Chapter 11

Rough integral equations

In this chapter we go back to the finite difference equation (3.19) in the rough setting $\alpha \in \left(\frac{1}{2}, \frac{1}{3}\right]$, and we discute its integral formulation that we already mentioned at the end of Section 7.2. Now that we have studied the rough integral in Chapter 10, we can indeed show that the equation

$$|Z_{st}^{[3]}| \lesssim |t-s|^{3\alpha}, \qquad Z_{st}^{[3]} = \delta Z_{st} - \sigma(Z_s)\,\mathbb{X}_{st}^1 - \sigma_2(Z_s)\,\mathbb{X}_{st}^2, \tag{11.1}$$

recall (3.18), can be interpreted in the context of controlled paths. Indeed, (11.1) suggests that, for any candidate solution Z, the pair $\boldsymbol{Z} = (Z, \sigma(Z))$ should be controlled by $\mathbb{X}$. At the same time, in order to apply Proposition 10.3 and interpret (11.1) as an integral equation, we are going to shows that $(\sigma(Z), \sigma_2(Z))$ is controlled by $\mathbb{X}$. This is guaranteed by the following

Lemma 11.1. *Let $\phi: \mathbb{R}^k \to \mathbb{R}^\ell$ be of class C^2 and $\boldsymbol{f} = (f, f^1) \in \mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)$. Set*

$$\phi(\boldsymbol{f}) := (\phi(f), \nabla\phi(f)\, f^1),$$

where $\phi(f): [0,T] \to \mathbb{R}^\ell$ is defined by $\phi(f)_t := \phi(f_t)$ and

$$\nabla\phi(f)\, f^1: [0,T] \to \mathbb{R}^\ell \otimes \mathbb{R}^d, \qquad (\nabla\phi(f)\, f^1)_t^{ab} = \sum_{j=1}^k \partial_j \phi^a(f_t) \cdot (f_t^1)^{jb}.$$

Then $\phi(\boldsymbol{f}) \in \mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^\ell)$.

Proof. Analogously to (3.22) we have for $\boldsymbol{f} = (f, f^1) \in \mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)$, setting $f_{st}^{[2]} := \delta f_{st} - f_s^1\, \mathbb{X}_{st}^1$ as in (10.1),

$$\begin{aligned} \phi(\boldsymbol{f})_{st}^{[2]} &:= \phi(f_t) - \phi(f_s) - \nabla\phi(f_s)\, f_s^1\, \mathbb{X}_{st}^1 \\ &= \nabla\phi(f_s)\, f_{st}^{[2]} + \int_0^1 [\nabla\phi(f_s + r\,\delta f_{st}) - \nabla\phi(f_s)]\, dr\, \delta f_{st} \\ &= \nabla\phi(f_s)\, f_{st}^{[2]} + \int_0^1 (1-u)\, \nabla^2\phi(f_s + u\,\delta f_{st})\, du\; \delta f_{st} \otimes \delta f_{st}. \end{aligned} \tag{11.2}$$

Then we can write using the estimate $|ab - \bar{a}\,\bar{b}| \leqslant |a-\bar{a}|\,|b| + |\bar{a}|\,|b-\bar{b}|$

$$\begin{aligned} |\nabla\phi(f_t)\, f_t^1 - \nabla\phi(f_s)\, f_s^1| &\leqslant c_{\phi,f}^{(1)}\, |f_t^1 - f_s^1| + c_{\phi,f}^{(2)}\, |f_t - f_s|\, \|f^1\|_\infty, \\ |\phi(\boldsymbol{f})_{st}^{[2]}| &\leqslant c_{\phi,f}^{(1)}\, |f_{st}^{[2]}| + c_{\phi,f}^{(2)}\, |\delta f_{st}|^2, \end{aligned} \tag{11.3}$$

where

$$c_{\phi,f}^{(1)} := \sup_{s\in[0,T]} |\nabla\phi(f_s)|, \qquad c_{\phi,f}^{(2)} := \sup_{s,t\in[0,T], u\in[0,1]} |\nabla^2\phi(f_s + u\,\delta f_{st})|. \tag{11.4}$$

