Chapter 10
Rough integration

10.1. Controlled paths

We fix $\alpha \in]\frac{1}{3}, \frac{1}{2}]$, $X \in \mathcal{C}^{\alpha}([0,T];\mathbb{R}^d)$. We recall that fixing a α-rough path $\mathbb{X}$ over X as in Definition 8.9 is equivalent to choosing a solution $(I, \mathbb{X}^2)$ to (8.17), with I and $\mathbb{X}^2$ representing our choices of the integrals, respectively,

$$I_t =: \int_0^t X_r \otimes \mathrm{d}X_r, \qquad \mathbb{X}^2_{st} =: \int_s^t (X_r - X_s) \otimes \mathrm{d}X_r = I_t - I_s - X_s \otimes (X_t - X_s).$$

The key point is that, having fixed a choice of $\mathbb{X}^2$, it is now possible to give *a canonical definition of the integral* $\int_0^{\cdot} Y\,\mathrm{d}X$ for a wide class of $Y \in \mathcal{C}^{\alpha}([0,T];\mathbb{R}^k \otimes (\mathbb{R}^d)^*)$, namely those *paths Y which are controlled by* $\mathbb{X}$. In order to motivate this notion, let us recall that, given $X \in \mathcal{C}^{\alpha}([0,T];\mathbb{R}^d)$ and $Y \in \mathcal{C}^{\beta}([0,T];\mathbb{R}^k \otimes (\mathbb{R}^d)^*)$, we look now for $J\colon [0,T] \to \mathbb{R}^k$ and $R^J\colon [0,T]^2_{\leqslant} \to \mathbb{R}^k$ such that, in analogy with (8.4),

$$J_0 = 0, \qquad \delta J_{st} = Y_s\,\delta X_{st} + R^J_{st}, \qquad |R^J_{st}| \lesssim |t-s|^{\alpha+\beta}.$$

In order to make this operation *iterable*, it is natural to require that *each component of* Y has an analogous property. This is exactly the motivation for the next

Definition 10.1. *Let* $\alpha \in]\frac{1}{3}, \frac{1}{2}]$, $\eta \in]0,1]$ *and* $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ *an* α*-rough path on* $\mathbb{R}^d$. *A pair* $\boldsymbol{Z} = (Z, Z^1)\colon [0,T] \to \mathbb{R}^k \times (\mathbb{R}^k \otimes (\mathbb{R}^d)^*)$ *with* Z *of class* $\mathcal{C}^{\alpha}$ *and* Z^1 *of class* C^{η} *is* a path $(\alpha+\eta)$-controlled *by* $\mathbb{X}$ *if*

$$\delta Z_{st} = Z^1_s\,\mathbb{X}^1_{st} + Z^{[2]}_{st}, \qquad |Z^{[2]}_{st}| \lesssim |t-s|^{\alpha+\eta}, \qquad (s,t) \in [0,T]^2_{\leqslant}. \tag{10.1}$$

The function Z^1 *is called a* derivative *of* Z *with respect to* $\mathbb{X}$ *and* $Z^{[2]}$ *is the* remainder *of the couple* (Z, Z^1).

For a fixed α*-rough path* $\mathbb{X}$ *on* $\mathbb{R}^d$, *we denote by* $\mathcal{D}^{\alpha+\eta}_{\mathbb{X}}(\mathbb{R}^k)$ *the space of paths* $(\alpha+\eta)$*-controlled by* $\mathbb{X}$ *with values in* $\mathbb{R}^k$.

Remark 10.2. Note that, if $\alpha + \eta \leqslant 1$, in general Z^1 is *not* determined by $(Z, \mathbb{X}^1)$, so that we say that Z^1 is *a* derivative rather than *the* derivative of Z. Viceversa, Z is *not* determined by $(Z^1, \mathbb{X}^1)$: if (Z, Z^1) is $(\alpha+\eta)$-controlled by $\mathbb{X}$ and $f \in \mathcal{C}^{\alpha+\eta}([0,T];\mathbb{R}^k)$ then $(Z+f, Z^1)$ is also $(\alpha+\eta)$-controlled by $\mathbb{X}$.

It is now clear from the definitions that, unlike rough paths, $(\alpha+\eta)$-controlled paths have a natural linear structure, in particular as a linear subspace of $\mathcal{C}^{\alpha} \times \mathcal{C}^{\eta}$.

Exercise 10.1. Show that for each $i,j=1,\ldots,d$, setting $[0,T]\ni t\mapsto(\mathbb{X}^1_{0t},\mathrm{Id})\in\mathbb{R}^d\times(\mathbb{R}^d\otimes(\mathbb{R}^d)^*)$ and $[0,T]\ni t\mapsto(\mathbb{X}^2_{0t},\mathbb{X}^1_{0t}\otimes\mathrm{Id})\in\mathbb{R}^d\otimes\mathbb{R}^d\times(\mathbb{R}^d\otimes\mathbb{R}^d\otimes(\mathbb{R}^d)^*)$ are paths 2α-controlled by $\mathbb{X}$.

