z \in [\varepsilon, 1/\varepsilon]$, and the proof is completed observing that $c^{-1}(a(n)) = b^{-1}(n)$, by the definition (6.11) of $c_n$. □

## 3. First part of the proof

**3.1. A fundamental expression.** We are going to use Fluctuation Theory to express the law of $S_n^+$ in a more useful way. For $x > 0$ and $n > 1$ we have

$$\begin{aligned} n\, \mathbf{P}\big(\mathcal{C}_n,\, S_n \in \mathrm{d}x\big) &\overset{(6.15)}{=} n\, \mathbf{P}\big(n \text{ is a ladder epoch},\, S_n \in \mathrm{d}x\big) \\ &= \sum_{r=1}^{\infty} n\, \mathbf{P}\big(T_r = n,\, S_n \in \mathrm{d}x\big) \overset{(6.16)}{=} \sum_{r=1}^{\infty} r\, \mathbf{P}\big(H_{r-1} < x \le H_r,\, S_n \in \mathrm{d}x\big)\,, \end{aligned}$$

where we have used both the combinatorial identities (6.15), (6.16). With a simple manipulation we get

$$\begin{aligned} \sum_{r=1}^{\infty} r\, \mathbf{P}\big(H_{r-1} < x \le H_r,\, S_n \in \mathrm{d}x\big) &= \sum_{r=1}^{\infty} \sum_{k=0}^{r-1} \mathbf{P}\big(H_{r-1} < x \le H_r,\, S_n \in \mathrm{d}x\big) \\ = \sum_{k=0}^{\infty} \sum_{r=k+1}^{\infty} \mathbf{P}\big(H_{r-1} < x \le H_r,\, S_n \in \mathrm{d}x\big) &= \sum_{k=0}^{\infty} \mathbf{P}\big(H_k < x,\, S_n \in \mathrm{d}x\big)\,, \end{aligned}$$

and using the Markov property

$$\mathbf{P}\big(H_k < x,\, S_n \in \mathrm{d}x\big) = \sum_{m=0}^{n-1} \int_{[0,x)} \mathbf{P}\big(T_k = m,\, H_k \in \mathrm{d}z\big)\, \mathbf{P}\big(S_{n-m} \in \mathrm{d}x - z\big)\,.$$

In conclusion we obtain the following relation (which is essentially the same as equation (10) in [**3**]):

$$\begin{aligned}
&\mathbf{P}\big(S_n/a_n \in \mathrm{d}x \,\big|\, \mathcal{C}_n\big) \\
&\quad = \frac{1}{n\mathbf{P}(\mathcal{C}_n)} \sum_{m=0}^{n-1} \int_{[0,a_n x)} \left( \sum_{k=0}^{\infty} \mathbf{P}\big(T_k = m,\, H_k \in \mathrm{d}z\big) \right) \mathbf{P}\big(S_{n-m} \in a_n \mathrm{d}x - z\big) \\
&\quad = \frac{b^{-1}(n)}{n\mathbf{P}(\mathcal{C}_n)} \int_{[0,1)\times[0,x)} \mathrm{d}\mu_n(\alpha,\beta)\, \mathbf{P}\left( \frac{S_{\lfloor n(1-\alpha)\rfloor}}{a_n} \in \mathrm{d}x - \beta \right), \qquad (6.19)
\end{aligned}$$

where $\mu_n$ is the finite measure on $[0,1) \times [0,\infty)$ defined by

$$\mu_n(A) \;:=\; \frac{1}{b^{-1}(n)} \sum_{k=0}^{\infty} \mathbf{P}\left( \left( \frac{T_k}{n}, \frac{H_k}{a_n} \right) \in A \right), \qquad (6.20)$$

for $n \in \mathbb{N}$ and for any Borel set $A \subseteq [0,1) \times [0,\infty)$. Notice that $\mu_n$ is nothing but a suitable rescaling of the renewal measure associated to the ladder variables process. Also observe that the sum defining $\mu_n$ can be stopped at $k = n-1$, since by definition $T_k \geq k$ for every $k$; hence $\mu_n$ is indeed a finite measure.

Before proceeding, we would like to stress the importance of equation (6.19), which is in a sense the core of our proof. The reason is that in the r.h.s. the conditioning on $\mathcal{C}_n$ has disappeared: we are left with a mixture, governed by the measure $\mu_n$, of the laws of $\{S_{\lfloor n(1-\alpha)\rfloor}\}_{\alpha\in[0,1)}$ *without conditioning*, and the asymptotic behavior of these laws can be controlled with Stone's Local Limit Theorem (6.2) (if we exclude the values of $\alpha$ close to 1).

In the following subsection we study the asymptotic behavior of the sequence of measures $\{\mu_n\}$, and in the next section we put together these preliminary results to conclude the proof of Theorem 6.1.

**3.2. A weak convergence result.** We are going to show that as $n \to \infty$ the sequence of measure $\{\mu_n\}$ converges weakly to the finite measure $\mu$ defined by

$$\mu(A) \;:=\; \int_A \mathrm{d}\alpha\, \mathrm{d}\beta\, \frac{\beta}{\sqrt{2\pi}\,\alpha^{3/2}}\, e^{-\beta^2/2\alpha}, \qquad (6.21)$$

for any Borel set $A \subseteq [0,1) \times [0,\infty)$ (it is easy to check that $\mu$ is really a finite measure, see below). Since we are not dealing with probability measures, we must be most precise: we mean weak convergence with respect to the class $C_b$ of bounded

and continuous functions on $\mathbb{R}^2$: $\mu_n \Rightarrow \mu$ iff $\int h\,\mathrm{d}\mu_n \to \int h\,\mathrm{d}\mu$ for every $h \in C_b$. If we introduce the distribution functions $F_n$, $F$ of the measures $\mu_n$, $\mu$:

$$F_n(a,b) := \mu_n\big([0,a]\times[0,b]\big) \qquad F(a,b) := \mu\big([0,a]\times[0,b]\big)\,,$$

then proving that $\mu_n \Rightarrow \mu$ as $n \to \infty$ is equivalent to showing that $F_n(a,b) \to F(a,b)$ for every $(a,b) \in [0,1]\times[0,\infty]$ (notice that $\infty$ is included, because the total mass of $\mu_n$ is not fixed).

