Chapter 1

The Sewing Bound

This first chapter is dedicated to an elementary but fundamental tool, the *Sewing Bound*, that will be applied extensively throughout the book. It is a general Hölder-type bound for functions of two real variables that can be understood by itself, see Theorem 1.9 below. To provide motivation, we present it as a natural a priori estimate for solutions of differential equations.

1.1. Controlled differential equation

Consider the following *controlled ordinary differential equation (ODE)*: given a continuously differentiable path $X:[0,T]\to\mathbb{R}^d$ and a continuous function $\sigma:\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$, we look for a differentiable path $Z:[0,T]\to\mathbb{R}^k$ such that

$$\dot{Z}_t=\sigma(Z_t)\,\dot{X}_t\,,\qquad t\in[0,T]. \tag{1.1}$$

By the fundamental theorem of calculus, this is equivalent to

$$Z_t=Z_0+\int_0^t\sigma(Z_s)\,\dot{X}_s\,\mathrm{d}s\,,\qquad t\in[0,T]. \tag{1.2}$$

In the special case $k=d=1$ and when $\sigma(x)=\lambda\,x$ is linear (with $\lambda\in\mathbb{R}$), we have the explicit solution $Z_t=z_0\exp(\lambda\,(X_t-X_0))$, which has the interesting property of being well-defined also when X is non differentiable.

For any dimensions $k,d\in\mathbb{N}$, if we assume that $\sigma(\cdot)$ is Lipschitz, classical results in the theory of ODEs guarantee that *equation (1.1)-(1.2) is well-posed for any continuously differentiable path X*, namely for any $Z_0\in\mathbb{R}^k$ there is one and only one solution Z (with no explicit formula, in general).

Our aim is to extend such a well-posedness result to a setting where X *is continuous but not differentiable* (also in cases where $\sigma(\cdot)$ may be non-linear). Of course, to this purpose it is first necessary to provide a generalized formulation of (1.1)-(1.2) where the derivative of X does not appear.

1.2. Controlled difference equation

Let us still suppose that X is continuously differentiable. We deduce by (1.1)-(1.2) that for $0\leqslant s\leqslant t\leqslant T$

$$Z_t-Z_s=\sigma(Z_s)\,(X_t-X_s)+\int_s^t(\sigma(Z_u)-\sigma(Z_s))\,\dot{X}_u\,\mathrm{d}u, \tag{1.3}$$

which implies that Z satisfies the following *controlled difference equation*:

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{1.4}$$

because $u \mapsto \sigma(Z_u)$ is continuous and $u \mapsto \dot{X}_u$ is (continuous, hence) bounded on $[0,T]$.

Remark 1.1. (UNIFORMITY) Whenever we write $o(t-s)$, as in (1.4), we always mean *uniformly for* $0 \leqslant s \leqslant t \leqslant T$, i.e.

$$\forall \varepsilon > 0\ \exists \delta > 0: \quad 0 \leqslant s \leqslant t \leqslant T,\ \ t - s \leq \delta \quad \text{implies} \quad |o(t-s)| \leq \varepsilon\,(t-s)\,. \tag{1.5}$$

This will be implicitly assumed in the sequel.

Let us make two simple observations.

- If X is continuously differentiable we deduced (1.4) from (1.1), but we can easily deduce (1.1) from (1.4): in other terms, the two equations (1.1) and (1.4) are *equivalent*.
- If X is *not* continuously differentiable, equation (1.4) is still *meaningful*, unlike equation (1.1) which contains explicitly $\dot{X}$.

For these reasons, henceforth we focus on the difference equation (1.4), which provides a generalized formulation of the differential equation (1.1) when X is continuous but not necessarily differentiable.

The problem is now to prove *well-posedness* for the difference equation (1.4). We are going to show that this is possible assuming a suitable *Hölder regularity* on X, but non trivial ideas are required. In this chapter we illustrate some key ideas, showing how to prove uniqueness of solutions via *a priori estimates* (existence of solutions will be studied in the next chapters). We start from a basic result, which ensures the continuity of solutions; more precise result will be obtained later.

LEMMA 1.2. (CONTINUITY OF SOLUTIONS) *Let X and σ be continuous. Then any solution Z of (1.4) is a continuous path, more precisely it satisfies*

$$|Z_t - Z_s| \leqslant C\,|X_t - X_s| + o(t-s)\,, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{1.6}$$

for a suitable constant $C < \infty$ which depends on Z.

Proof. Relation (1.6) follows by (1.4) with $C := \|\sigma(Z)\|_\infty = \sup_{0 \leqslant t \leqslant T} |\sigma(Z_t)|$, renaming $|o(t-s)|$ as $o(t-s)$. We only have to prove that $C < \infty$. Since σ is continuous by assumption, it is enough to show that Z is *bounded*.

Since $o(t-s)$ is uniform, see (1.5), we can fix $\bar{\delta} > 0$ such that $|o(t-s)| \leqslant 1$ for all $0 \leqslant s \leqslant t \leqslant T$ with $|t-s| \leqslant \bar{\delta}$. It follows that Z is bounded in any interval $[\bar{s}, \bar{t}]$ with $|\bar{t} - \bar{s}| \leqslant \bar{\delta}$, because by (1.4) we can bound

$$\sup_{t \in [\bar{s}, \bar{t}]} |Z_t| \leqslant |Z_{\bar{s}}| + |\sigma(Z_{\bar{s}})| \sup_{t \in [\bar{s}, \bar{t}]} |X_t - X_{\bar{s}}| + 1 < \infty\,.$$

We conclude that Z is bounded in the whole interval $[0,T]$, because we can write $[0,T]$ as a finite union of intervals $[\bar{s}, \bar{t}]$ with $|\bar{t} - \bar{s}| \leqslant \bar{\delta}$. ☐

Remark 1.3. (COUNTEREXAMPLES) The weaker requirement that (1.4) holds *for any fixed* $s \in [0,T]$ *as* $t \downarrow s$ is not enough for our purposes, since in this case Z *needs not be continuous*. An easy conterexample is the following: given any continuous path $X\colon [0,2] \to \mathbb{R}$, we define $Z\colon [0,2] \to \mathbb{R}$ by

$$Z_t := \begin{cases} X_t & \text{if} \quad 0 \leqslant t < 1, \\ X_t + 1 & \text{if} \quad 1 \leqslant t \leqslant 2. \end{cases}$$

Note that $Z_t - Z_s = X_t - X_s$ when either $0 \leqslant s \leqslant t < 1$ or $1 \leqslant s \leqslant t \leqslant 2$, hence Z satisfies the difference equation (1.4) with $\sigma(\cdot) \equiv 1$ *for any fixed* $s \in [0,2)$ *as* $t \downarrow s$, but *not uniformly for* $0 \leqslant s \leqslant t \leqslant 2$, since Z is discontinuous at $t = 1$.