10.2. The rough integral

Now we can finally show how to modify the germ $Y_s(X_t-X_s)$ in order to obtain a well-defined integration theory.

Proposition 10.3. *Let $\alpha\in\,]\frac{1}{3},\frac{1}{2}]$, $\eta\in\,]0,1]$ and $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ a α-rough path on $\mathbb{R}^d$. If $\boldsymbol{Z}=(Z,Z^1)\colon[0,T]\to\mathbb{R}^k\times(\mathbb{R}^k\otimes(\mathbb{R}^d)^*)$ is $(\alpha+\eta)$-controlled by $\mathbb{X}$ as in Definition 10.1, then the germ*

$$A_{st}=Z_s\,\mathbb{X}^1_{st}+Z^1_s\,\mathbb{X}^2_{st}$$

satisfies $\delta A\in C^{2\alpha+\eta}_3$. .

Therefore if $2\alpha+\eta>1$ we can canonically *define $J_t=$ "$\int_0^t Z\,\mathrm{d}\mathbb{X}$" as the unique function $J\colon[0,T]\to\mathbb{R}^k$ such that $J_0=0$ and $\delta J-A\in C^{2\alpha+\eta}_2$, namely*

$$|J_t-J_s-Z_s\,\mathbb{X}^1_{st}-Z^1_s\,\mathbb{X}^2_{st}|\lesssim|t-s|^{2\alpha+\eta},$$

and we have

$$J_t=\lim_{|\mathcal{P}|\to0}\sum_{i=0}^{\#\mathcal{P}-1}(Z_{t_i}\mathbb{X}^1_{t_it_{i+1}}+Z^1_{t_i}\mathbb{X}^2_{t_it_{i+1}})$$

along arbitrary partitions $\mathcal{P}$ of $[0,t]$ with vanishing mesh $|\mathcal{P}|\to0$.

Proof. We compute by (8.20)

$$\begin{aligned}\delta A_{sut}&=-\delta Z_{su}\,\mathbb{X}^1_{ut}+Z^1_s\,\delta\mathbb{X}^2_{sut}-\delta Z^1_{su}\,\mathbb{X}^2_{ut}\\&=-(\delta Z_{su}-Z^1_s\,\mathbb{X}^1_{su})\,\mathbb{X}^1_{ut}-\delta Z^1_{su}\,\mathbb{X}^2_{ut}\\&=-Z^{[2]}_{su}\,\mathbb{X}^1_{ut}-\delta Z^1_{su}\,\mathbb{X}^2_{ut}.\end{aligned}\tag{10.2}$$

Then by (2.8)

$$\begin{aligned}|\delta A_{sut}|&\leqslant\|Z^{[2]}\|_{\alpha+\eta}|u-s|^{\alpha+\eta}\|\mathbb{X}^1\|_\alpha|t-u|^\alpha+\|\delta Z^1\|_\eta|u-s|^\eta\|\mathbb{X}^2\|_{2\alpha}|t-u|^{2\alpha}\\&\leqslant(\|Z^{[2]}\|_{\alpha+\eta}\|\mathbb{X}^1\|_\alpha+\|\delta Z^1\|_\eta\|\mathbb{X}^2\|_{2\alpha})|t-s|^{2\alpha+\eta}.\end{aligned}\tag{10.3}$$

Since $\delta A\in C^{2\alpha+\eta}_3$, if $2\alpha+\eta>1$ we can apply the Sewing Lemma and define $J^{[3]}:=-\Lambda(\delta A)$ and $J\colon[0,T]\to\mathbb{R}^k$ such that $J_0=0$ and $\delta J=A+J^{[3]}$ where Λ is the Sewing Map of Theorem 6.11, namely

$$J_0=0\,,\qquad\delta J_{st}=Z_s\,\mathbb{X}^1_{st}+Z^1_s\,\mathbb{X}^2_{st}+J^{[3]}_{st}\,,\qquad|J^{[3]}_{st}|\lesssim|t-s|^{2\alpha+\eta}\,.\tag{10.4}$$

The last assertion on the convergence of the generalised Riemann sums follows from (6.13). □

We have in particular proved by (6.14) and (10.3) that

$$\|J^{[3]}\|_{2\alpha+\eta}\leqslant K_{2\alpha+\eta}(\|Z^{[2]}\|_{\alpha+\eta}\|\mathbb{X}^1\|_\alpha+\|\delta Z^1\|_\eta\|\mathbb{X}^2\|_{2\alpha}).\tag{10.5}$$

We stress that the function J depends on $(\boldsymbol{Z}, \mathbb{X})$, in particular on Z^1 as well. We use the following notations

$$\boldsymbol{J} := (J, Z), \qquad \int_0^t \boldsymbol{Z}\, \mathrm{d}\mathbb{X} := (J_t, Z_t) = \boldsymbol{J}_t. \tag{10.6}$$

We shall see in Proposition 10.5 below that $\boldsymbol{J}\colon [0,T] \to \mathbb{R}^k \times (\mathbb{R}^k \otimes (\mathbb{R}^d)^*)$ is 2α-controlled by $\mathbb{X}$, i.e. Z is a derivative of J with respect to $\mathbb{X}$ as in Definition 10.1.