Therefore $(\phi(f), \nabla\phi(f)\, f^1)$ is controlled by $\mathbb{X}$. □

This suggests that we can reinterpret the finite difference equation (11.1) as follows: we look for $Z: [0,T] \to \mathbb{R}^k$ such that $\boldsymbol{Z} = (Z, \sigma(Z))$ is controlled by $\mathbb{X}$ (namely it belongs to $\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)$) and

$$\boldsymbol{Z}_t = (Z_0, 0) + \int_0^t \sigma(\boldsymbol{Z})\, d\mathbb{X}, \qquad \forall\, t \in [0,T]. \tag{11.5}$$

By Lemma 11.1, $\sigma(\boldsymbol{Z}) = (\sigma(Z), \nabla\sigma(Z)\, Z^1)$, but here $Z^1 = \sigma(Z)$, so that

$$\sigma(\boldsymbol{Z}) = (\sigma(Z), \nabla\sigma(Z)\, \sigma(Z)) = (\sigma(Z), \sigma_2(Z)),$$

is controlled by $\mathbb{X}$, where we use the notation $\sigma_2: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^d)^*$

$$\sigma_2(y) := \nabla\sigma(y)\, \sigma(y), \qquad [\sigma_2(y)]^i_{j\ell} := \sum_{a=1}^k \partial_a \sigma^i_j(y)\, \sigma^a_\ell(y).$$

By Proposition 10.3, the integral equation in (11.5) is equivalent to

$$|Z^{[3]}_{st}| \lesssim |t-s|^{3\alpha}, \qquad Z^{[3]}_{st} = \delta Z_{st} - \sigma(Z_s)\, \mathbb{X}^1_{st} - \sigma_2(Z_s)\, \mathbb{X}^2_{st}. \tag{11.6}$$

Viceversa, if $Z \in \mathcal{C}^\alpha([0,T]; \mathbb{R}^k)$ is such that $Z^{[3]} \in C_2^{3\alpha}$, then setting $Z^1 := \sigma(Z)$ the path $\boldsymbol{Z} = (Z, Z^1)$ is controlled by $\mathbb{X}$ and satisfies (11.5). Therefore, the integral equation (11.5) is equivalent to the finite difference equation (11.6).

11.1. Localization argument

PROPOSITION 11.2. *If we can prove local existence for the rough differential equation (11.6) under the assumption that σ is of class C^3 and $\sigma, \nabla\sigma, \nabla^2\sigma, \nabla^3\sigma$ are bounded, then we can prove local existence for (11.6) assuming only that σ is of class C^3.*

Proof. Let σ be of class C^3. Note that σ and its derivatives are bounded on the closed unit ball $B := \{z \in \mathbb{R}^k: |z - Z_0| \leq 1\}$, which is a compact subset of $\mathbb{R}^k$. Then we can find a function $\hat{\sigma}$ of class C^3 which is bounded with all its derivatives up to the third on the whole $\mathbb{R}^k$ and coincides with σ on B. By local existence for $\hat{\sigma}$, there is a solution $\hat{Z}: [0,T] \to \mathbb{R}^k$ of the RDE (11.6) with σ replaced by $\hat{\sigma}$. Since Z is continuous with $Z_0 \in B$, we can find $T' > 0$ such that $Z_t \in B$ for all $t \in [0,T']$. Then $\sigma(Z_t) = \hat{\sigma}(Z_t)$ and $\sigma_2(Z_t) = \hat{\sigma}_2(Z_t)$ for all $t \in [0,T']$, so that Z is a solution of the original RDE (11.6) on the shorter time interval $[0,T']$. We have proved *local existence* assuming only that σ is of class C^3. □