For another counterexample, which is even unbounded, consider

$$Z_t := \begin{cases} \dfrac{1}{1-t} & \text{if} \quad 0 \leqslant t < 1, \\ 0 & \text{if} \quad 1 \leqslant t \leqslant 2, \end{cases}$$

which satisfies (1.4) as $t \downarrow s$ for any fixed $s \in [0,2]$, for $X_t \equiv t$ and $\sigma(z) = z^2$.

1.3. SOME USEFUL FUNCTION SPACES

For $n \geqslant 1$ we define the simplex

$$[0,T]^n_{\leqslant} := \{(t_1, \ldots, t_n)\colon \quad 0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant T\} \tag{1.7}$$

(note that $[0,T]^1_{\leqslant} = [0,T]$). We then write $C_n = C([0,T]^n_{\leqslant}, \mathbb{R}^k)$ as a shorthand for the space of *continuous functions from* $[0,T]^n_{\leqslant}$ *to* $\mathbb{R}^k$:

$$C_n := C([0,T]^n_{\leqslant}, \mathbb{R}^k) := \{F\colon [0,T]^n_{\leqslant} \to \mathbb{R}^k\colon \ F \text{ is continuous}\}. \tag{1.8}$$

We are going to work with functions of one (f_s), two (F_{st}) or three (G_{sut}) ordered variables in $[0,T]$, hence we focus on the spaces C_1, C_2, C_3.

- On the spaces C_2 and C_3 we introduce a Hölder-like structure: given any $\eta \in (0,\infty)$, we define for $F \in C_2$ and $G \in C_3$

  $$\|F\|_\eta := \sup_{0 \leqslant s < t \leqslant T} \frac{|F_{st}|}{(t-s)^\eta}, \qquad \|G\|_\eta := \sup_{\substack{0 \leqslant s \leqslant u \leqslant t \leqslant T \\ s<t}} \frac{|G_{sut}|}{(t-s)^\eta}, \tag{1.9}$$

  and we denote by C_2^η and C_3^η the corresponding function spaces:

  $$C_2^\eta := \{F \in C_2\colon \ \|F\|_\eta < \infty\}, \qquad C_3^\eta := \{G \in C_3\colon \ \|G\|_\eta < \infty\}, \tag{1.10}$$

  which are Banach spaces endowed with the norm $\|\cdot\|_\eta$ (exercise).

- On the space C_1 of continuous functions $f\colon [0,T] \to \mathbb{R}^k$ we consider the usual Hölder structure. We first introduce the *increment* δf by

  $$(\delta f)_{st} := f_t - f_s, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{1.11}$$

and note that $\delta f \in C_2$ for any $f \in C_1$. Then, for $\alpha \in (0,1]$, we define the classical space $\mathcal{C}^\alpha = \mathcal{C}^\alpha([0,T],\mathbb{R}^k)$ of α-Hölder functions

$$\mathcal{C}^\alpha := \left\{ f: [0,T] \to \mathbb{R}^k: \quad \|\delta f\|_\alpha = \sup_{0 \le s < t \le T} \frac{|f_t - f_s|}{(t-s)^\alpha} < \infty \right\} \tag{1.12}$$

(for $\alpha = 1$ it is the space of Lipschitz functions). Note that $\|\delta f\|_\alpha$ in (1.12) is consistent with (1.11) and (1.9).

Remark 1.4. (HÖLDER SEMI-NORM) We stress that $f \mapsto \|\delta f\|_\alpha$ is a semi-norm on $\mathcal{C}^\alpha$ (it vanishes on constant functions). The standard norm on $\mathcal{C}^\alpha$ is

$$\|f\|_{\mathcal{C}^\alpha} := \|f\|_\infty + \|\delta f\|_\alpha , \tag{1.13}$$

where we define the standard sup norm

$$\|f\|_\infty := \sup_{t \in [0,T]} |f_t|. \tag{1.14}$$

For $f: [0,T] \to \mathbb{R}^k$ we can bound $\|f\|_\infty \le |f(0)| + T^\alpha \|\delta f\|_\alpha$ (see (1.39) below), hence

$$\|f\|_{\mathcal{C}^\alpha} \le |f(0)| + (1 + T^\alpha) \|\delta f\|_\alpha . \tag{1.15}$$

This explains why it is often enough to focus on the semi-norm $\|\delta f\|_\alpha$.

Remark 1.5. (HÖLDER EXPONENTS) We only consider the Hölder space $\mathcal{C}^\alpha$ for $\alpha \in (0,1]$ because for $\alpha > 1$ the only functions in $\mathcal{C}^\alpha$ are constant functions (note that $\|\delta f\|_\alpha < \infty$ for $\alpha > 1$ implies $\dot{f}_t = 0$ for every $t \in [0,T]$).

On the other hand, the spaces C_2^η and C_3^η in (1.10) are interesting *for any exponent* $\eta \in (0,\infty)$. For instance, the condition $\|F\|_\eta < \infty$ for a function $F \in C_2$ means that $|F_{st}| \leqslant C\,(t-s)^\eta$, which does not imply $F \equiv 0$ when $\eta > 1$ (unless $F = \delta f$ is the increment of some function $f \in C_1$).

In our results below we will have to assume that the non-linearity $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ belongs to classes of Hölder functions, in the following sense.

DEFINITION 1.6. *Let $\gamma > 0$. A function $F: \mathbb{R}^k \to \mathbb{R}^N$ is said to be globally γ-Hölder (or globally of class $\mathcal{C}^\gamma$) if*

- *for $\gamma \in (0,1]$ we have*

$$[F]_{\mathcal{C}^\gamma} := \sup_{x,y \in \mathbb{R}^k, x \neq y} \frac{|F(x) - F(y)|}{|x-y|^\gamma} < +\infty$$

- *for $\gamma \in (n, n+1]$ and $n = \{1,2,\ldots\}$, F is n times continuously differentiable and*

$$[D^{(n)}F]_{\mathcal{C}^\gamma} := \sup_{x,y \in \mathbb{R}^k, x \neq y} \frac{|D^{(n)}F(x) - D^{(n)}F(y)|}{|x-y|^{\gamma - n}} < +\infty$$

where $D^{(n)}$ is the n-fold differential of F.

