# RANDOM GRAPHS AND COMPLEX NETWORKS


Federico Bassetti and Francesco Caravenna

*Università degli Studi di Pavia e di Milano-Bicocca*


# *Goals*

- Introduction to the mathematical modeling of random graphs
- Ideas and techniques of modern probability theory in an accessible and relatively non-technical framework
- Only a basic probabilistic background is required (finite and countable probability, discrete random variables)

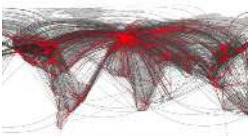
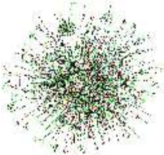
# *Schedule*

- 30 hours, held in Pavia and Milano-Bicocca
- Either $2 \times 2$ hours or $1 \times 3$ hours per week (to be discussed with students)
- Days: Tuesday and/or Thursday
- First lecture: Tuesday 18 March 2014 h.14.00 in Pavia

# *Why random graphs?*

Increasing interest in recent years for real-world networks:

- World Wide Web
- communication networks (road network, flight network)
- collaboration and citation networks, social relations
- biological and ecological networks (food webs, transcription networks, protein networks...)


# *Key features*

Different examples display surprisingly similar large-scale features:

- "Small Worlds": distances among nodes are much smaller than the size of the network

  $\rightsquigarrow$ *six degrees of separation*

- "Scale Free": the number of nodes with $k$ connections decays slowly (polynomially) in $k$

  $\rightsquigarrow$ *some nodes ("hubs") have many connections*

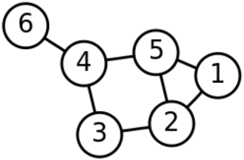
Goal: describe/explain such features through mathematical models

# *Graphs*

From a mathematical point of view a graph is a couple

$$G = (\mathcal{N}, \mathcal{E})$$

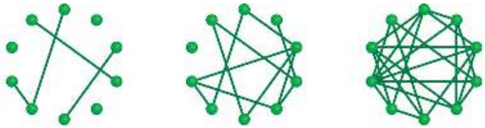
- $\mathcal{N} = \{1, \ldots, n\}$ is the set of nodes
- $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of edges (links, connections)



# *Random Graphs*

Fix the set of nodes $\mathcal{N} = \{1, \ldots, n\}$

A random graph is a random way to choose the edges for $\mathcal{N}$



i.e. a probability measure on the possible choices of edges $\mathcal{E}$ (subsets of $\mathcal{N} \times \mathcal{N}$)

- Which kind of probability measure?
- Asymptotic properties as $n \to \infty$?

One of the simplest random graphs is the

### *Erdös-Rényi random graph* $\mathcal{G}(n, p)$

- $\mathcal{N} = \{1, \ldots, n\}$ is the set of nodes
- For each couple $(i, j)$ of nodes, toss a coin and put an edge if a head comes up

  (use independent coins with the same head probability $p$)

# Phase transition in the Erdös-Rényi graph

Take

$$p = \frac{\lambda}{n}$$

Let $|C_{max}|$ be the size of the largest connected subgraph

## Phase transition as $n \to +\infty$

- If $\lambda < 1$

$$|C_{max}| \sim k_\lambda \log(n) \qquad \text{in probab.}$$

$\rightsquigarrow$ *many "small" disconnected ilands*

- If $\lambda > 1$

$$|C_{max}| \sim z_\lambda \, n \qquad \text{in probab.}$$

$\rightsquigarrow$ *one macroscopic giant component*

($k_\lambda$ and $z_\lambda$ are explicit constants)

# *More recent models*

The Erdös-Rényi random graph is not scale free

Popular alternative models (scale free and small worlds)

- Preferential Attachment: graph built dynamically, adding nodes in a sequential fashion; connections among nodes with large degrees are favored
- Configuration Model: degrees are fixed beforehand; nodes are connected in the most random way (uniformly)

These will be the object of the second part of the course.

# *Program*

1. Probabilistic Tools and Techniques:
   random variables, coupling, inequalities
2. Branching Processes as Random Trees:
   asymptotic properties
3. The Erdos-Renyi random graph:
   phase transition and study of the giant connected component
4. The configuration model:
   asymptotics of distances and small world phenomenon
5. Preferential attachment model:
   emergence of a scale free network

# *REFERENCES*

- Lecture notes Random Graphs and Complex Networks
  by R. van der Hofstad
  available at http://www.win.tue.nl/ rhofstad/NotesRGCN.pdf
- Book Random Graph Dynamics
  by R. Durrett
  Cambridge Unversity Press.