Introduction

The problem of interest in this book is the study of differential equations driven by *irregular functions* (more specifically: continuous but not differentiable). This will be achieved through the powerful and elegant theory of *rough paths*. A key motivation comes from stochastic differential equations driven by Brownian motion, but the goal is to develop a general theory which does not rely on probability.

We will focus on *controlled differential equations*: given a *driving path* $X = (X_t)_{t \geqslant 0}$ and a function $\sigma(\cdot)$, we look for a *solution* $Z = (Z_t)_{t \geqslant 0}$ of

$$\dot{Z}_t = \sigma(Z_t)\, \dot{X}_t\,. \tag{1}$$

The challenge is to make sense of this equation *when X is not differentiable*. We consider in particular the case when X is a generic α-Hölder continuous path, i.e. it satfies $|X_t - X_s| = O(|t-s|^\alpha)$ for some $\alpha \in]0,1[$.

The basic idea is to reformulate the ill-posed differential equation (1) in ways that do not contain the derivative $\dot{X}$. In Part I of this book (Chapters 1 to 4) we rewrite the differential equation (1) as a *finite difference equation* on some interval $[0,T]$:

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + \{\cdots\} + o(t-s) \qquad \text{for} \quad 0 \leqslant s < t \leqslant T, \tag{2}$$

where $\{\cdots\}$ denote suitable additional terms (to be described below). For such a difference equation we prove *well-posedness*, i.e. existence, uniqueness, regularity of solutions and continuous dependence on initial data, when X is a generic α-Hölder path with $\alpha > \frac{1}{3}$ and $\sigma(\cdot)$ is a sufficiently regular function. More precisely:

- in Chapter 1 we introduce our key tool, that we call the *Sewing Bound*[1];
- in Chapter 2 we consider the so-called *Young case* $\alpha > \frac{1}{2}$: we prove well-posedness for the difference equation (2) where we simply take $\{\cdots\} = 0$;
- in Chapter 3 we consider the so-called *Rough case* $\frac{1}{3} < \alpha \leqslant \frac{1}{2}$: in this regime we need to *enrich the path X* with a notion of *iterated integral* $\int X_u \, \mathrm{d}X_u$, leading to a so-called *rough path*, then we prove well-posedness for the difference equation (2) for a natural additional term $\{\cdots\} \neq 0$;
- in Chapter 4 we apply the theory of Chapter 3 when $X = B$ is a typical path of *Brownian motion*: the solution of the difference equation (2) coincides with the solution of the *stochastic differential equation* $\mathrm{d}Z_t = \sigma(Z_t)\,\mathrm{d}B_t$ in the Ito or Stratonovich sense, depending on the choice of iterated integral $\int B_u \, \mathrm{d}B_u$.

1. This may be viewed as "half" of the celebrated *Sewing Lemma*, to be discussed in Chapter 6.

Altogether, we show that the difference equation (2) driven by a rough path (i.e. a Hölder continuous path X enriched by an iterated integral) provides an elementary yet powerful reformulation of the ill-posed differential equation (1), which sheds a "pathwise light" on stochastic differential equations driven by Brownian motion.[2]

Part II of this book is devoted to the alternative (and possibly more customary) reformulation of the differential equation (1) as an integral equation:

$$Z_t = Z_0 + \mathcal{I}_t(\sigma(Z), X), \qquad \text{where} \quad \mathcal{I}_t(Y, X) = \int_0^t Y_s \dot{X}_s \, \mathrm{d}s. \tag{3}$$

The natural strategy to solve it (for given initial datum Z_0) is via the following steps:

- define a notion of *integral* $\mathcal{I}_t(Y, X)$ *with respect to a non-differentiable* X, for a suitable class $\mathcal{H}$ of integrands $Y = (Y_s)_{s \geqslant 0}$;
- show that the maps $Y \mapsto (\mathcal{I}_t(Y, X))_{t \geqslant 0}$ and $Z \mapsto \sigma(Z)$ are *continuous* with respect to a suitable metric on the space $\mathcal{H}$ of integrands;
- solve (3) as a *fixed point equation*, by showing that for small time horizon $T > 0$ the map $Z \mapsto \Phi(Z) := (\mathcal{I}_t(\sigma(Z), X))_{t \in [0,T]}$ is a contraction.

This is indeed the standard strategy to solve stochastic differential equations driven by Brownian motion $X = B$, in which case $\mathcal{I}_t(Y, B) = \int_0^t Y_s \, \mathrm{d}B_s$ makes sense as a stochastic inegral (e.g. Ito or Stratonovich).

We show that the same strategy can be applied in the theory of *rough paths*: one can define a notion of *rough integral* $\mathcal{I}_t(Y, X)$ for a wide class of integrands Y, known as *controlled paths*, and then solve the equation (3) by fixed point arguments. This is the approach in the book by Hairer and Friz **[Insert reference]**.

To be completed...

NOTATION

We fix a time horizon $T > 0$ and two dimensions $k, d \in \mathbb{N}$. We use "path" as a synonymous of "function defined on $[0, T]$" with values in $\mathbb{R}^d$. We denote by $|\cdot|$ the Euclidean norm.

The space of linear maps from $\mathbb{R}^d$ to $\mathbb{R}^k$ is denoted by $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^k) = \mathbb{R}^k \otimes (\mathbb{R}^d)^*$. We can identify it with the space $\mathbb{R}^{k \times d}$ of $k \times d$ real matrices, that we equip with the Hilbert-Schmidt norm $|\cdot|$ (i.e. the Euclidean norm on $\mathbb{R}^{k \times d}$). Note that for a linear map $A \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k)$ and a vector $v \in \mathbb{R}^d$ we have $|A v| \leqslant |A| \, |v|$.

2. The restriction to α-Hölder paths X with $\alpha > \frac{1}{3}$ is made for simplicity: the theory can be extended to any $\alpha > 0$ (for $\frac{1}{n+1} < \alpha \leqslant \frac{1}{n}$ with $n \in \mathbb{N}$ one needs to enrich the path X with n-order iterated integrals).