Moreover $F:\mathbb{R}^k\to\mathbb{R}^N$ *is said to be locally* γ*-Hölder (or locally of class* $\mathcal{C}^\gamma$*) if*

- *for* $\gamma\in(0,1]$ *we have for all* $R>0$

$$\sup_{\substack{x,y\in\mathbb{R}^k,x\neq y\\|x|,|y|\leqslant R}}\frac{|F(x)-F(y)|}{|x-y|^\gamma}<+\infty$$

- *for* $\gamma\in(n,n+1]$ *and* $n=\{1,2,\ldots\}$*,* F *is* n *times continuously differentiable and*

$$\sup_{\substack{x,y\in\mathbb{R}^k,x\neq y\\|x|,|y|\leqslant R}}\frac{|D^{(n)}F(x)-D^{(n)}F(y)|}{|x-y|^{\gamma-n}}<+\infty.$$

We stress that in the previous definition we do not assume F of $D^{(n)}F$ to be bounded. The case $\gamma=1$ corresponds to the classical *Lipschitz* condition.

1.4. LOCAL UNIQUENESS OF SOLUTIONS

We prove *uniqueness of solutions* for the controlled difference equation (1.4) when $X\in\mathcal{C}^\alpha$ *is an Hölder path of exponent* $\alpha>\frac{1}{2}$. For simplicity, we focus on the case when $\sigma:\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$ is a *linear* application: $\sigma\in(\mathbb{R}^k\otimes(\mathbb{R}^d)^*)\otimes(\mathbb{R}^k)^*$, and we write σZ instead of $\sigma(Z)$ (we discuss non linear $\sigma(\cdot)$ in Chapter 2).

THEOREM 1.7. (LOCAL UNIQUENESS OF SOLUTIONS, LINEAR CASE) *Fix a path* $X:[0,T]\to\mathbb{R}^d$ *in* $\mathcal{C}^\alpha$*, with* $\alpha\in\left]\frac{1}{2},1\right]$*, and a linear map* $\sigma:\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$*. If* $T>0$ *is small enough (depending on* X,α,σ*), then for any* $z_0\in\mathbb{R}^k$ *there is at most one path* $Z:[0,T]\to\mathbb{R}^k$ *with* $Z_0=z_0$ *which solves the linear controlled difference equation (1.4), that is (recalling (1.11))*

$$\delta Z_{st}-(\sigma Z_s)\,\delta X_{st}=o(t-s),\qquad 0\leqslant s\leqslant t\leqslant T. \tag{1.16}$$

Proof. Suppose that we have two paths $Z,\bar{Z}:[0,T]\to\mathbb{R}^k$ satisfying (1.16) with $Z_0=\bar{Z}_0$ and define $Y:=Z-\bar{Z}$. Our goal is to show that $Y=0$.

Let us introduce the function $R\in C_2=C([0,T]^2_\leqslant,\mathbb{R}^k)$ defined by

$$R_{st}:=\delta Y_{st}-(\sigma Y_s)\,\delta X_{st}\,,\qquad 0\leqslant s\leqslant t\leqslant T, \tag{1.17}$$

and note that by (1.16) and linearity we have

$$R_{st}=o(t-s)\,. \tag{1.18}$$

Recalling (1.9), we can estimate

$$\|\delta Y\|_\alpha\leqslant|\sigma|\,\|Y\|_\infty\,\|\delta X\|_\alpha+\|R\|_\alpha\,,$$

and since $R_{st}=o(t-s)=o((t-s)^\alpha)$, we have $\|R\|_\alpha<+\infty$ and therefore $\|\delta Y\|_\alpha<+\infty$. Since $Y_0=0$, we can bound

$$\|Y\|_\infty\leqslant|Y_0|+\sup_{0\leqslant t\leqslant T}|Y_t-Y_0|\leqslant T^\alpha\,\|\delta Y\|_\alpha\,.$$

Since $1 \leqslant T^\alpha (t-s)^{-\alpha}$ for $0 \leqslant s < t \leqslant T$, we can also bound

$$\|R\|_\alpha \leqslant T^\alpha \|R\|_{2\alpha},$$

so that

$$\|\delta Y\|_\alpha \leqslant T^\alpha (|\sigma| \, \|\delta Y\|_\alpha \, \|\delta X\|_\alpha + \|R\|_{2\alpha}).$$

Suppose we can prove that, for some constant $C = C(X, \alpha, \sigma) < \infty$,

$$\|R\|_{2\alpha} \leqslant C \, \|\delta Y\|_\alpha. \tag{1.19}$$

Then we obtain

$$\|\delta Y\|_\alpha \leqslant T^\alpha (|\sigma| \, \|\delta X\|_\alpha + C) \, \|\delta Y\|_\alpha.$$

If we fix T small enough, so that $T^\alpha (|\sigma| \, \|\delta X\|_\alpha + C) < 1$, we get $\|\delta Y\|_\alpha = 0$, hence $\delta Y \equiv 0$. This means that $Y_t = Y_s$ for all $s, t \in [0, T]$, and since $Y_0 = 0$ we obtain $Y \equiv 0$, namely our goal $Z \equiv \bar{Z}$. This completes the proof *assuming the estimate (1.19)* (where the hypothesis $\alpha > \frac{1}{2}$ will play a key role). □

To actually complete the proof of Theorem 1.7, it remains to show that the inequality (1.19) holds. This is performed in the next two sections:

- in Section 1.5 we present a fundamental estimate, the *Sewing Bound*, which applies to *any function* $R_{st} = o(t-s)$ (recall (1.18));
- in Section 1.6 we apply the Sewing Bound to R_{st} in (1.17) and we prove the desired estimate (1.19) for $\alpha > \frac{1}{2}$ (see the assumptions of Theorem 1.7).

1.5. The Sewing bound

Let us fix an arbitrary function $R \in C_2 = C([0, T]^2_\leqslant, \mathbb{R}^k)$ with $R_{st} = o(t-s)$. Our goal is to bound $|R_{ab}|$ for any given $0 \leqslant a < b \leqslant T$.