We define a norm $\|\cdot\|_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}$ and a seminorm $[\cdot]_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}$ on the space $\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}$ of paths $(\alpha+\eta)$-controlled by $\mathbb{X}$, defined as follows:

$$\begin{aligned} \|\boldsymbol{Z}\|_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}} &:= |Z_0| + |Z_0^1| + [\boldsymbol{Z}]_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}, \qquad \boldsymbol{Z} = (Z, Z^1) \\ [\boldsymbol{Z}]_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}} &:= \|\delta Z^1\|_\eta + \|Z^{[2]}\|_{\alpha+\eta}, \qquad Z_{st}^{[2]} = \delta Z_{st} - Z_s^1\, \mathbb{X}_{st}^1, \end{aligned} \tag{10.7}$$

as in (10.1). Recall that we defined the standard norm $\|f\|_{\mathcal{C}^\alpha} = \|f\|_\infty + \|\delta f\|_\alpha$ in (1.13).

Lemma 10.4. *We have the equivalence of norms for all* $\boldsymbol{Z} = (Z, Z^1) \in \mathcal{D}_{\mathbb{X}}^{\alpha+\eta}$

$$\|\boldsymbol{Z}\|_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}} \leqslant \|Z\|_{\mathcal{C}^\alpha} + \|Z^1\|_{\mathcal{C}^\eta} + \|Z^{[2]}\|_{\alpha+\eta} \leqslant C\, \|\boldsymbol{Z}\|_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}, \tag{10.8}$$

where $C > 0$ *is an explicit constant which depends only on* $(\mathbb{X}, T, \alpha, \eta)$. *In particular,* $(\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}, \|\cdot\|_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}})$ *is a Banach space.*

Proof. The first inequality in (10.8) is obvious by the definition of the norm $\|\cdot\|_{\mathcal{C}^\alpha}$. In order to prove the second one, first we note that by (1.15)

$$\|f\|_{\mathcal{C}^\eta} = \|f\|_\infty + \|\delta f\|_\eta \leqslant (1 + T^\eta)(|f_0| + \|\delta f\|_\eta).$$

This shows that $\|Z^1\|_{\mathcal{C}^\eta} \lesssim \|\boldsymbol{Z}\|_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}$ for $(Z, Z^1) \in \mathcal{D}_{\mathbb{X}}^{\alpha+\eta}$. Now, since $\delta Z_{st} = Z_s^1\, \mathbb{X}_{st}^1 + Z_{st}^{[2]}$ by (10.1),

$$\begin{aligned} \|\delta Z\|_\alpha &\leqslant \|Z^1\|_\infty \|\mathbb{X}^1\|_\alpha + \|Z^{[2]}\|_\alpha \\ &\leqslant C_{T,\eta}(|Z_0^1| + \|\delta Z^1\|_\eta)\|\mathbb{X}^1\|_\alpha + T^\eta \|Z^{[2]}\|_{\alpha+\eta}, \end{aligned}$$

namely $\|Z\|_{\mathcal{C}^\alpha} \lesssim \|\boldsymbol{Z}\|_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}$. Finally $\|Z^{[2]}\|_{\alpha+\eta} \leqslant \|\boldsymbol{Z}\|_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}$. The proof is complete. □

10.3. Continuity properties of the rough integral

We wrote before Definition 10.1 that the notion of controlled path aimed at making the rough integral map $(Z, Z^1) \mapsto (J, Z)$ iterable, where we use the notation of Proposition 10.3. In order to make this precise, we need the following important

Proposition 10.5. *Let* $\mathbb{X}$ *be a* α*-rough path on* $\mathbb{R}^d$ *with* $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$, $\eta \in]1 - 2\alpha, 1]$ *and* $\boldsymbol{Z} \in \mathcal{D}_{\mathbb{X}}^{\alpha+\eta}$ *a path* $(\alpha+\eta)$*-controlled by* $\mathbb{X}$. *Then, in the notation of (10.6),*

- $\boldsymbol{J} = \int_0^\cdot \boldsymbol{Z}\, \mathrm{d}\mathbb{X}$ *is* 2α*-controlled by* $\mathbb{X}$