PROPOSITION 6.7. *The sequence of measures $\{\mu_n\}$ converges weakly to the measure $\mu$.*

**Proof.** We start checking the convergence of the total mass:

$$F_n(1,\infty) = \frac{1}{b^{-1}(n)}\sum_{k=0}^{\infty}\mathbf{P}\big(T_k \le n\big) =: \frac{1}{b^{-1}(n)}\,G(n)\,,$$

where $G(n)$ is the distribution function of the renewal measure associated to the ladder epochs process $\{T_k\}$. There is a sharp link between the asymptotic behavior as $n \to \infty$ of $G(n)$ and that of $\mathbf{P}(T_1 > n)$, given by [**29**, Lem. in §XIV.3]:

$$G(n) \sim \frac{2}{\pi}\,\frac{1}{\mathbf{P}(T_1 > n)} \qquad (n \to \infty)\,. \tag{6.22}$$

Since from relation (6.13) we have that

$$\mathbf{P}\big(T_1 > n\big) \sim \sqrt{\frac{2}{\pi}}\,\frac{1}{b^{-1}(n)} \qquad (n \to \infty)\,,$$

it follows that $F_n(1,\infty) \to \sqrt{2/\pi}$ as $n \to \infty$. On the other hand, the check that $F(1,\infty) = \sqrt{2/\pi}$ is immediate:

$$F(1,\infty) = \frac{1}{\sqrt{2\pi}}\int_0^1 \mathrm{d}\alpha\,\frac{1}{\alpha^{3/2}}\int_0^\infty \mathrm{d}\beta\,\beta\,e^{-\beta^2/2\alpha} = \frac{1}{\sqrt{2\pi}}\int_0^1 \mathrm{d}\alpha\,\frac{1}{\sqrt{\alpha}} = \sqrt{\frac{2}{\pi}}\,.$$

Since the total mass converges, we claim that it suffices to show that

$$\liminf_{n\to\infty}\mu_n\big((a_1,a_2]\times(b_1,b_2]\big) \;\ge\; \mu\big((a_1,a_2]\times(b_1,b_2]\big) \tag{6.23}$$

for all $0 < a_1 < a_2 < 1$, $0 < b_1 < b_2 < \infty$, and weak convergence will be proved. The (simple) proof of this claim can be found in § 5.1.

Directly from the definition of $\mu_n$ we have

$$\mu_n\big((a_1,a_2]\times(b_1,b_2]\big) = \frac{1}{b^{-1}(n)}\sum_{k=0}^{\infty}\mathbf{P}\left(\frac{T_k}{n}\in(a_1,a_2],\,\frac{H_k}{a_n}\in(b_1,b_2]\right).$$

We simply restrict the sum to the set of $k$ such that $k/b^{-1}(n) \in (b_1+\varepsilon, b_2-\varepsilon]$, $\varepsilon$ being a small fixed positive number, getting

$$\mu_n\big((a_1,a_2]\times(b_1,b_2]\big) \;\geq\; \frac{1}{b^{-1}(n)} \sum_{s\in \frac{\mathbb{Z}}{b^{-1}(n)}\cap(b_1+\varepsilon,b_2-\varepsilon]} \xi_n(s)\,, \tag{6.24}$$

where

$$\xi_n(s) := \mathbf{P}\left( \frac{T_{\lfloor sb^{-1}(n)\rfloor}}{n} \in (a_1,a_2],\; \frac{H_{\lfloor sb^{-1}(n)\rfloor}}{a_n} \in (b_1,b_2] \right).$$

By the definition (6.11) of $c_n$, we have that $a_n = c(b^{-1}(n))$: then, using the weak convergence (6.12) and the uniform convergence property of regularly varying sequences (6.9), it is not difficult to check that

$$\xi_n(s) \to \mathbf{P}\left( Y \in \left( \frac{a_1}{s^2}, \frac{a_2}{s^2} \right] \right) =: \xi(s) \qquad (n\to\infty)\,,$$

*uniformly* for $s \in (b_1+\varepsilon, b_2-\varepsilon]$.

Observe that the term in the r.h.s. of (6.24) is a Riemann sum of the function $\xi_n(s)$ over the bounded interval $(b_1+\varepsilon, b_2-\varepsilon]$. Since the sequence of functions $\{\xi_n(s)\}$ is clearly equibounded and converges uniformly to $\xi(s)$, it is immediate to check that the r.h.s. of (6.24) does converge to the integral of $\xi(s)$ over $(b_1+\varepsilon, b_2-\varepsilon]$. Therefore

$$\begin{aligned}
\liminf_{n\to\infty} \mu_n\big((a_1,a_2]\times(b_1,b_2]\big) \;&\geq\; \int_{b_1+\varepsilon}^{b_2-\varepsilon} \mathrm{d}s\; \mathbf{P}\left( Y \in \left( \frac{a_1}{s^2}, \frac{a_2}{s^2} \right] \right) \\
&=\; \int_{b_1+\varepsilon}^{b_2-\varepsilon} \mathrm{d}s \int_{a_1/s^2}^{a_2/s^2} \mathrm{d}z\; \frac{e^{-1/2z}}{\sqrt{2\pi}\, z^{3/2}} \;=\; \int_{b_1+\varepsilon}^{b_2-\varepsilon} \mathrm{d}s \int_{a_1}^{a_2} \mathrm{d}t\; \frac{s\, e^{-s^2/2t}}{\sqrt{2\pi}\, t^{3/2}} \\
&=\; \mu\big((a_1,a_2]\times(b_1+\varepsilon,b_2-\varepsilon]\big)\,,
\end{aligned}$$

and letting $\varepsilon \to 0$ relation (6.23) follows. □

## 4. Second part of the proof

**4.1. General strategy.** Now we are ready to put together the results obtained in the last section. We start by rephrasing relation (6.4), which is our final goal, in terms of $S_n/a_n$, a form that is more convenient for our purposes: we have to prove that

$$\forall K>0 \quad \limsup_{n\to\infty}\; a_n \left[ \sup_{x\in\mathbb{R}^+,\, h\leq K/a_n} \Big| \mathbf{P}\big(S_n/a_n \in x+I_h \,\big|\, \mathcal{C}_n\big) \;-\; h\,\varphi^+(x) \Big| \right] = 0\,, \tag{6.25}$$

where $I_h := [0,h)$, and $x+I_h := [x, x+h)$.