We first show that we can express R_{ab} via "Riemann sums" along partitions $\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_m = b\}$ of $[a, b]$. These are defined by

$$I_{\mathcal{P}}(R) := \sum_{i=1}^{\#\mathcal{P}} R_{t_{i-1} t_i}, \tag{1.20}$$

where we denote by $\#\mathcal{P} := m$ the number of intervals of the partition $\mathcal{P}$. Let us denote by $|\mathcal{P}| := \max_{1 \leqslant i \leqslant m} (t_i - t_{i-1})$ the *mesh* of $\mathcal{P}$.

LEMMA 1.8. (RIEMANN SUMS) *Given any* $R \in C_2$ *with* $R_{st} = o(t-s)$, *for any* $0 \leqslant a < b \leqslant T$ *and for any sequence* $(\mathcal{P}_n)_{n \geqslant 0}$ *of partitions of* $[a, b]$ *with vanishing mesh* $\lim_{n\to\infty} |\mathcal{P}_n| = 0$ *we have*

$$\lim_{n\to\infty} I_{\mathcal{P}_n}(R) = 0.$$

If furthermore $\mathcal{P}_0 = \{a, b\}$ is the trivial partition, then we can write

$$R_{ab} = \sum_{n=0}^{\infty} (I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)), \qquad 0 \leqslant a < b \leqslant T. \tag{1.21}$$

Proof. Writing $\mathcal{P}_n = \{a = t_0^n < t_1^n < \ldots < t_{\#\mathcal{P}_n}^n = b\}$, we can estimate

$$|I_{\mathcal{P}_n}(R)| \leqslant \sum_{i=1}^{\#\mathcal{P}_n} |R_{t_{i-1}^n t_i^n}| \leqslant \left\{ \max_{j=1,\ldots,\#\mathcal{P}_n} \frac{|R_{t_{j-1}^n t_j^n}|}{(t_j^n - t_{j-1}^n)} \right\} \sum_{j=1}^{\#\mathcal{P}_n} (t_j^n - t_{j-1}^n),$$

hence $|I_{\mathcal{P}_n}(R)| \to 0$ as $n \to \infty$, because the final sum equals $b - a$ and the bracket vanishes (since $R_{st} = o(t - s)$ and $|\mathcal{P}_n| = \max_{1 \leqslant j \leqslant \#\mathcal{P}_n} (t_j^n - t_{j-1}^n) \to 0$).

We deduce relation (1.21) by the telescopic sum

$$I_{\mathcal{P}_0}(R) - I_{\mathcal{P}_N}(R) = \sum_{n=0}^{N-1} (I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)),$$

because $\lim_{N \to \infty} I_{\mathcal{P}_N}(R) = 0$ while $I_{\mathcal{P}_0}(R) = R_{ab}$ for $\mathcal{P}_0 = \{a, b\}$. □

If we remove a single point t_i from a partition $\mathcal{P} = \{t_0 < t_1 < \ldots < t_m\}$, we obtain a new partition $\mathcal{P}'$ for which, recalling (1.20), we can write

$$I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R) = R_{t_{i-1} t_{i+1}} - R_{t_{i-1} t_i} - R_{t_i t_{i+1}}. \tag{1.22}$$

The expression in the RHS deserves a name: given any two-variables function $F \in C_2$, we define its increment $\delta F \in C_3$ as the three-variables function

$$\delta F_{sut} := F_{st} - F_{su} - F_{ut}, \qquad 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{1.23}$$

We can then rewrite (1.22) as

$$I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R) = \delta R_{t_{i-1} t_i t_{i+1}}, \tag{1.24}$$

and recalling (1.9) we obtain the following estimate, for any $\eta > 0$:

$$|I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R)| \leqslant \|\delta R\|_\eta \, |t_{i+1} - t_{i-1}|^\eta. \tag{1.25}$$

We are now ready to state and prove the Sewing Bound.

THEOREM 1.9. (SEWING BOUND) *Given any $R \in C_2$ with $R_{st} = o(t - s)$, the following estimate holds for any $\eta \in (1, \infty)$ (recall (1.9)):*

$$\|R\|_\eta \leqslant K_\eta \, \|\delta R\|_\eta \qquad \text{where} \qquad K_\eta := (1 - 2^{1-\eta})^{-1}. \tag{1.26}$$

Proof. Fix $R \in C_2$ such that $\|\delta R\|_\eta < \infty$ for some $\eta > 1$ (otherwise there is nothing to prove). Also fix $0 \leqslant a < b \leqslant T$ and consider for $n \geqslant 0$ the dyadic partitions $\mathcal{P}_n := \{t_i^n := a + \frac{i}{2^n}(b - a) \colon 0 \le i \le 2^n\}$ of $[a, b]$. Since $\mathcal{P}_0 = \{a, b\}$ is the trivial partition, we can apply (1.21) to bound

$$|R_{ab}| \leqslant \sum_{n=0}^{\infty} |I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)|. \tag{1.27}$$

If we remove from $\mathcal{P}_{n+1}$ all the "odd points" t_{2j+1}^{n+1}, with $0 \le j \le 2^n - 1$, we obtain $\mathcal{P}_n$. Then, iterating relations (1.24)-(1.25), we have

$$\begin{aligned} |I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R)| &\leqslant \sum_{j=0}^{2^n-1} |\delta R_{t_{2j}^{n+1} t_{2j+1}^{n+1} t_{2j+2}^{n+1}}| \\ &\leqslant 2^n \, \|\delta R\|_\eta \left(\frac{2(b-a)}{2^{n+1}}\right)^\eta \\ &= 2^{-(\eta-1)n} \, \|\delta R\|_\eta \, (b-a)^\eta . \end{aligned} \tag{1.28}$$

Plugging this into (1.27), since $\sum_{n=0}^\infty 2^{-(\eta-1)n} = (1-2^{1-\eta})^{-1}$, we obtain

$$|R_{ab}| \leqslant (1-2^{1-\eta})^{-1} \, \|\delta R\|_\eta \, (b-a)^\eta, \qquad 0 \leqslant a < b \leqslant T, \tag{1.29}$$

which proves (1.26). □

Remark 1.10. Recalling (1.11) and (1.23), we have defined linear maps

$$C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_3 \tag{1.30}$$

which satisfy $\delta \circ \delta = 0$. Indeed, for any $f \in C_1$ we have

$$\delta(\delta f)_{sut} = (f_t - f_s) - (f_u - f_s) - (f_t - f_u) = 0.$$

Intuitively, $\delta F \in C_3$ measures how much a function $F \in C_2$ differs from being the increment δf of some $f \in C_1$, because $\delta F \equiv 0$ *if and only if* $F = \delta f$ *for some* $f \in C_1$ (it suffices to define $f_t := F_{0t}$ and to check that $\delta f_{st} = \delta F_{0st} + F_{st} = F_{st}$).