11.2. Invariance

In this section we prepare the ground for a contraction argument to be proved in the next section. We start with an estimate of $[\sigma(\boldsymbol{f})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^\ell)}$ in terms of $[\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)}$, under the assumption that σ is of class C^2 with bounded first and second derivative. We fix $D > 0$ such that

$$D \geq \max\{\|\nabla\sigma\|_\infty, \|\nabla^2\sigma\|_\infty\}.$$

LEMMA 11.3. *Let* $\sigma\colon \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ *be of class* C^2 *with* $\|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty \leqslant D$, *for some* $D < +\infty$. *Then for some* $C > 0$ *and any* $\boldsymbol{f} = (f, f^1) \in \mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)$

$$[\sigma(\boldsymbol{f})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k\otimes\mathbb{R}^d)} \leqslant D([\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} + \|f^1\|_\infty \|\delta f\|_\alpha + \|\delta f\|_\alpha^2). \tag{11.7}$$

Proof. By (11.3) we have

$$\|\delta(\nabla\sigma(f)\, f^1)\|_\alpha \leqslant D(\|\delta f^1\|_\alpha + \|f^1\|_\infty \|\delta f\|_\alpha),$$

$$\|\sigma(\boldsymbol{f})^{[2]}\|_{2\alpha} \leqslant D(\|f^{[2]}\|_{2\alpha} + \|\delta f\|_\alpha^2).$$

Therefore, recalling (10.7),

$$\begin{aligned}[\sigma(\boldsymbol{f})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k\otimes\mathbb{R}^d)} &= \|\delta(\nabla\sigma(f)\, f^1)\|_\alpha + \|\sigma(\boldsymbol{f})^{[2]}\|_{2\alpha} \\ &\leqslant D([\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} + \|f^1\|_\infty \|\delta f\|_\alpha + \|\delta f\|_\alpha^2).\end{aligned}$$

where, in the last inequality, we apply (10.8). □

We define $\Gamma\colon \mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k) \to \mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)$

$$\Gamma(\boldsymbol{f}) := (Z_0, 0) + \int_0^{\cdot} \sigma(\boldsymbol{f})\, d\mathbb{X},$$

(we know that indeed Γ maps $\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)$ into $\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)$ by Lemma 11.1). In other words, $\Gamma(f, f^1)$ is equal to the only $(J, J^1) \in \mathcal{D}_{\mathbb{X}}^{2\alpha}$ such that

$$J_0 = Z_0, \quad J_s^1 = \sigma(f_s), \quad \delta J_{st} - \sigma(f_s)\, \mathbb{X}_{st}^1 - \nabla\sigma(f_s)\, f_s^1\, \mathbb{X}_{st}^2 \in C_2^{3\alpha}. \tag{11.8}$$

We want to construct solutions to (11.6) by a fixed point argument for T small enough. Let $M > 0$ and $\mathbb{X}$ such that $\|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha} \leqslant M$ and

$$\mathcal{B} := \{\boldsymbol{f} = (f, f^1) \in \mathcal{D}_{\mathbb{X}}^{2\alpha} \colon (f_0, f_0^1) = (Z_0, \sigma(Z_0)),\ [\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \leqslant 4C\}, \tag{11.9}$$

where

$$C := (1 + M) D \|\sigma\|_\infty. \tag{11.10}$$

LEMMA 11.4. *If* $T^\alpha \leqslant \varepsilon_0$ *given by*

$$\varepsilon_0 := \frac{1}{8(1 + K_{3\alpha})(1 + D)(1 + \|\sigma\|_\infty)(1 + M)^2}, \tag{11.11}$$

then $\Gamma(\mathcal{B}) \subseteq \mathcal{B}$. *Moreover, setting*

$$L := 2(1 + M)\|\sigma\|_\infty = \frac{2C}{D}, \tag{11.12}$$

for any $\boldsymbol{f} = (f, f^1) \in \mathcal{B}$ *we have*

$$\max\{\|\delta f\|_\alpha, \|f^1\|_\infty\} \leqslant L. \tag{11.13}$$

Proof. Let $\boldsymbol{f} \in \mathcal{B}$. Setting $\varepsilon := T^\alpha$, if $\varepsilon \leqslant \varepsilon_0$ then in particular

$$\varepsilon C \leqslant \frac{\|\sigma\|_\infty}{8(1 + K_{3\alpha})(1 + \|\sigma\|_\infty)(1 + M)} \leqslant \frac{\|\sigma\|_\infty}{8}.$$