Altough the idea behind the proof is quite simple, our arguments depend on an approximation parameter $\varepsilon$ and there are a number of somewhat technical points. In order to keep the exposition as transparent as possible, it is convenient to introduce the following notation: given two real functions $f(n,x,h,\varepsilon)$ and $g(n,x,h,\varepsilon)$ of the variables $n \in \mathbb{N}$, $x \in \mathbb{R}^+$, $h \in \mathbb{R}^+$ and $\varepsilon \in (0,1)$, we say that $f \stackrel{\star}{\sim} g$ if and only if

$$\forall K > 0 \quad \limsup_{\varepsilon \to 0} \limsup_{n \to \infty} a_n \left[ \sup_{x \in \mathbb{R}^+,\, h \le K/a_n} \big| f(n,x,h,\varepsilon) - g(n,x,h,\varepsilon) \big| \right] = 0 \,.$$

With this terminology we can reformulate (6.25) as

$$\mathbf{P}\big(S_n/a_n \in x + I_h \,\big|\, \mathcal{C}_n\big) \;\stackrel{\star}{\sim}\; h\,\varphi^+(x)\,. \tag{6.26}$$

To obtain a more explicit expression of the l.h.s. of (6.26), we resort to equation (6.19): with an easy integration we get

$$\mathbf{P}\big(S_n/a_n \in x + I_h \,\big|\, \mathcal{C}_n\big) \;=\; \frac{b^{-1}(n)}{n\mathbf{P}(\mathcal{C}_n)} \int_{D_1^{x+h}} \mathrm{d}\mu_n(\alpha,\beta)\, \widehat{G}_n^{x,h}(\alpha,\beta)\,, \tag{6.27}$$

where we have introduced the notation $D_a^b := [0,a) \times [0,b)$, and

$$\widehat{G}_n^{x,h}(\alpha,\beta) \;:=\; \mathbf{P}\left( \frac{S_{\lfloor n(1-\alpha) \rfloor}}{a_n} \in \big\{ (x-\beta) + I_h \big\} \cap [0,\infty) \right). \tag{6.28}$$

In order to determine the asymptotic behavior of the r.h.s. of (6.27), we recall that:

- from (6.14) we have
$$\frac{b^{-1}(n)}{n\mathbf{P}(\mathcal{C}_n)} \to \sqrt{2\pi}\,;$$
- from Proposition 6.7 we have that $\mu_n \Rightarrow \mu$;
- from Stone's LLT (6.2) it follows that, for large $n$, $\widehat{G}_n^{x,h}(\alpha,\beta)$ is close to
$$\mathcal{G}^{x,h}(\alpha,\beta) \;:=\; h\,\frac{1}{\sqrt{1-\alpha}}\,\varphi\!\left(\frac{x-\beta}{\sqrt{1-\alpha}}\right), \tag{6.29}$$
where we have used that $a_{n(1-\alpha)} \sim \sqrt{1-\alpha}\, a_n$ as $n \to \infty$, by (6.9).

In fact, the rest of this section is devoted to showing that

$$\mathbf{P}\big(S_n/a_n \in x + I_h \,\big|\, \mathcal{C}_n\big) \;\stackrel{\star}{\sim}\; \sqrt{2\pi} \int_{D_1^x} \mathrm{d}\mu(\alpha,\beta)\, \mathcal{G}^{x,h}(\alpha,\beta)\,. \tag{6.30}$$

It may not be a priori obvious whether this coincides with our goal (6.26), that is whether

$$\varphi^+(x) \;=\; \sqrt{2\pi} \int_{D_1^x} \mathrm{d}\mu(\alpha,\beta)\, \frac{1}{\sqrt{1-\alpha}}\,\varphi\!\left(\frac{x-\beta}{\sqrt{1-\alpha}}\right). \tag{6.31}$$

Indeed this relation holds true: in fact (6.30) implies the weak convergence of $S_n/a_n$ under $\mathbf{P}(\,\cdot\,|\,\mathcal{C}_n)$ towards a limiting law with the r.h.s. of (6.31) as density, and we already know from (6.3) that $S_n/a_n$ under $\mathbf{P}(\,\cdot\,|\,\mathcal{C}_n)$ converges weakly to $\varphi^+(x)\,\mathrm{d}x$. Anyway, a more direct verification of (6.31) is also given in § 5.2.

Thus we are left with proving (6.30), or equivalently

$$\int_{D_1^{x+h}} \mathrm{d}\mu_n(\alpha,\beta)\,\widehat{G}_n^{x,h}(\alpha,\beta) \;\stackrel{\star}{\sim}\; \int_{D_1^x} \mathrm{d}\mu(\alpha,\beta)\,\mathcal{G}^{x,h}(\alpha,\beta)\,.$$

Since $\stackrel{\star}{\sim}$ is an equivalence relation, this will be done through a sequence of intermediate equivalences:

$$\int_{D_1^{x+h}} \mathrm{d}\mu_n\,\widehat{G}_n^{x,h} \;\stackrel{\star}{\sim}\; \ldots \;\stackrel{\star}{\sim}\; \ldots \;\stackrel{\star}{\sim}\; \ldots \;\stackrel{\star}{\sim}\; \int_{D_1^x} \mathrm{d}\mu\,\mathcal{G}^{x,h}\,,$$

and for ease of exposition the proof has been accordingly split in four steps. The idea is quite simple: we first restrict the domain from $D_1^{x+h}$ to $D_{1-\varepsilon}^x$ (steps 1–2), then we will be able to apply Stone's LLT and Proposition 6.7 to pass from $(\widehat{G}_n^{x,h},\mu_n)$ to $(\mathcal{G}^{x,h},\mu)$ (step 3), and finally we come back to the domain $D_1^x$ (step 4).

Before proceeding, we define a slight variant $G_n^{x,h}$ of $\widehat{G}_n^{x,h}$:

$$G_n^{x,h}(\alpha,\beta) \;:=\; \mathbf{P}\left(\frac{S_{\lfloor n(1-\alpha)\rfloor}}{a_n} \in (x-\beta)+I_h\right) \tag{6.32}$$

(notice that we have simply removed the set $[0,\infty)$, see (6.28)) and we establish a preliminary lemma.