Remark 1.11. The assumption $R_{st} = o(t-s)$ in Theorem 1.9 cannot be avoided: if $R := \delta f$ for a non constant $f \in C_1$, then $\delta R = 0$ while $\|R\|_\eta > 0$.

1.6. END OF PROOF OF UNIQUENESS

In this section, we apply the Sewing Bound (1.26) to the function R_{st} defined in (1.17), in order to prove the estimate (1.19) for $\alpha > \frac{1}{2}$.

We first determine the increment δR through a simple and instructive computation: by (1.17), since $\delta(\delta Z) = 0$ (see Remark 1.10), we have

$$\begin{aligned} \delta R_{sut} &:= R_{st} - R_{su} - R_{ut} \\ &= (Y_t - Y_s) - (Y_u - Y_s) - (Y_t - Y_u) \\ &\quad -(\sigma Y_s)(X_t - X_s) + (\sigma Y_s)(X_u - X_s) + (\sigma Y_u)(X_t - X_u) \\ &= [\sigma (Y_u - Y_s)] (X_t - X_u). \end{aligned} \tag{1.31}$$

Recalling (1.9), this implies

$$\|\delta R\|_{2\alpha} \leqslant |\sigma| \, \|\delta Y\|_\alpha \, \|\delta X\|_\alpha.$$

We next note that if $\alpha > \frac{1}{2}$ (as it is assumed in Theorem 1.7) we can apply the Sewing Bound (1.26) for $\eta = 2\alpha > 1$ to obtain

$$\|R\|_{2\alpha} \leqslant K_{2\alpha} \, \|\delta R\|_{2\alpha} \leqslant K_{2\alpha} \, |\sigma| \, \|\delta Y\|_\alpha \, \|\delta X\|_\alpha .$$

This is precisely our goal (1.19) with $C = C(X, \alpha, \sigma) := K_{2\alpha} |\sigma| \, \|\delta X\|_\alpha$.

Summarizing: thanks to the Sewing bound (1.26), we have obtained the estimate (1.19) and completed the proof of Theorem 1.7, showing uniqueness of solutions to the difference equation (1.4) for any $X \in \mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2}, 1\right]$. In the next chapters we extend this approach to non-linear $\sigma(\cdot)$ and to situations where $X \in \mathcal{C}^\alpha$ with $\alpha \leqslant \frac{1}{2}$.

Remark 1.12. For later purpose, let us record the computation (1.31) withouth σ: given any (say, real) paths X and Y, if

$$A_{st} = Y_s \, \delta X_{st}, \qquad \forall 0 \leqslant s \leqslant t \leqslant T,$$

then

$$\delta A_{sut} = -\delta Y_{su} \, \delta X_{ut}, \qquad \forall 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{1.32}$$

1.7. WEIGHTED NORMS

We conclude this chapter defining *weighted versions* $\|\cdot\|_{\eta,\tau}$ of the norms $\|\cdot\|_\eta$ introduced in (1.9): given $F \in C_2$ and $G \in C_3$, we set for $\eta, \tau \in (0, \infty)$

$$\|F\|_{\eta,\tau} := \sup_{0 \leqslant s \leqslant t \leqslant T} \mathbb{1}_{\{0 < t-s \leqslant \tau\}} \, \mathrm{e}^{-\frac{t}{\tau}} \frac{|F_{st}|}{(t-s)^\eta}, \tag{1.33}$$

$$\|G\|_{\eta,\tau} := \sup_{0 \leqslant s \leqslant u \leqslant t \leqslant T} \mathbb{1}_{\{0 < t-s \leqslant \tau\}} \, \mathrm{e}^{-\frac{t}{\tau}} \frac{|G_{sut}|}{(t-s)^\eta}, \tag{1.34}$$

where C_2 and C_3 are the spaces of continuous functions from $[0, T]^2_\leqslant$ and $[0, T]^3_\leqslant$ to $\mathbb{R}^k$, see (1.8). Note that as $\tau \to \infty$ we recover the usual norms:

$$\|\cdot\|_\eta = \lim_{\tau \to \infty} \|\cdot\|_{\eta,\tau}. \tag{1.35}$$

Remark 1.13. (NORMS VS. SEMI-NORMS) While $\|\cdot\|_\eta$ is a norm, $\|\cdot\|_{\eta,\tau}$ is a norm for $\tau \geqslant T$ but *it is only a semi-norm for* $\tau < T$ (for instance, $\|F\|_{\eta,\tau} = 0$ for $F \in C_2$ implies $F_{st} = 0$ only for $t - s \leqslant \tau$: no constraint is imposed on F_{st} for $t - s > \tau$).

However, if $F = \delta f$, that is $F_{st} = f_t - f_s$ for some $f \in C_1$, we have the equivalence

$$\|\delta f\|_{\eta,\tau} \leqslant \|\delta f\|_\eta \leqslant \left(1 + \frac{T}{\tau}\right) \mathrm{e}^{\frac{T}{\tau}} \|\delta f\|_{\eta,\tau}. \tag{1.36}$$

The first inequality is clear. For the second one, given $0 \leqslant s < t \leqslant T$, we can write $s = t_0 < t_1 < \cdots < t_N = t$ with $t_i - t_{i-1} \leqslant \tau$ and $N \leqslant 1 + \frac{T}{\tau}$ (for instance, we can consider $t_i = s + i \frac{t-s}{N}$ where $N := \lceil \frac{t-s}{\tau} \rceil$); we then obtain $\delta f_{st} = \sum_{i=1}^N \delta f_{t_{i-1} t_i}$ and $|\delta f_{t_{i-1} t_i}| \leqslant \|\delta f\|_{\eta,\tau} \, \mathrm{e}^{t_i/\tau} (t_i - t_{i-1})^\eta \leqslant \|\delta f\|_{\eta,\tau} \, \mathrm{e}^{T/\tau} (t-s)^\eta$, which yields (1.36).