We obtain

$$\|f^1\|_\infty \leqslant |\sigma(Z_0)| + \varepsilon \|\delta f^1\|_\alpha \leqslant \|\sigma\|_\infty + \varepsilon_0 [\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \leqslant 2\|\sigma\|_\infty \leqslant L,$$

since $\varepsilon_0 4C \leqslant \|\sigma\|_\infty$. Similarly

$$\begin{aligned} \|\delta f\|_\alpha &\leqslant \varepsilon \|f^{[2]}\|_{2\alpha} + \|f^1\|_\infty \|\mathbb{X}^1\|_\alpha \leqslant \varepsilon_0 4C + (\|\sigma\|_\infty + \varepsilon_0 4C) M \\ &= \varepsilon_0 4C(1+M) + \|\sigma\|_\infty M \leqslant 2(1+M)\|\sigma\|_\infty = L. \end{aligned}$$

Therefore (11.13) is proved.

We prove now that $\Gamma(\boldsymbol{f}) \in \mathcal{B}$. We recall that $\Gamma(\boldsymbol{f}) = (J, \sigma(f))$, where J is uniquely determined by (11.8). By (10.9)

$$[\Gamma(\boldsymbol{f})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \leqslant 2(1+M)(|\nabla\sigma(Z_0)\,\sigma(Z_0)| + \varepsilon(1+K_{3\alpha})[\sigma(\boldsymbol{f})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)}).$$

By (11.7) and (11.13) we obtain

$$[\Gamma(\boldsymbol{f})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \leqslant 2(1+M)(D\|\sigma\|_\infty + \varepsilon(1+K_{3\alpha}) D([\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} + 2L^2)).$$

Now $(1+M)D\|\sigma\|_\infty = C$, and

$$D([\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} + 2L^2) \leqslant D\left(4C + 2\frac{4C^2}{D^2}\right) \leqslant 8C\left(D + \frac{C}{D}\right).$$

Note that

$$D + \frac{C}{D} = D + (1+M)\|\sigma\|_\infty \leqslant (1+M)(1+D)(1+\|\sigma\|_\infty), \tag{11.14}$$

so that by (11.11)

$$[\Gamma(\boldsymbol{f})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \leqslant 2C + 2C = 4C.$$

Therefore, $\Gamma(\boldsymbol{f}) \in \mathcal{B}$. □

11.3. LOCAL LIPSCHITZ CONTINUITY

We suppose that σ is of class C^3, with $\|\sigma\|_\infty + \|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty + \|\nabla^3\sigma\|_\infty < +\infty$ and we fix $D > 0$ such that

$$D \geqslant \|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty + \|\nabla^3\sigma\|_\infty.$$

LEMMA 11.5. (LOCAL LIPSCHITZ ESTIMATE) *If $T^\alpha \in \,]0, \varepsilon_0]$ where ε_0 is as in (11.11), then for $\boldsymbol{f}, \bar{\boldsymbol{f}} \in \mathcal{B}$, with $\mathcal{B}$ defined in (11.9), we have the local Lipschitz estimate*

$$[\sigma(\boldsymbol{f}) - \sigma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k \otimes (\mathbb{R}^d)^*)} \leqslant (2 + D + \|\sigma\|_\infty)\,[\boldsymbol{f} - \bar{\boldsymbol{f}}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \tag{11.15}$$

Proof. By Lemma 11.4 we have for $\boldsymbol{f} = (f, f^1)$, $\bar{\boldsymbol{f}} = (\bar{f}, \bar{f}^1)$

$$\max\{\|\delta f\|_\alpha, \|\delta \bar{f}\|_\alpha, \|\bar{f}^1\|_\infty\} \leqslant L,$$

with L as in (11.12). Now, we want to estimate

$$[\sigma(\boldsymbol{f})-\sigma(\bar{\boldsymbol{f}})]_{\mathcal{D}^{2\alpha}_{\mathbb{X}}(\mathbb{R}^k\otimes(\mathbb{R}^d)^*)} = \underbrace{\|\delta(\nabla\sigma(f)\, f^1-\nabla\sigma(\bar{f})\, \bar{f}^1)\|_\alpha}_{A} + \underbrace{\|\sigma(\boldsymbol{f})^{[2]}-\sigma(\bar{\boldsymbol{f}})^{[2]}\|_{2\alpha}}_{B}.$$