LEMMA 6.8. *For every $K>0$ there exists a positive constant $\mathtt{C}=\mathtt{C}(K)$ such that*

$$G_n^{x,h}(\alpha,\beta) \le \frac{\mathtt{C}}{a_{\lfloor(1-\alpha)n\rfloor}} \qquad \forall n\in\mathbb{N},\ \forall x,\beta\in\mathbb{R},\ \forall\alpha\in[0,1),\ \forall h\le K/a_n\,,$$

*and the same relation holds also for $\widehat{G}_n^{x,h}(\alpha,\beta)$.*

**Proof.** Since by definition $\widehat{G}_n^{x,h}(\alpha,\beta)\le G_n^{x,h}(\alpha,\beta)$, it suffices to prove the relation for $G_n^{x,h}$. However, this is a simple consequence of Stone's LLT (6.2), that we can rewrite in terms of $S_n/a_n$ as

$$\forall K>0 \quad \limsup_{l\to\infty}\, a_l\left[\sup_{y\in\mathbb{R},\,h'\le K/a_l}\Big|\,\mathbf{P}\big(S_l/a_l\in y+I_{h'}\big)\;-\;h'\,\varphi(y)\,\Big|\right]=0\,. \tag{6.33}$$

In fact from this relation, using the triangle inequality and the fact that $\sup_{x\in\mathbb{R}} |\varphi(x)| < \infty$, it follows easily that for every $K > 0$

$$a_l \, \mathbf{P}\big(S_l/a_l \in y + I_{h'}\big) \le \mathtt{C} \qquad \forall l \in \mathbb{N},\ \forall y \in \mathbb{R},\ \forall h' \le K/a_l\,, \tag{6.34}$$

for some positive constant $\mathtt{C} = \mathtt{C}(K)$. Now it suffices to observe that $G_n^{x,h}$ can be written as

$$G_n^{x,h}(\alpha,\beta) \;=\; \mathbf{P}\left(\frac{S_{\lfloor n(1-\alpha)\rfloor}}{a_{\lfloor n(1-\alpha)\rfloor}} \in \frac{a_n}{a_{\lfloor n(1-\alpha)\rfloor}}(x-\beta) + I_{\frac{h\,a_n}{a_{\lfloor n(1-\alpha)\rfloor}}}\right), \tag{6.35}$$

so that we can apply (6.34) with $l = \lfloor n(1-\alpha)\rfloor$ and analogous substitutions. □

**4.2. First step.** In the first intermediate equivalence we pass from the domain $D_1^{x+h}$ to $D_{1-\varepsilon}^{x+h}$, that is we are going to show that

$$\int_{D_1^{x+h}} \mathrm{d}\mu_n \, \widehat{G}_n^{x,h} \;\overset{\star}{\sim}\; \int_{D_{1-\varepsilon}^{x+h}} \mathrm{d}\mu_n \, \widehat{G}_n^{x,h}\,.$$

This means by definition that for every $K > 0$

$$\limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \, R_n^\varepsilon = 0\,, \tag{6.36}$$

where $R_n^\varepsilon := \sup_{\{x\in\mathbb{R}^+,\, h\le K/a_n\}} r_n^\varepsilon(x,h)$ and

$$r_n^\varepsilon(x,h) := a_n \int_{[1-\varepsilon,1)\times[0,x+h)} \mathrm{d}\mu_n(\alpha,\beta)\, \widehat{G}_n^{x,h}(\alpha,\beta)\,.$$

Applying Lemma 6.8 and enlarging the domain of integration, we get

$$\begin{aligned} R_n^\varepsilon &\le \mathtt{C}\, a_n \int_{[1-\varepsilon,1)\times[0,\infty)} \mathrm{d}\mu_n(\alpha,\beta)\, \frac{1}{a_{\lfloor(1-\alpha)n\rfloor}} \\ &= \mathtt{C}\, a_n \sum_{m=\lfloor(1-\varepsilon)n\rfloor}^{n-1} \left[\frac{1}{b^{-1}(n)} \sum_{k=0}^{\infty} \mathbf{P}\big(T_k = m\big)\right] \frac{1}{a_{n-m}} \\ &= \mathtt{C}\, \frac{a_n}{b^{-1}(n)} \sum_{m=\lfloor(1-\varepsilon)n\rfloor}^{n-1} \frac{u(m)}{a_{n-m}}\,, \end{aligned} \tag{6.37}$$

where in the second line we have applied the definition (6.20) of $\mu_n$, and in the third line we have introduced $u(m) := \sum_{k=0}^{\infty} \mathbf{P}(T_k = m)$, which is the mass function of the renewal measure associated to the ladder epochs process $\{T_k\}$. In the proof of Proposition 6.7 we have encountered the asymptotic behavior of the distribution

function $G(n) := \sum_{m=1}^{n} u(m)$, see (6.22). The corresponding local asymptotic behavior for $u(m)$ follows since the sequence $u(m)$ is decreasing in $m$ (this is a simple consequence of the Duality Lemma (6.15), see also [**27**, Th.4]): hence

$$u(m) \sim \frac{1}{\pi} \frac{1}{m\mathbf{P}(T_1 > m)} \sim \frac{1}{\sqrt{2\pi}} \frac{b^{-1}(m)}{m} \qquad (m \to \infty)\,,$$

having used (6.13). It follows that $u(m) \le C_1\, b^{-1}(m)/m$ for every $m$, for some positive constant $C_1$. Recalling that $b^{-1}(\cdot)$ is increasing, from (6.37) we get

$$\begin{aligned} R_n^\varepsilon \;&\le\; \mathtt{C}\, C_1 \frac{a_n}{b^{-1}(n)} \sum_{m=\lfloor(1-\varepsilon)n\rfloor}^{n-1} \frac{b^{-1}(m)}{m\, a_{n-m}} \\ &\le\; \mathtt{C} C_1 \, \frac{a_n}{\lfloor(1-\varepsilon)n\rfloor} \sum_{k=1}^{\lfloor\varepsilon n\rfloor} \frac{1}{a_k} \;\le\; \mathtt{C} C_1 C_2 \, \frac{\varepsilon}{1-\varepsilon} \, \frac{a_n}{a_{\lfloor\varepsilon n\rfloor}}\,, \end{aligned}$$

for some positive constant $C_2$: in the last inequality we have used (6.10), since $a_n \in R_{1/2}$. Now from (6.9) we have that $a_n/a_{\lfloor\varepsilon n\rfloor} \to 1/\sqrt{\varepsilon}$ as $n \to \infty$, hence

$$\limsup_{n\to\infty} R_n^\varepsilon \le C\, \frac{\sqrt{\varepsilon}}{1-\varepsilon}\,,$$

with $C := \mathtt{C} C_1 C_2$, and (6.36) follows.