Remark 1.14. (FROM LOCAL TO GLOBAL) The weighted semi-norms $\|\cdot\|_{\eta,\tau}$ will be useful to transform *local* results in *global* results. Indeed, using the standard norms $\|\cdot\|_\eta$ often requires the size $T > 0$ of the time interval $[0, T]$ to be *small*, as in Theorem 1.7, which can be annoying. Using $\|\cdot\|_{\eta,\tau}$ will allow us to *keep* $T > 0$ *arbitrary*, by choosing a sufficiently small $\tau > 0$.

Recalling the supremum norm $\|f\|_\infty$ of a function $f \in C_1$, see (1.14), we define the corresponding weighted version

$$\|f\|_{\infty,\tau} := \sup_{0\leqslant t\leqslant T} \mathrm{e}^{-\frac{t}{\tau}} |f_t| \,. \tag{1.37}$$

We stress that $\|\cdot\|_{\infty,\tau}$ *is a norm equivalent to* $\|\cdot\|_\infty$ *for any* $\tau > 0$, since

$$\|\cdot\|_{\infty,\tau} \leqslant \|\cdot\|_\infty \leqslant \mathrm{e}^{\frac{T}{\tau}} \|\cdot\|_{\infty,\tau} \,. \tag{1.38}$$

Remark 1.15. (EQUIVALENT HÖLDER NORM) It follows by (1.36) and (1.38) that $\|\cdot\|_{\infty,\tau} + \|\cdot\|_{\alpha,\tau}$ *is a norm equivalent to* $\|\cdot\|_{\mathcal{C}^\alpha} := \|\cdot\|_\infty + \|\cdot\|_\alpha$ on the space $\mathcal{C}^\alpha$ of Hölder functions, see Remark 1.4, for any $\tau > 0$.

We will often use the Hölder semi-norms $\|\delta f\|_\alpha$ and $\|\delta f\|_{\alpha,\tau}$ to bound the supremum norms $\|f\|_\infty$ and $\|f\|_{\infty,\tau}$, thanks to the following result.

LEMMA 1.16. (SUPREMUM-HÖLDER BOUND) *For any* $f \in C_1$ *and* $\eta \in (0,\infty)$

$$\|f\|_\infty \leqslant |f_0| + T^\eta \|\delta f\|_\eta \,, \tag{1.39}$$

$$\|f\|_{\infty,\tau} \leqslant |f_0| + 3\,(\tau \wedge T)^\eta \|\delta f\|_{\eta,\tau}, \qquad \forall \tau > 0 \,. \tag{1.40}$$

Proof. Let us prove (1.39): for any $f \in C_1$ and for $t \in \,]0,T]$ we have

$$|f_t| \leqslant |f_0| + |f_t - f_0| = |f_0| + t^\eta \frac{|f_t - f_0|}{t^\eta} \leqslant |f_0| + T^\eta \|\delta f\|_\eta .$$

The proof of (1.40) is slightly more involved. If $t \in \,]0, \tau \wedge T]$, then

$$\mathrm{e}^{-\frac{t}{\tau}} |f_t| \leqslant |f_0| + t^\eta \,\mathrm{e}^{-\frac{t}{\tau}} \frac{|f_t - f_0|}{t^\eta} \leqslant |f_0| + (\tau \wedge T)^\eta \|\delta f\|_{\eta,\tau},$$

which, in particular, implies (1.40) when $\tau \geqslant T$. When $\tau < T$, it remains to consider $\tau < t \leqslant T$: in this case, we define $N := \min\{n \in \mathbb{N}: \ n\tau \geq t\} \geq 2$ so that $\frac{t}{N} \leqslant \tau$. We set $t_k = k\,\frac{t}{N}$ for $k \geq 0$, so that $t_N = t$. Then

$$\begin{aligned}
\mathrm{e}^{-\frac{t}{\tau}} |f_t| &\leqslant |f_0| + \sum_{k=1}^{N} (t_k - t_{k-1})^\eta \,\mathrm{e}^{-\frac{t-t_k}{\tau}} \left[\mathrm{e}^{-\frac{t_k}{\tau}} \frac{|f_{t_k} - f_{t_{k-1}}|}{(t_k - t_{k-1})^\eta} \right] \\
&\leqslant |f_0| + (\tau \wedge T)^\eta \|\delta f\|_{\eta,\tau} \sum_{k=1}^{N} \mathrm{e}^{-\frac{t-t_k}{\tau}}.
\end{aligned}$$

By definition of N we have $(N-1)\tau < t$; since $\tau < t$ we obtain $N\tau < 2t$ and therefore $\frac{t}{N\tau} \geq \frac{1}{2}$. Since $t - t_k = (N-k)\frac{t}{N}$, renaming $\ell := N - k$ we obtain

$$\sum_{k=1}^{N} \mathrm{e}^{-\frac{t-t_k}{\tau}} = \sum_{\ell=0}^{N-1} \mathrm{e}^{-\ell \frac{t}{N\tau}} = \frac{1 - \mathrm{e}^{-\frac{t}{\tau}}}{1 - \mathrm{e}^{-\frac{t}{N\tau}}} \leqslant \frac{1}{1 - \mathrm{e}^{-\frac{1}{2}}} \leqslant 3.$$

The proof is complete. □

We finally show that the Sewing Bound (1.26) still holds if we replace $\|\cdot\|_\eta$ by $\|\cdot\|_{\eta,\tau}$, for any $\tau > 0$.