- *the map* $\mathcal{D}_{\mathbb{X}}^{\alpha+\eta} \ni \boldsymbol{Z} \mapsto \boldsymbol{J} \in \mathcal{D}_{\mathbb{X}}^{2\alpha}$ *is linear and for all* $\boldsymbol{Z} \in \mathcal{D}_{\mathbb{X}}^{\alpha+\eta}$

$$[\boldsymbol{J}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}} \leqslant 2(1+\|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}})[|Z_0^1| + T^\eta(1+K_{2\alpha+\eta})[\boldsymbol{Z}]_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}]. \tag{10.9}$$

Proof. Recall first (10.5), so that in particular $\|J^{[3]}\|_{2\alpha+\eta} < +\infty$. Now $J_{st}^{[2]} = Z_s^1\,\mathbb{X}_{st}^2 + J_{st}^{[3]}$ satisfies

$$\|J^{[2]}\|_{2\alpha} \leqslant \|Z^1\|_\infty \|\mathbb{X}^2\|_{2\alpha} + \|J^{[3]}\|_{2\alpha} \leqslant \|Z^1\|_\infty \|\mathbb{X}^2\|_{2\alpha} + T^\eta \|J^{[3]}\|_{2\alpha+\eta}. \tag{10.10}$$

Finally $\delta J_{st} = Z_s\,\mathbb{X}_{st}^1 + J_{st}^{[2]}$ and therefore

$$\|\delta J\|_\alpha \leqslant \|Z\|_\infty \|\mathbb{X}^1\|_\alpha + \|Z^1\|_\infty \|\mathbb{X}^2\|_{2\alpha} + T^{\alpha+\eta}\|J^{[3]}\|_{2\alpha+\eta}.$$

Therefore $(J, Z, J^{[2]}) \in \mathcal{C}^\alpha \times \mathcal{C}^\alpha \times C_2^{2\alpha}$ and we obtain that (J, Z) is 2α-controlled by $\mathbb{X}$.

We prove now the second assertion. Since $\delta Z_{st} = Z_s^1\,\mathbb{X}_{st}^1 + Z_{st}^{[2]}$, by (1.39)

$$\begin{aligned}\|\delta Z\|_\alpha &\leqslant \|Z^1\|_\infty \|\mathbb{X}^1\|_\alpha + T^\eta \|Z^{[2]}\|_{\alpha+\eta}\\ &\leqslant (\|\mathbb{X}^1\|_\alpha + 1)(|Z_0^1| + T^\eta[\boldsymbol{Z}]_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}).\end{aligned}$$

Now, analogously to (10.10), again by (1.39)

$$\begin{aligned}\|J^{[2]}\|_{2\alpha} &\leqslant \|Z^1\|_\infty \|\mathbb{X}^2\|_{2\alpha} + \|J^{[3]}\|_{2\alpha}\\ &\leqslant T^\eta \|J^{[3]}\|_{2\alpha+\eta} + \|\mathbb{X}^2\|_{2\alpha}(|Z_0^1| + T^\eta \|\delta Z^1\|_\eta).\end{aligned}$$

Therefore, since $\|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha} = \|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}}$, recall (8.23),

$$\|\delta Z\|_\alpha + \|J^{[2]}\|_{2\alpha} \leqslant T^\eta \|J^{[3]}\|_{2\alpha+\eta} + (1+\|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}})[|Z_0^1| + T^\eta[\boldsymbol{Z}]_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}].$$

By (10.5) we obtain

$$\begin{aligned}[\boldsymbol{J}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}} &= \|\delta Z\|_\alpha + \|J^{[2]}\|_{2\alpha} \leqslant\\ &\leqslant 2\,(1+\|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}})[|Z_0^1| + (1+K_{2\alpha+\eta})T^\eta[\boldsymbol{Z}]_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}]\end{aligned}$$

The proof is complete. □

We note that the estimate of the seminorm $[\boldsymbol{J}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}$ in terms of $[\boldsymbol{Z}]_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}$ rather than of the norm $\|\boldsymbol{J}\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}$ in terms of $\|\boldsymbol{Z}\|_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}$ plays an important role in Chapter 11 (with $\eta = \alpha$), see in particular (11.9). In any case, from (10.9) it is easy to obtain an estimate of $\|\boldsymbol{J}\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}$: since $J_0 = 0$ and $J_0^1 = Z_0$, we obtain

$$\begin{aligned}\|\boldsymbol{J}\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}} &= |Z_0| + [\boldsymbol{J}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}} \leqslant\\ &\leqslant 2(1+K_{2\alpha+\eta})(1+\|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}})(1+T^\eta)\|\boldsymbol{Z}\|_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}.\end{aligned}$$