**4.3. Second step.** Now we show that we can restrict the domain from $D_{1-\varepsilon}^{x+h}$ to $D_{1-\varepsilon}^{x}$:

$$\int_{D_{1-\varepsilon}^{x+h}} \mathrm{d}\mu_n\, \widehat{G}_n^{x,h} \;\overset{\star}{\sim}\; \int_{D_{1-\varepsilon}^{x}} \mathrm{d}\mu_n\, \widehat{G}_n^{x,h} \;=\; \int_{D_{1-\varepsilon}^{x}} \mathrm{d}\mu_n\, G_n^{x,h}\,,$$

where the equality simply follows from the fact that by definition (see (6.28) and (6.32))

$$\widehat{G}_n^{x,h}(\alpha,\beta) = G_n^{x,h}(\alpha,\beta) \qquad \text{for } \beta \le x\,.$$

We have to show that for every $K > 0$

$$\limsup_{\varepsilon\to 0} \limsup_{n\to\infty} Q_n^\varepsilon = 0\,, \tag{6.38}$$

where $Q_n^\varepsilon := \sup_{\{x\in\mathbb{R}^+,\, h\le K/a_n\}} q_n^\varepsilon(x,h)$ and

$$q_n^\varepsilon(x,h) := a_n \int_{[0,1-\varepsilon)\times[x,x+h)} \mathrm{d}\mu_n(\alpha,\beta)\, \widehat{G}_n^{x,h}(\alpha,\beta)\,.$$

From Lemma 6.8 and from the fact that $a_n$ is increasing we easily get

$$q_n^\varepsilon(x,h) \le \mathtt{C}\, \frac{a_n}{a_{\lfloor\varepsilon n\rfloor}}\, \mu_n\big([0,1-\varepsilon)\times[x,x+h)\big)\,.$$

As $a_n \in R_{1/2}$, we have $a_n/a_{\lfloor \varepsilon n \rfloor} \to 1/\sqrt{\varepsilon}$ as $n \to \infty$ by (6.9), hence for fixed $\varepsilon > 0$ we can find a positive constant $C_1 = C_1(\varepsilon)$ such that for all $n \in \mathbb{N}$

$$q_n^\varepsilon(x,h) \le \mathtt{C}\, C_1\, \mu_n\big([0,1-\varepsilon) \times [x, x+h)\big)\,.$$

However the term in the r.h.s. can be easily estimated: using the definition (6.20) of $\mu_n$, for $h \le K/a_n$ we get

$$\begin{aligned}\mu_n\big([0,1-\varepsilon) \times [x,x+h)\big) \;&=\; \frac{1}{b^{-1}(n)} \sum_{k=0}^{\infty} \mathbf{P}\big(T_k < (1-\varepsilon)n,\, H_k \in [a_n x, a_n x + a_n h)\big) \\ &\le\; \frac{1}{b^{-1}(n)} \sum_{k=0}^{\infty} \mathbf{P}\big(H_k \in [a_n x, a_n x + K)\big) \;\le\; \frac{1}{b^{-1}(n)} \sup_{z \in \mathbb{R}^+} U\big([z, z+K)\big)\,,\end{aligned}$$

where $U(\mathrm{d}x) := \sum_{k=0}^{\infty} \mathbf{P}(H_k \in \mathrm{d}x)$ is the renewal measure associated to the ladder heights process $\{H_k\}$, that we have already encountered in the proof of Theorem 6.5. Notice that

$$\forall K > 0 \qquad \sup_{z \in \mathbb{R}^+} U\big([z, z+K)\big) =: C_2 < \infty\,,$$

which holds whenever $\{H_k\}$ is a transient random walk, cf. [**29**, Th.1 in §VI.10]. Thus for every fixed $\varepsilon > 0$

$$Q_n^\varepsilon = \sup_{x \in \mathbb{R}^+,\, h \le K/a_n} q_n^\varepsilon(x,h) \;\le\; \mathtt{C} C_1 C_2\, \frac{1}{b^{-1}(n)} \;\to\; 0 \qquad (n \to \infty)\,,$$

and (6.38) follows.

**4.4. Third step.** This is the central step: we prove that

$$\int_{D^x_{1-\varepsilon}} \mathrm{d}\mu_n\, G_n^{x,h} \;\overset{\star}{\sim}\; \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu\, \mathcal{G}^{x,h}\,,$$

that is for every $K > 0$

$$\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \sup_{x \in \mathbb{R}^+,\, h \le K/a_n} a_n \left| \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu_n\, G_n^{x,h} \;-\; \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu\, \mathcal{G}^{x,h} \right| = 0\,. \tag{6.39}$$

By the triangle inequality

$$\begin{aligned} a_n &\left| \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu_n\, G_n^{x,h} \;-\; \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu\, \mathcal{G}^{x,h} \right| \\ &\le\; a_n \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu_n \left| G_n^{x,h} - \mathcal{G}^{x,h} \right| \;+\; a_n \left| \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu_n\, \mathcal{G}^{x,h} \;-\; \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu\, \mathcal{G}^{x,h} \right|, \end{aligned} \tag{6.40}$$

and we study separately the two terms in the r.h.s. above.