We set $\Delta := f-\bar{f}$, $\Delta^1 := f^1-\bar{f}^1$, $\Delta^{[2]} := f^{[2]}-\bar{f}^{[2]}$. We first estimate A:

$$\begin{aligned}&|\delta(\nabla\sigma(f)\, f^1-\nabla\sigma(\bar{f})\, \bar{f}^1)_{st}| = \\ &= |\delta(\nabla\sigma(f))_{st}\, f^1_t + \nabla\sigma(f_s)\delta f^1_{st} - \delta(\nabla\sigma(\bar{f}))_{st}\, \bar{f}^1_t - \nabla\sigma(\bar{f}_s)\delta\bar{f}^1_{st}| \\ &\leqslant |\delta(\nabla\sigma(f)-\nabla\sigma(\bar{f}))_{st}\, f^1_t| + |\delta(\nabla\sigma(\bar{f}))_{st}\, (f^1_t-\bar{f}^1_t)| + \\ &\quad + |(\nabla\sigma(f_s)-\nabla\sigma(\bar{f}_s))\delta f^1_{st}| + |\nabla\sigma(\bar{f}_s)(\delta f-\delta\bar{f})_{st}|.\end{aligned}$$

By Lemma 2.8 and (1.39) we have for $\varepsilon = T^\alpha$

$$\begin{aligned}A &\leqslant D[\|f^1\|_\infty(\|\delta\Delta\|_\alpha + (\|\delta f\|_\alpha+\|\delta\bar{f}\|_\alpha)\|\Delta\|_\infty) + \|\delta\bar{f}\|_\alpha\|\Delta^1\|_\infty + \\ &\quad +\|\Delta\|_\infty\|\delta f^1\|_\alpha + \|\delta\Delta^1\|_\alpha] \\ &\leqslant D[\,(\,(\|\delta f\|_\alpha+\|\delta\bar{f}\|_\alpha)\|f^1\|_\infty+\|\delta f^1\|_\alpha)\|\Delta\|_\infty + \|f^1\|_\infty\|\delta\Delta\|_\alpha + \\ &\quad +(1+\varepsilon\|\delta\bar{f}\|_\alpha)\|\delta\Delta^1\|_\alpha] \\ &\leqslant D[(2L^2+\|\delta f^1\|_\alpha)\|\Delta\|_\infty + L\|\delta\Delta\|_\alpha + (1+\varepsilon L)\|\delta\Delta^1\|_\alpha]\end{aligned}$$

We show now that

$$\begin{aligned}B &\leqslant D\,((\|f^{[2]}\|_{2\alpha}+3\|\delta f\|^2_\alpha)\|\Delta\|_\infty + (\|\delta f\|_\alpha+\|\delta\bar{f}\|_\alpha)\|\delta\Delta\|_\alpha + \|\Delta^{[2]}\|_{2\alpha}) \\ &\leqslant D[(\|f^{[2]}\|_{2\alpha}+3L^2)\|\Delta\|_\infty + 2L\|\delta\Delta\|_\alpha + \|\Delta^{[2]}\|_{2\alpha}]. \qquad (11.16)\end{aligned}$$

We have by (11.2)

$$\begin{aligned}B &\leqslant \|\nabla\sigma(f)\, f^{[2]}-\nabla\sigma(\bar{f})\, \bar{f}^{[2]}\|_{2\alpha} + \\ &\quad + \int_0^1 \|\nabla^2\sigma\,(f+u\delta f)\,\delta f\otimes\delta f - \nabla^2\sigma\,(\bar{f}+u\delta\bar{f})\,\delta\bar{f}\otimes\delta\bar{f}\|_{2\alpha}\, \mathrm{d}u.\end{aligned}$$