Therefore the linear operator $\mathcal{D}_{\mathbb{X}}^{\alpha+\eta} \ni \boldsymbol{Z} \mapsto \int_0^\cdot \boldsymbol{Z}\,\mathrm{d}\mathbb{X} \in \mathcal{D}_{\mathbb{X}}^{2\alpha}$ is continuous. In fact a stronger property holds: we have continuity of the map $(\mathbb{X}, \boldsymbol{Z}) \mapsto \int_0^\cdot \boldsymbol{Z}\,\mathrm{d}\mathbb{X}$. In order to prove this, we need to introduce the following space

$$\mathcal{S}_{\alpha,\eta} := \{(\mathbb{X}, \boldsymbol{Z}) \colon \mathbb{X} \text{ is a } \alpha\text{-rough path}, \boldsymbol{Z} \in \mathcal{D}_{\mathbb{X}}^{\alpha+\eta}\},$$

and the following quantity for $\boldsymbol{Z}\in\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}$ and $\bar{\boldsymbol{Z}}\in\mathcal{D}_{\bar{\mathbb{X}}}^{\alpha+\eta}$

$$[\boldsymbol{Z};\bar{\boldsymbol{Z}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\eta}:=\|\delta Z^1-\delta\bar{Z}^1\|_\eta+\|Z^{[2]}-\bar{Z}^{[2]}\|_{\alpha+\eta},$$

where $Z^{[2]}=\delta Z-Z^1\mathbb{X}^1$ and $\bar{Z}^{[2]}=\delta\bar{Z}-\bar{Z}^1\bar{\mathbb{X}}^1$, recall (10.7). We endow $\mathcal{S}_{\alpha,\eta}$ with the distance (see (8.24) for the definition of $d_{\mathcal{R}_{\alpha,d}}$)

$$\mathrm{d}_{\alpha,\eta}((\mathbb{X},\boldsymbol{Z}),(\bar{\mathbb{X}},\bar{\boldsymbol{Z}}))=d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}})+|Z_0-\bar{Z}_0|+|Z_0^1-\bar{Z}_0^1|+[\boldsymbol{Z};\bar{\boldsymbol{Z}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\eta}.$$

Let us note that in the case $\mathbb{X}=\bar{\mathbb{X}}$, we have

$$[\boldsymbol{Z};\bar{\boldsymbol{Z}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\eta}=[\boldsymbol{Z}-\bar{\boldsymbol{Z}}]_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}},\qquad \mathrm{d}_{\alpha,\eta}((\mathbb{X},\boldsymbol{Z}),(\mathbb{X},\bar{\boldsymbol{Z}}))=\|\boldsymbol{Z}-\bar{\boldsymbol{Z}}\|_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}},$$

see the definition (10.7) of the norm $\|\cdot\|_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}$. Note that $[\boldsymbol{Z};\bar{\boldsymbol{Z}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\eta}$ is *not* a function of $\boldsymbol{Z}-\bar{\boldsymbol{Z}}$ when $\mathbb{X}\neq\bar{\mathbb{X}}$.

Proposition 10.6. (Local Lipschitz estimate) *Let* $\alpha\in]\frac{1}{3},\frac{1}{2}]$ *and* $\eta\in]1-2\alpha,1]$. *The function* $\mathcal{S}_{\alpha,\eta}\ni(\mathbb{X},\boldsymbol{Z})\mapsto(\mathbb{X},\int_0^\cdot\boldsymbol{Z}\,\mathrm{d}\mathbb{X})\in\mathcal{S}_{\alpha,\alpha}$ *is continuous with respect to the distances* $\mathrm{d}_{\alpha,\eta}$ *and* $\mathrm{d}_{\alpha,\alpha}$.

More precisely, for every $M\geqslant 0$ *there is* $K_{M,\alpha,\eta}\geqslant 0$ *such that for all* $(\mathbb{X},\boldsymbol{Z}),(\bar{\mathbb{X}},\bar{\boldsymbol{Z}})\in\mathcal{S}_{\alpha,\eta}$ *satisfying*

$$1+T^\eta+\|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}}+\|\bar{\boldsymbol{Z}}\|_{\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}}\leqslant M,$$

setting $\boldsymbol{J}:=\int_0^\cdot\boldsymbol{Z}\,\mathrm{d}\mathbb{X}$ *and* $\bar{\boldsymbol{J}}:=\int_0^\cdot\bar{\boldsymbol{Z}}\,\mathrm{d}\bar{\mathbb{X}}$ *we have*

$$\begin{aligned}&\mathrm{d}_{\alpha,\alpha}((\mathbb{X},\boldsymbol{J}),(\bar{\mathbb{X}},\bar{\boldsymbol{J}}))\leqslant\\&\leqslant 2M^2(1+K_{2\alpha+\eta})[d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}})+|Z_0-\bar{Z}_0|+|Z_0^1-\bar{Z}_0^1|+T^\eta[\boldsymbol{Z};\bar{\boldsymbol{Z}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\eta}]\\&\leqslant 2M^3(1+K_{2\alpha+\eta})\,\mathrm{d}_{\alpha,\eta}((\mathbb{X},\boldsymbol{Z}),(\bar{\mathbb{X}},\bar{\boldsymbol{Z}})).\end{aligned}$$