4.4.1. *First term.* With a rough estimate we have

$$\begin{aligned} a_n \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu_n \left|G_n^{x,h} - \mathcal{G}^{x,h}\right| \\ \leq \left[\sup_{n\in\mathbb{N}} \mu_n\big(D_1^\infty\big)\right] \left(\sup_{(\alpha,\beta)\in D^\infty_{1-\varepsilon}} a_n \left|G_n^{x,h}(\alpha,\beta) - \mathcal{G}^{x,h}(\alpha,\beta)\right|\right), \end{aligned} \tag{6.41}$$

and notice the prefactor in the r.h.s. is bounded since $\mu_n(D_1^\infty) \to \mu(D_1^\infty)$. For the remaining term, we use the triangle inequality and the definition (6.29) of $\mathcal{G}^{x,h}$, getting

$$\begin{aligned} a_n \left|G_n^{x,h}(\alpha,\beta) - \mathcal{G}^{x,h}(\alpha,\beta)\right| \\ \leq \left(\frac{a_n}{a_{\lfloor(1-\alpha)n\rfloor}}\right) a_{\lfloor(1-\alpha)n\rfloor} \left|G_n^{x,h}(\alpha,\beta) - \frac{h\,a_n}{a_{\lfloor(1-\alpha)n\rfloor}}\,\varphi\!\left(\frac{a_n\,(x-\beta)}{a_{\lfloor(1-\alpha)n\rfloor}}\right)\right| \\ + \ (h\,a_n)\left|\frac{a_n}{a_{\lfloor(1-\alpha)n\rfloor}}\,\varphi\!\left(\frac{a_n\,(x-\beta)}{a_{\lfloor(1-\alpha)n\rfloor}}\right) - \frac{1}{\sqrt{1-\alpha}}\,\varphi\!\left(\frac{x-\beta}{\sqrt{1-\alpha}}\right)\right|. \end{aligned} \tag{6.42}$$

Let us look at the first term in the r.h.s. above: by the by the uniform convergence property of regularly varying sequences (6.9) we have

$$\sup_{\alpha\in(0,1-\varepsilon)} \left|\frac{a_n}{a_{\lfloor(1-\alpha)n\rfloor}} - \frac{1}{\sqrt{1-\alpha}}\right| \to 0 \qquad (n\to\infty)\,, \tag{6.43}$$

hence the prefactor is uniformly bounded. For the remaining part, from the expression (6.35) for $G_n^{x,h}$ it is clear that one can apply Stone's LLT, see (6.33), yielding

$$\sup_{(\alpha,\beta)\in D^\infty_{1-\varepsilon},\, x\in\mathbb{R}^+,\, h\leq K/a_n} a_{\lfloor(1-\alpha)n\rfloor} \left|G_n^{x,h}(\alpha,\beta) - \frac{h\,a_n}{a_{\lfloor(1-\alpha)n\rfloor}}\,\varphi\!\left(\frac{a_n\,(x-\beta)}{a_{\lfloor(1-\alpha)n\rfloor}}\right)\right| \ \to \ 0$$

as $n\to\infty$.

For the second term in the r.h.s. of (6.42), notice that the prefactor $(h\,a_n)$ gives no problem since $h \leq K/a_n$ in our limit. On the other hand, it is easily seen that the absolute value is vanishing as $n\to\infty$, uniformly for $(\alpha,\beta)\in D^\infty_{1-\varepsilon}$ and for $x\in\mathbb{R}^+$: this is thanks to relation (6.43) and to the fact that the function $\varphi(x)$ is uniformly continuous. Coming back to equation (6.41), we have shown that

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{R}^+,\, h\leq K/a_n} a_n \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu_n \left|G_n^{x,h} - \mathcal{G}^{x,h}\right| \ = \ 0\,. \tag{6.44}$$

4.4.2. *Second term.* Using the definition (6.29) of $\mathcal{G}^{x,h}$, the second term in the r.h.s. of equation (6.40) can be written as

$$\begin{aligned} a_n \left| \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu_n \, \mathcal{G}^{x,h} \; - \; \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu \, \mathcal{G}^{x,h} \right| \\ = \; (h\, a_n) \left| \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu_n \, \Psi(\alpha, x-\beta) \; - \; \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu \, \Psi(\alpha, x-\beta) \right| \end{aligned} \tag{6.45}$$

where we have introduced the shorthand

$$\Psi(s,t) := \frac{1}{\sqrt{1-s}} \, \varphi\!\left( \frac{t}{\sqrt{1-s}} \right) \mathbf{1}_{(t \geq 0)} \, .$$

As usual, for us $(h\, a_n) \leq K$ and we can thus concentrate on the absolute value in the r.h.s. of (6.45). Observe that, for fixed $x \geq 0$, the function $(\alpha, \beta) \mapsto \Psi(\alpha, x-\beta)$ on the domain $D^\infty_{1-\varepsilon}$ is bounded, and continuous except on the line $\beta = x$: since $\mu_n \Rightarrow \mu$, it follows that for fixed $x$ the r.h.s. of (6.45) is vanishing as $n \to \infty$. However, we would like the convergence to be uniform in $x \in \mathbb{R}^+$: this stronger result holds true too, as one can verify by approximating $\Psi$ with a sequence of uniformly continuous functions (the details are carried out in § 5.3). The net result is

$$\limsup_{n \to \infty} \sup_{x \in \mathbb{R}^+,\, h \leq K/a_n} a_n \left| \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu_n \, \mathcal{G}^{x,h} \; - \; \int_{D^x_{1-\varepsilon}} \mathrm{d}\mu \, \mathcal{G}^{x,h} \right| \; = \; 0 \, . \tag{6.46}$$

Putting together relations (6.40), (6.44) and (6.46) it is easily seen that (6.39) holds (even without taking the limit in $\varepsilon$), and the step is completed.