THEOREM 1.17. (WEIGHTED SEWING BOUND) *Given any $R \in C_2$ with $R_{st} = o(t-s)$, the following estimate holds for any $\eta \in (1,\infty)$ and $\tau > 0$:*

$$\|R\|_{\eta,\tau} \leqslant K_\eta \, \|\delta R\|_{\eta,\tau} \qquad \textit{where} \qquad K_\eta := (1-2^{1-\eta})^{-1}. \tag{1.41}$$

Proof. Given $0 \leqslant a \leqslant b \leqslant T$, let us define

$$\|\delta R\|_{\eta,[a,b]} := \sup_{\substack{s,u,t \in [a,b]:\\ s \leqslant u \leqslant t,\ s<t}} \frac{|\delta R_{sut}|}{(t-s)^\eta}. \tag{1.42}$$

Following the proof of Theorem 1.9, we can replace $\|\delta R\|_\eta$ by $\|\delta R\|_{\eta,[a,b]}$ in (1.28) and in (1.29), hence we obtain $|R_{ab}| \leqslant K_\eta \, \|\delta R\|_{\eta,[a,b]} \, (b-a)^\eta$. Then for $b-a \leqslant \tau$ we can estimate

$$e^{-\frac{b}{\tau}} \frac{|R_{ab}|}{(b-a)^\eta} \leqslant e^{-\frac{b}{\tau}} K_\eta \, \|\delta R\|_{\eta,[a,b]} \leqslant K_\eta \, \|\delta R\|_{\eta,\tau},$$

and (1.41) follows taking the supremum over $0 \leqslant a \leqslant b \leqslant T$ with $b-a \leqslant \tau$. □

1.8. A DISCRETE SEWING BOUND

We can prove a version of the Sewing Bound for functions $R = (R_{st})_{s<t \in \mathbb{T}}$ defined on a *finite set of points* $\mathbb{T} := \{0 = t_1 < \cdots < t_{\#\mathbb{T}}\} \subseteq \mathbb{R}_+$ (this will be useful to construct solutions to difference equations via Euler schemes, see Sections 2.6 and 3.9). The condition $R_{st} = o(t-s)$ from Theorem 1.9 is now replaced by the requirement that *R vanishes on consecutive points of $\mathbb{T}$*, i.e. $R_{t_i t_{i+1}} = 0$ for all $1 \leqslant i < \#\mathbb{T}$.

We define versions $\|\cdot\|_{\eta,\tau}^{\mathbb{T}}$ of the norms $\|\cdot\|_{\eta,\tau}$ restricted on $\mathbb{T}$ for $\tau > 0$, recall (1.33)-(1.34):

$$\|A\|_{\eta,\tau}^{\mathbb{T}} := \sup_{\substack{0 \leqslant s < t\\ s,t \in \mathbb{T}}} \mathbb{1}_{\{0 < t-s \leqslant \tau\}} \, \mathrm{e}^{-\frac{t}{\tau}} \frac{|A_{st}|}{|t-s|^\eta}, \tag{1.43}$$

$$\|B\|_{\eta,\tau}^{\mathbb{T}} := \sup_{\substack{0 \leqslant s \leqslant u \leqslant t\\ s,u,t \in \mathbb{T},\ s<t}} \mathbb{1}_{\{0 < t-s \leqslant \tau\}} \, \mathrm{e}^{-\frac{t}{\tau}} \frac{|B_{sut}|}{|t-s|^\eta} \tag{1.44}$$

for $A: \{(s,t) \in \mathbb{T}^2 : 0 \leqslant s < t\} \to \mathbb{R}$ and $B: \{(s,u,t) \in \mathbb{T}^3 : 0 \leqslant s \leqslant u \leqslant t, s < t\} \to \mathbb{R}$.

THEOREM 1.18. (DISCRETE SEWING BOUND) *If a function $R = (R_{st})_{s<t \in \mathbb{T}}$ vanishes on consecutive points of $\mathbb{T}$ (i.e. $R_{t_i t_{i+1}} = 0$), then for any $\eta > 1$ and $\tau > 0$ we have*

$$\|R\|_{\eta,\tau}^{\mathbb{T}} \leqslant C_\eta \, \|\delta R\|_{\eta,\tau}^{\mathbb{T}} \qquad \textit{with} \qquad C_\eta := 2^\eta \sum_{n \geq 1} \frac{1}{n^\eta} = 2^\eta \zeta(\eta) < \infty. \tag{1.45}$$

Proof. We fix $s, t \in \mathbb{T}$ with $s < t$ and we start by proving that

$$|R_{st}| \leqslant C_\eta \, \|\delta R\|_\eta^{\mathbb{T}} \, (t-s)^\eta.$$

We have $s = t_k$ and $t = t_{k+m}$ and we may assume that $m \geqslant 2$ (otherwise there is nothing to prove, since for $m = 1$ we have $R_{t_i t_{i+1}} = 0$).

Consider the partition $\mathcal{P} = \{s = t_k < t_{k+1} < \ldots < t_{k+m} = t\}$ with m intervals. Note that for some index $i \in \{k+1, \ldots, k+m-1\}$ we must have $t_{i+1} - t_{i-1} \leq \frac{2\,(t-s)}{m-1}$, otherwise we would get the contradiction

$$2\,(t-s) \geq \sum_{i=k+1}^{k+m-1} (t_{i+1} - t_{i-1}) > \sum_{i=k+1}^{k+m-1} \frac{2\,(t-s)}{m-1} = 2\,(t-s)\,.$$

Removing the point t_i from $\mathcal{P}$ we obtain a partition $\mathcal{P}'$ with $m-1$ intervals. If we define $I_{\mathcal{P}}(R) := \sum_{i=k}^{k+m-1} R_{t_i t_{i+1}}$ as in (1.20), as in (1.24) we have

$$|I_{\mathcal{P}}(R) - I_{\mathcal{P}'}(R)| = |\delta R_{t_{i-1} t_i t_{i+1}}| \leqslant \frac{2^\eta\,(t-s)^\eta}{(m-1)^\eta} \sup_{\substack{s \leqslant u < v < w \leqslant t \\ u,v,w \in \mathbb{T}}} \frac{|\delta R_{uvw}|}{|w-u|^\eta}\,.$$