Proof. Let $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ and $\bar{\mathbb{X}}=(\bar{\mathbb{X}}^1,\bar{\mathbb{X}}^2)$ be α-rough paths and $\boldsymbol{Z}\in\mathcal{D}_{\mathbb{X}}^{\alpha+\eta}$, $\bar{\boldsymbol{Z}}\in\mathcal{D}_{\bar{\mathbb{X}}}^{\alpha+\eta}$. We argue as in the proof of (10.9), using furthermore a number of times the simple estimate

$$|a\,b-\bar{a}\,\bar{b}|\leqslant|a-\bar{a}|\,|b|+|\bar{a}|\,|b-\bar{b}|. \tag{10.11}$$

We set for notational convenience $\varepsilon:=T^\eta$. Then, since $\delta Z_{st}=Z_s^1\,\mathbb{X}_{st}^1+Z_{st}^{[2]}$, by (1.39)

$$\begin{aligned}\|\delta Z-\delta\bar{Z}\|_\alpha&\leqslant\|Z^1-\bar{Z}^1\|_\infty\|\mathbb{X}^1\|_\alpha+\|\bar{Z}^1\|_\infty\|\mathbb{X}^1-\bar{\mathbb{X}}^1\|_\alpha+\varepsilon\|Z^{[2]}-\bar{Z}^{[2]}\|_{\alpha+\eta}\\&\leqslant(\|\mathbb{X}^1\|_\alpha+1)(|Z_0^1-\bar{Z}_0^1|+\varepsilon[\boldsymbol{Z};\bar{\boldsymbol{Z}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\eta})+M^2\|\mathbb{X}^1-\bar{\mathbb{X}}^1\|_\alpha,\end{aligned}$$

since by assumption

$$\|\bar{Z}^1\|_\infty\leqslant|\bar{Z}_0^1|+\varepsilon\|\delta\bar{Z}^1\|_\eta\leqslant(1+\varepsilon)(|\bar{Z}_0^1|+\|\delta\bar{Z}^1\|_\eta)\leqslant M^2.$$

Now $J_{st}^{[2]}=Z_s^1\,\mathbb{X}_{st}^2+J_{st}^{[3]}$, so that arguing similarly

$$\begin{aligned}&\|J^{[2]}-\bar{J}^{[2]}\|_{2\alpha}\leqslant\|J^{[3]}-\bar{J}^{[3]}\|_{2\alpha}+\|Z^1\,\mathbb{X}^2-\bar{Z}^1\,\bar{\mathbb{X}}^2\|_{2\alpha}\leqslant\\&\leqslant\varepsilon\|J^{[3]}-\bar{J}^{[3]}\|_{2\alpha+\eta}+\|\mathbb{X}^2\|_{2\alpha}(|Z_0^1-\bar{Z}_0^1|+\varepsilon\|\delta Z^1-\delta\bar{Z}^1\|_\eta)+M^2\|\mathbb{X}^2-\bar{\mathbb{X}}^2\|_{2\alpha}.\end{aligned}$$

Therefore, since $1+\|\mathbb{X}^1\|_\alpha+\|\mathbb{X}^2\|_{2\alpha}=1+\|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}}\leqslant M$,

$$\begin{aligned}&\|\delta Z-\delta\bar{Z}\|_\alpha+\|J^{[2]}-\bar{J}^{[2]}\|_{2\alpha}\leqslant\\&\leqslant\varepsilon\|J^{[3]}-\bar{J}^{[3]}\|_{2\alpha+\eta}+M^2(|Z_0^1-\bar{Z}_0^1|+\varepsilon[\boldsymbol{Z};\bar{\boldsymbol{Z}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\eta}+d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}})).\end{aligned}$$

We can estimate in the same way

$$\begin{aligned}\|\delta A-\delta\bar{A}\|_{2\alpha+\eta}&\leqslant\|Z^{[2]}-\bar{Z}^{[2]}\|_{\alpha+\eta}\|\mathbb{X}^1\|_\alpha+\|\bar{Z}^{[2]}\|_{\alpha+\eta}\|\mathbb{X}^1-\bar{\mathbb{X}}^1\|_\alpha+\\&\quad+\|\delta Z^1-\delta\bar{Z}^1\|_\eta\|\mathbb{X}^2\|_{2\alpha}+\|\delta\bar{Z}^1\|_\eta\|\mathbb{X}^2-\bar{\mathbb{X}}^2\|_{2\alpha}\\&\leqslant[\boldsymbol{Z};\bar{\boldsymbol{Z}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\eta}\|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}}+[\bar{\boldsymbol{Z}}]_{\mathcal{D}_{\bar{\mathbb{X}}}^{\alpha+\eta}}d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}})\\&\leqslant M([\boldsymbol{Z};\bar{\boldsymbol{Z}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\eta}+d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}})).\end{aligned}$$