**4.5. Fourth step.** We finally show that

$$\int_{D^x_{1-\varepsilon}} \mathrm{d}\mu \, \mathcal{G}^{x,h} \; \overset{\star}{\sim} \; \int_{D^x_1} \mathrm{d}\mu \, \mathcal{G}^{x,h} \, ,$$

that is, for every $K > 0$

$$\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \sup_{x \in \mathbb{R}^+,\, h \leq K/a_n} a_n \int_{[1-\varepsilon, 1) \times [0,x)} \mathrm{d}\mu(\alpha, \beta) \, \mathcal{G}^{x,h}(\alpha, \beta) = 0 \, . \tag{6.47}$$

This is very easy: observe that

$$\mathcal{G}^{x,h}(\alpha, \beta) \leq \frac{h}{\sqrt{2\pi}\, \sqrt{1-\alpha}} \, ,$$

as one can check from the explicit expressions for $\mathcal{G}^{x,h}$ (6.29) and $\varphi(x)$ (6.1). Hence

$$a_n \int_{[1-\varepsilon, 1) \times [0,x)} \mathrm{d}\mu(\alpha, \beta) \, \mathcal{G}^{x,h}(\alpha, \beta) \; \leq \; \frac{(h a_n)}{\sqrt{2\pi}} \int_{[1-\varepsilon, 1) \times [0, \infty)} \mathrm{d}\mu(\alpha, \beta) \frac{1}{\sqrt{1-\alpha}} \, ,$$

and (6.47) follows, because the function

$$\big\{(\alpha,\beta)\mapsto(1-\alpha)^{-1/2}\big\}\in\mathbb{L}^1\big(D_1^\infty,\,\mathrm{d}\mu\big)\,,$$

as on can easily verify. This completes the proof of Theorem 6.1.

## 5. Appendix

**5.1. An elementary fact.** We prove the claim stated in the proof of Proposition 6.7, in a slightly more general context. Namely, let $\mu_n$, $\mu$ be finite measures on the domain $D := [0,1)\times[0,\infty)$, with $\mu(\partial D)=0$. Assume that $\mu_n(D)\to\mu(D)$ as $n\to\infty$, and that

$$\liminf_{n\to\infty}\mu_n\big((a_1,a_2]\times(b_1,b_2]\big)\;\geq\;\mu\big((a_1,a_2]\times(b_1,b_2]\big)\,,\tag{6.48}$$

for all $0<a_1<a_2<1$, $0<b_1<b_2<\infty$. What we are going to show is that

$$\exists\;\lim_{n\to\infty}\mu_n\big((a_1,a_2]\times(b_1,b_2]\big)\;=\;\mu\big((a_1,a_2]\times(b_1,b_2]\big)\,,\tag{6.49}$$

for all $0<a_1<a_2<1$, $0<b_1<b_2<\infty$, and this implies that $\mu_n\Rightarrow\mu$.

Suppose that (6.49) does not hold: then for some rectangle $Q:=(x_1,x_2]\times(y_1,y_2]$ contained in the interior of $D$ and for some $\varepsilon>0$ one has

$$\limsup_{n\to\infty}\mu_n(Q)\;\geq\;\mu(Q)+\varepsilon\,.\tag{6.50}$$

We introduce for $\eta\in(0,1/2)$ the rectangle $W:=(\eta,1-\eta]\times(\eta,1/\eta]$: by choosing $\eta$ sufficiently small we can assume that $W\supseteq Q$ and that

$$\mu(W)\geq\mu(D)-\varepsilon/2\tag{6.51}$$

(we recall that by hypothesis $\mu(\partial D)=0$). The rectangle $W$ can be easily written as a *disjoint* union

$$W=Q\cup\bigcup_{i=1}^{4}Q_i\,,$$

where the rectangles $Q_i$ (whose exact definition however is immaterial) are defined by

$$\begin{aligned}Q_1&:=(\eta,1-\eta]\times(\eta,y_1] & Q_2&:=(\eta,x_1]\times(y_1,y_2]\\ Q_3&:=(x_2,1-\eta]\times(y_1,y_2] & Q_4&:=(\eta,1-\eta]\times(y_2,1/\eta]\,.\end{aligned}$$

Now, on the one hand we have

$$\limsup_{n\to\infty}\mu_n(W)\;\leq\;\limsup_{n\to\infty}\mu_n(D)\;=\;\mu(D)\,,$$

but on the other hand

$$\begin{aligned}\limsup_{n\to\infty}\mu_n(W) \;&=\; \limsup_{n\to\infty}\mu_n\Big(Q\cup\bigcup_{i=1}^4 Q_i\Big) \;\ge\; \limsup_{n\to\infty}\mu_n(Q)+\liminf_{n\to\infty}\mu_n\Big(\bigcup_{i=1}^4 Q_i\Big)\\ &\overset{(6.50)}{\ge}\; \mu(Q)+\varepsilon+\sum_{i=1}^4\liminf_{n\to\infty}\mu_n(Q_i) \;\overset{(6.48)}{\ge}\; \mu(Q)+\varepsilon+\sum_{i=1}^4\mu(Q_i) \;=\; \varepsilon+\mu(W)\\ &\overset{(6.51)}{\ge}\; \mu(D)+\varepsilon/2\,,\end{aligned}$$

which evidently is absurd, hence (6.49) holds true.

**5.2. An integral.** We are going to give a more direct proof of relation (6.31): substituting the explicit expressions for $\varphi(x)$, $\varphi^+(x)$, $\mu$ given in equations (6.1), (6.3), (6.21) and performing an elementary change of variable, we can rewrite it as

$$x\,e^{-x^2/2} \;=\; \frac{x^2}{\sqrt{2\pi}}\int_0^1 \mathrm{d}w\int_0^1 \mathrm{d}z\;\frac{w}{z^{3/2}(1-z)^{1/2}}\,e^{-\frac{x^2}{2}\left[\frac{w^2}{z}+\frac{(1-w)^2}{(1-z)}\right]}\,. \tag{6.52}$$

Altough it is possible to perform explicitly the integration in the r.h.s. above, it is easier to proceed in a different way. Let $\{B_t\}$ be a standard Brownian motion and let $T_a := \inf\{t : B_t = a\}$ be its first passage time: then the law of $T_a$ is given by

$$\mathbb{P}\big(T_a\in\mathrm{d}t\big) = g(a,t)\,\mathrm{d}t\,,\qquad g(a,t) := \frac{a}{\sqrt{2\pi}\,t^{3/2}}e^{-a^2/2t}\,.$$

By the strong Markov property, for $x>0$ and $w\in(0,1)$ we have the equality in law $T_x \sim T_{wx}+T_{(1-w)x}$, where we mean that $T_{wx}$ and $T_{(1-w)x}$ are independent. Therefore

$$g(x,1) = \int_0^1 \mathrm{d}z\, g\big(wx,z\big)\,g\big((1-w)x,1-z\big)\,,$$

and integrating over $w\in(0,1)$ we get

$$g(x,1) = \int_0^1\mathrm{d}w\int_0^1 \mathrm{d}z\, g\big(wx,z\big)\,g\big((1-w)x,1-z\big)\,. \tag{6.53}$$