Iterating this argument, until we arrive at the trivial partition $\{s, t\}$, we get

$$|I_{\mathcal{P}}(R) - R_{st}| \leq C_\eta\,(t-s)^\eta \sup_{\substack{s \leqslant u < v < w \leqslant t \\ u,v,w \in \mathbb{T}}} \frac{|\delta R_{uvw}|}{|w-u|^\eta}, \tag{1.46}$$

with $C_\eta := \sum_{n \geq 1} \frac{2^\eta}{n^\eta} < \infty$ because $\eta > 1$. We finally note that $I_{\mathcal{P}}(R) = 0$ by the assumption $R_{t_i t_{i+1}} = 0$. Finally if $t - s \leqslant \tau$ then $w - u \leqslant \tau$ in the supremum in (1.46) and since $\mathrm{e}^{-\frac{t}{\tau}} \leqslant \mathrm{e}^{-\frac{w}{\tau}}$ we obtain

$$\mathrm{e}^{-\frac{t}{\tau}}\,|R_{st}| \leqslant C_\eta\,(t-s)^\eta\,\|\delta R\|_{\eta,\tau}^{\mathbb{T}},$$

and the proof is complete. □

We also have an analog of Lemma 1.16. We set for $f: \mathbb{T} \to \mathbb{R}$ and $\tau > 0$

$$\|f\|_{\infty,\tau}^{\mathbb{T}} := \sup_{t \in \mathbb{T}} \mathrm{e}^{-\frac{t}{\tau}}\,|f_t|\,.$$

LEMMA 1.19. (DISCRETE SUPREMUM-HÖLDER BOUND) *For* $\mathbb{T} := \{0 = t_1 < \cdots < t_{\#\mathbb{T}}\} \subseteq \mathbb{R}_+$ *set*

$$M := \max_{i=2,\ldots,\#\mathbb{T}} |t_i - t_{i-1}|.$$

Then for all $f: \mathbb{T} \to \mathbb{R}$, $\tau \geqslant 2M$ *and* $\eta > 0$

$$\|f\|_{\infty,\tau}^{\mathbb{T}} \leqslant |f_0| + 5\,\tau^\eta\,\|\delta f\|_{\eta,\tau}^{\mathbb{T}}\,. \tag{1.47}$$

Proof. We define $T_0 := 0$ and for $i \geqslant 1$, as long as $\mathbb{T} \cap (T_{i-1}, T_{i-1} + \tau]$ is not empty, we set

$$T_i := \max \mathbb{T} \cap (T_{i-1}, T_{i-1} + \tau], \qquad i = 1, \ldots, N,$$

so that $T_N = \max \mathbb{T}$. We have by construction $T_i + M > T_{i-1} + \tau$ for all $i = 1, \ldots, N-1$, and since $M \leqslant \frac{\tau}{2}$

$$T_i - T_{i-1} \geqslant \tau - M \geqslant \frac{\tau}{2}\,.$$

For $i=N$ we have only $T_N > T_{N-1}$. Therefore for $i=1,\dots N$

$$\begin{aligned}
e^{-\frac{T_i}{\tau}}|f_{T_i}| &\leqslant |f_0| + \sum_{k=1}^{i} (T_k - T_{k-1})^{\eta} e^{-\frac{T_i - T_k}{\tau}} \left[e^{-\frac{T_k}{\tau}} \frac{|f_{T_k} - f_{T_{k-1}}|}{(T_k - T_{k-1})^{\eta}} \right] \\
&\leqslant |f_0| + \tau^{\eta} \|\delta f\|_{\eta,\tau}^{\mathbb{T}} \sum_{k=1}^{i} e^{-\frac{T_i - T_k}{\tau}} \\
&\leqslant |f_0| + \tau^{\eta} \|\delta f\|_{\eta,\tau}^{\mathbb{T}} \left(1 + \sum_{k=0}^{\infty} e^{-\frac{k}{2}} \right) \\
&\leqslant |f_{t_0}| + 4\tau^{\eta} \|\delta f\|_{\eta,\tau}^{\mathbb{T}}.
\end{aligned}$$

Now for $t \in \mathbb{T} \backslash \{T_i\}_i$ we have $T_i < t < T_{i+1}$ for some i and then

$$\begin{aligned}
e^{-\frac{t}{\tau}}|f_t| &\leqslant e^{-\frac{t}{\tau}}|f_{T_i}| + (t - T_i)^{\eta} e^{-\frac{t}{\tau}} \frac{|f_t - f_{T_i}|}{(t - T_i)^{\eta}} \leqslant e^{-\frac{T_i}{\tau}}|f_{T_i}| + \tau^{\eta} \|\delta f\|_{\eta,\tau}^{\mathbb{T}} \\
&\leqslant |f_0| + 5\tau^{\eta} \|\delta f\|_{\eta,\tau}^{\mathbb{T}}.
\end{aligned}$$

The proof is complete. □

1.9. Extra (to be completed)

We also introduce the usual supremum norm, for $F \in C_2$ and $G \in C_3$:

$$\|F\|_{\infty} := \sup_{0 \leqslant s \leqslant t \leqslant T} |F_{st}|, \qquad \|G\|_{\infty} := \sup_{0 \leqslant s \leqslant u \leqslant t \leqslant T} |G_{sut}|,$$

and a corresponding weighted version, for $\tau \in (0, \infty)$:

$$\|F\|_{\infty,\tau} := \sup_{0 \leqslant s \leqslant t \leqslant T} e^{-\frac{t}{\tau}} |F_{st}|, \qquad \|G\|_{\infty,\tau} := \sup_{0 \leqslant s \leqslant u \leqslant t \leqslant T} e^{-\frac{t}{\tau}} |G_{sut}|. \tag{1.48}$$

Note that

$$\lim_{\tau \to +\infty} \|F\|_{\infty,\tau} = \|F\|_{\infty}, \quad \lim_{\tau \to +\infty} \|G\|_{\eta,\tau} = \|G\|_{\eta}, \quad \lim_{\tau \to +\infty} \|H\|_{\eta,\tau} = \|H\|_{\eta}.$$

We have

$$\|F\|_{\eta,\tau} \leqslant \|G\|_{\infty,\tau} \|H\|_{\eta}, \qquad (F_{sut} = G_{su} H_{ut}), \tag{1.49}$$

Note that $\|\cdot\|_{\eta,\tau}$ is only a semi-norm on C_n^{η} if $\tau < T$; we have at least

$$\|\cdot\|_{\eta,\tau} \leqslant \|\cdot\|_{\eta} \leqslant e^{\frac{T}{\tau}} \left(\|\cdot\|_{\eta,\tau} + \frac{1}{\tau^{\eta}} \|\cdot\|_{\infty,\tau} \right). \tag{1.50}$$

However, if $\tau \geq T$ we have again equivalence of norms

$$\|\cdot\|_{\eta,\tau} \leqslant \|\cdot\|_{\eta} \leqslant e^{\frac{T}{\tau}} \|\cdot\|_{\eta,\tau}, \qquad \tau \geq T. \tag{1.51}$$