By the Sewing bound (1.41)

$$\|J^{[3]}-\bar{J}^{[3]}\|_{2\alpha+\eta}\leqslant K_{2\alpha+\eta}M([\boldsymbol{Z};\bar{\boldsymbol{Z}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\eta}+d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}})).$$

We obtain

$$\begin{aligned}&[\boldsymbol{J};\bar{\boldsymbol{J}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\alpha}=\|\delta Z-\delta\bar{Z}\|_\alpha+\|J^{[2]}-\bar{J}^{[2]}\|_{2\alpha}\leqslant\\&\qquad\leqslant M^2(1+K_{2\alpha+\eta})[|Z_0^1-\bar{Z}_0^1|+d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}})+\varepsilon[\boldsymbol{Z};\bar{\boldsymbol{Z}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\eta}].\end{aligned}$$

Since $J_0-\bar{J}_0=0$, $J_0^1-\bar{J}_0^1=Z_0-\bar{Z}_0$, we obtain

$$\begin{aligned}&\mathrm{d}_{\alpha,\alpha}((\mathbb{X},\boldsymbol{J}),(\bar{\mathbb{X}},\bar{\boldsymbol{J}}))=d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}})+|Z_0-\bar{Z}_0|+[\boldsymbol{J};\bar{\boldsymbol{J}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\alpha}\\&\leqslant 2M^2(1+K_{2\alpha+\eta})[|Z_0-\bar{Z}_0|+|Z_0^1-\bar{Z}_0^1|+d_{\mathcal{R}_{\alpha,d}}(\mathbb{X},\bar{\mathbb{X}})+\varepsilon[\boldsymbol{Z};\bar{\boldsymbol{Z}}]_{\mathbb{X},\bar{\mathbb{X}},\alpha,\eta}].\end{aligned}$$

The second estimate follows since we have assumed that $1+\varepsilon\leqslant M$. □

10.4. Stochastic and rough integrals

In this section we explore the connections between Itô integrals and Young or rough integrals. We fix $\alpha\in\left]0,\frac{1}{2}\right[$ and a realisation of the Itô rough path $\mathbb{B}$ defined in (4.3) satisfying a.s. (4.4). We consider an adapted process $h:[0,T]\to(\mathbb{R}^d)^*$ with continuous paths and its Itô integral

$$I_t:=\int_0^t h_s\,\mathrm{d}B_s,\qquad t\in[0,T].$$

Let us suppose also that a.s. h is of class $\mathcal{C}^\beta$ with $\beta\in]0,1[$. By (4.7) we have

$$|\delta I_{st}-h_s\mathbb{B}_{st}^1|=\left|\int_s^t\delta h_{sr}\,\mathrm{d}B_r\right|\lesssim(t-s)^{\alpha+\beta},\qquad\forall 0\leqslant s\leqslant t\leqslant T.$$

By Theorem 4.3, this means a.s. the Itô integral in (4.6) is a generalised integral of h in $\mathrm{d}B$ in the sense of Definition 8.1.

The situation is different according to the value of $\alpha+\beta$. If $\alpha+\beta>1$, then we can apply Theorem 7.1 and we obtain that I_t is equal to the integral $\int_0^t h_s\,\mathrm{d}B_s$ also in the Young sense. In this regime, we have uniqueness of generalised integrals in the sense of Definition 8.1. Moreover, by (6.13) we have a.s.

$$I_t = \lim_{|\mathcal{P}|\to 0} \sum_{i=0}^{\#\mathcal{P}-1} h_{t_i}(B_{t_{i+1}} - B_{t_i}),$$

where $\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_k = t\}$ is a partition of $[0,t]$.

If $\alpha+\beta \leqslant 1$, then I is indeed only one of the generalised integrals as in Definition 8.1: for any $f\colon [0,T]\to\mathbb{R}$ of class $\mathcal{C}^{\alpha+\beta}$, then $I+f$ is also such a generalised integral. In this setting, in order to characterise uniquely the Itô integral among all generalised integrals, one can use (4.9): if we assume that, almost surely,

$$|\delta h_{sr} - h_s^1\,\mathbb{B}_{sr}^1| \lesssim (r-s)^{\eta+\alpha},$$

for some adapted process $h^1 = (h_t^1)_{t\in[0,T]}$ of class $\mathcal{C}^\eta$ with $\eta\in]0,1]$, then a.s.

$$|\delta I_{st} - h_s\,\mathbb{B}_{st}^1 - h_s^1\,\mathbb{B}_{st}^2| \;=\; \left|\int_s^t (\delta h_{sr} - h_s^1\,\mathbb{B}_{sr}^1)\,\mathrm{d}B_r\right| \lesssim (t-s)^{2\alpha+\eta}.$$