Now observe that relation (6.52) can be written as

$$\begin{aligned} g(x,1) &= \int_0^1\mathrm{d}w\int_0^1\mathrm{d}z\,\frac{1-z}{1-w}\,g\big(wx,z\big)\,g\big((1-w)x,1-z\big)\\ &= \int_0^1\mathrm{d}w\int_0^1\mathrm{d}z\,\frac{z}{w}\,g\big(wx,z\big)\,g\big((1-w)x,1-z\big)\,,\end{aligned}$$

and comparing with (6.53) we are left with showing that

$$\int_0^1\mathrm{d}w\int_0^1\mathrm{d}z\,\Big(1-\frac{z}{w}\Big)\,g\big(wx,z\big)\,g\big((1-w)x,1-z\big) \;=\; 0$$

However, the l.h.s. above can be decomposed in

$$\int_0^1 \mathrm{d}w \int_w^1 \mathrm{d}z \, \big(\dots\big) \, + \, \int_0^1 \mathrm{d}w \int_0^w \mathrm{d}z \, \big(\dots\big) \, =: \, I_1 \, + \, I_2 \,,$$

and with a change of variable one easily verifies that $I_1 = -I_2$.

**5.3. A uniformity result.** We are going to show that

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{R}^+} \left| \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu_n \, \Psi(\alpha, x-\beta) \, - \, \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu \, \Psi(\alpha, x-\beta) \right| \, = \, 0 \,, \tag{6.54}$$

where we recall that $D^b_a := [0,a) \times [0,b)$ and the function $\Psi$ is defined by

$$\Psi(s,t) := \frac{1}{\sqrt{1-s}} \, \varphi\!\left(\frac{t}{\sqrt{1-s}}\right) \mathbf{1}_{(t\geq 0)} \,.$$

Let us consider the fixed domain $T := [0, 1-\varepsilon] \times \mathbb{R}$. Here the function $\Psi$ is bounded, $\|\Psi\|_{\infty,T} = 1/\sqrt{2\pi\varepsilon}$, and continuous except on the line $t = 0$. We can easily build a family of approximations $\{\Psi_\delta\}$ of $\Psi$ that are bounded and uniformly continuous on the whole $T$, setting for $\delta > 0$

$$\Psi_\delta(s,t) := \begin{cases} \Psi(s,t) & t \geq 0 \\ \Psi(s,0) \cdot (1 + t/\delta) & t \in [-\delta, 0] \\ 0 & t \leq -\delta \end{cases} .$$

Notice that $\|\Psi_\delta\|_{\infty,T} = \|\Psi\|_{\infty,T}$, and that for $(s,t) \in T$

$$\big|\Psi(s,t) - \Psi_\delta(s,t)\big| \leq \|\Psi\|_{\infty,T} \, \mathbf{1}_{[-\delta,0]}(t) \,. \tag{6.55}$$

Let us introduce for short the notation $\Psi^x(\alpha,\beta) := \Psi(\alpha, x-\beta)$, and analogously for $\Psi_\delta$. From the triangle inequality we get

$$\begin{aligned} \left| \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu_n \, \Psi^x \, - \, \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu \, \Psi^x \right| \, \leq \, & \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu_n \, \big|\Psi^x - \Psi^x_\delta\big| \\ & + \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu \, \big|\Psi^x - \Psi^x_\delta\big| \, + \, \left| \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu_n \, \Psi^x_\delta \, - \, \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu \, \Psi^x_\delta \right| . \end{aligned} \tag{6.56}$$

Using relation (6.55), the first two terms in the r.h.s. above can be estimated by

$$\|\Psi\|_{\infty,T} \Big( \mu_n\big([0,1-\varepsilon] \times [x, x+\delta]\big) + \mu\big([0,1-\varepsilon] \times [x, x+\delta]\big) \Big) \,.$$

Since $\mu$ is an absolutely continuous and finite measure, its distribution function is uniformly continuous: therefore for every $\eta > 0$ we can take $\delta_0$ sufficiently small so that

$$\sup_{x\in\mathbb{R}^+} \mu\big([0,1-\varepsilon]\times[x,x+\delta_0]\big) \le \frac{\eta}{4\|\Psi\|_{\infty,T}}\,.$$

On the other hand, we know that for every $x \ge 0$

$$\mu_n\big([0,1-\varepsilon]\times[x,x+\delta_0]\big) \to \mu\big([0,1-\varepsilon]\times[x,x+\delta_0]\big) \qquad (n\to\infty)\,,$$

and this convergence is uniform for $x \in \mathbb{R}^+$, as it can be easily checked. Hence by the triangle inequality we can choose $n_0$ so large that

$$\sup_{n\ge n_0}\sup_{x\in\mathbb{R}^+} \mu_n\big([0,1-\varepsilon]\times[x,x+\delta_0]\big) \le \frac{\eta}{2\|\Psi\|_{\infty,T}}\,.$$

Finally, observe that for fixed $\delta_0$ the family of functions $\{\Psi^x_{\delta_0}\}_{x\in\mathbb{R}^+}$ is equibounded and equicontinuous: since $\mu_n \Rightarrow \mu$, from a classical result [**29**, Cor. in §VIII.1] we have that the third term in the r.h.s. of (6.56) with $\delta = \delta_0$ is vanishing as $n \to \infty$ uniformly for $x \in \mathbb{R}^+$. Therefore we can assume that $n_0$ has been chosen so large that

$$\sup_{n\ge n_0}\sup_{x\in\mathbb{R}^+} \left| \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu_n\, \Psi^x_{\delta_0} \;-\; \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu\, \Psi^x_{\delta_0} \right| \le \frac{\eta}{4}\,.$$

Applying the preceding bounds to equation (6.56) with $\delta = \delta_0$, we have shown that for every $\eta > 0$ we can find $n_0$ such that for every $n \ge n_0$

$$\sup_{x\in\mathbb{R}^+} \left| \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu_n\, \Psi^x \;-\; \int_{D^\infty_{1-\varepsilon}} \mathrm{d}\mu\, \Psi^x \right| \le \eta\,,$$

and equation (6.54) is proved.

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