By Proposition 10.3, if $2\alpha+\eta>1$ then (I,h) is the rough integral of (h,h^1) with respect to $\mathbb{B}$, namely

$$(I_t, h_t) = \int_0^t (h,h^1)\,\mathrm{d}\mathbb{B}, \qquad t\geqslant 0,$$

as in (10.6). Moreover, by (6.13) we have a.s.

$$I_t = \lim_{|\mathcal{P}|\to 0} \sum_{i=0}^{\#\mathcal{P}-1} [h_{t_i}\,\mathbb{B}_{t_i t_{i+1}}^1 + h_{t_i}^1\,\mathbb{B}_{t_i t_{i+1}}^2],$$

where $\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_k = t\}$.

10.5. Properties in the geometric case

We have seen in Proposition 7.7 that the Young integral satisfies the classical integration by parts formula. We consider now a weakly geometric rough path $\mathbb{X}$ and two paths $\boldsymbol{f} = (f, f^1)$, $\boldsymbol{g} = (g, g^1)$ which are 2α-controlled by $\mathbb{X}$. We set

$$F_t := F_0 + \int_0^t f_s\,\mathrm{d}\mathbb{X}_s, \qquad G_t := G_0 + \int_0^t g_s\,\mathrm{d}\mathbb{X}_s, \qquad t\geqslant 0.$$

We want to show that, under the assumption that $\mathbb{X}$ is weakly geometric, an analogous integration by parts formula holds, namely:

$$F_t G_t \;=\; F_0\,G_0 + \underbrace{\int_0^t F_s\,g_s\,\mathrm{d}\mathbb{X}_s + \int_0^t G_s\,f_s\,\mathrm{d}\mathbb{X}_s}_{I_t}.$$

We start by showing that $(F_s\, g_s, F_s\, g_s^1 + f_s g_s)_{s\in[0,T]}$ is 2α-controlled by $\mathbb{X}$:

$$\begin{aligned} F_t\, g_t - F_s\, g_s &= F_t\, \delta g_{st} + g_s\, \delta F_{st} \\ &= F_s\, \delta g_{st} + g_s\, \delta F_{st} + \delta F_{st}\, \delta g_{st} \\ &= (F_s\, g_s^1 + f_s\, g_s)\, \mathbb{X}_{st}^1 + O(|t-s|^{2\alpha}). \end{aligned}$$

The same holds of course for $(f_s\, G_s, G_s f_s^1 + f_s g_s)_{s\in[0,T]}$. Now we know that I_t is the integral uniquely associated with the germ

$$A_{st} = (F_s\, g_s + G_s\, f_s)\mathbb{X}_{st}^1 + (F_s\, g_s^1 + G_s f_s^1 + 2 f_s\, g_s)\mathbb{X}_{st}^2.$$

By the weakly geometric condition, we have $2\mathbb{X}_{st}^2 = (\mathbb{X}_{st}^1)^2$ and therefore we obtain

$$A_{st} = (F_s\, g_s + G_s\, f_s)\mathbb{X}_{st}^1 + (F_s\, g_s^1 + G_s f_s^1)\mathbb{X}_{st}^2 + f_s\, g_s (\mathbb{X}_{st}^1)^2.$$

Now we write

$$\begin{aligned} \delta(FG)_{st} &= \delta F_{st} G_t + F_s \delta G_{st} \\ &= G_s\, \delta F_{st} + F_s\, \delta G_{st} + \delta F_{st}\, \delta G_{st} \\ &= (F_s\, g_s + G_s\, f_s)\mathbb{X}_{st}^1 + (F_s\, g_s^1 + G_s f_s^1)\mathbb{X}_{st}^2 + \delta F_{st}\, \delta G_{st} + O(|t-s|^{3\alpha}). \end{aligned}$$

Now since $\mathbb{X}^2 \in C_2^{2\alpha}$

$$\begin{aligned} \delta F_{st}\, \delta G_{st} &= (f_s \mathbb{X}_{st}^1 + f_s^1 \mathbb{X}_{st}^2)(g_s \mathbb{X}_{st}^1 + g_s^1 \mathbb{X}_{st}^2) + O(|t-s|^{3\alpha}) \\ &= f_s\, g_s (\mathbb{X}_{st}^1)^2 + O(|t-s|^{3\alpha}). \end{aligned}$$

Then we obtain that

$$\delta(FG)_{st} = A_{st} + O(|t-s|^{3\alpha}).$$

Since $3\alpha > 1$, it follows that $F_t G_t - F_0 G_0 = I_t$ for all $t \geqslant 0$.

Example 10.7. It is well known that the Stratonovich stochastic integral satisfies the above integration by parts formula. This section extends this result to all (weakly) geometric rough paths.