With the usual estimate $|a\,b-\bar{a}\,\bar{b}| \leqslant |a-\bar{a}|\,|b| + |\bar{a}|\,|b-\bar{b}|$ we can write

$$\begin{aligned}&\|\nabla\sigma(f)\, f^{[2]}-\nabla\sigma(\bar{f})\bar{f}^{[2]}\|_{2\alpha} \leqslant \\ &\leqslant \|\nabla\sigma(f)-\nabla\sigma(\bar{f})\|_\infty\|f^{[2]}\|_{2\alpha} + \|\nabla\sigma(\bar{f})\|_\infty\|\Delta^{[2]}\|_{2\alpha} \\ &\leqslant \|\nabla^2\sigma\|_\infty\,\|\Delta\|_\infty\|f^{[2]}\|_{2\alpha} + \|\nabla\sigma\|_\infty\,\|\Delta^{[2]}\|_{2\alpha} \\ &\leqslant D(\|\Delta\|_\infty\|f^{[2]}\|_{2\alpha} + \|\Delta^{[2]}\|_{2\alpha}).\end{aligned}$$

For the other term

$$\begin{aligned}&\int_0^1 \|\nabla^2\sigma(f+u\delta f)\cdot\delta f\otimes\delta f - \nabla^2\sigma(\bar{f}+u\delta\bar{f})\cdot\delta\bar{f}\otimes\delta\bar{f}\|_{2\alpha}\, \mathrm{d}u \leqslant \\ &\leqslant \|\nabla^3\sigma\|_\infty\,\|\delta f\|^2_\alpha(\|\Delta\|_\infty + \|\delta\Delta\|_\infty) + \|\nabla^2\sigma\|_\infty(\|\delta f\|_\alpha+\|\delta\bar{f}\|_\alpha)\|\delta\Delta\|_\alpha \\ &\leqslant D(\|\delta f\|^2_\alpha(\|\Delta\|_\infty + \|\delta\Delta\|_\infty) + (\|\delta f\|_\alpha+\|\delta\bar{f}\|_\alpha)\|\delta\Delta\|_\alpha).\end{aligned}$$

Recalling that $\|\delta\Delta\|_\infty \leqslant 2\|\Delta\|_\infty$, we have finished the proof of (11.16).

Since $\Delta_0 = f_0 - \bar{f}_0 = 0$, we have $\|\Delta\|_\infty \leqslant \varepsilon \|\delta\Delta\|_\alpha$. Summing up, we obtain

$$\begin{aligned}&[\sigma(\boldsymbol{f}) - \sigma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k \otimes (\mathbb{R}^d)^*)} = A + B\\ &\leqslant \{(3L + \varepsilon(5L^2 + [\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)}))\|\delta\Delta\|_\alpha + (1 + \varepsilon L)\,[\boldsymbol{f} - \bar{\boldsymbol{f}}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)}\}.\end{aligned}$$

On the other hand

$$\begin{aligned}\|\delta\Delta\|_\alpha &\leqslant \varepsilon\|\Delta^{[2]}\|_{2\alpha} + \|\Delta^1\|_\infty \|\mathbb{X}^1\|_\alpha\\ &\leqslant \varepsilon\|\Delta^{[2]}\|_{2\alpha} + \varepsilon M\|\delta\Delta^1\|_\alpha\\ &\leqslant \varepsilon(1 + M)\,[\boldsymbol{f} - \bar{\boldsymbol{f}}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)}.\end{aligned}$$

Therefore

$$[\sigma(\boldsymbol{f}) - \sigma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k \otimes (\mathbb{R}^d)^*)} \leqslant (\varepsilon(1 + M)c_1 + c_2)\,[\boldsymbol{f} - \bar{\boldsymbol{f}}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)},$$

where we set

$$c_1 := D\,(3L + \varepsilon([\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} + 5L^2)), \qquad c_2 := D(1 + \varepsilon L).$$

Since $[\boldsymbol{f}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \leqslant 4C$ we obtain, recalling that $DL = 2C$ by (11.12),

$$\begin{aligned}c_1 &\leqslant D\,(3L + \varepsilon(4C + 5L^2)) \leqslant 6C + 20\varepsilon C\left(D + \frac{C}{D}\right)\\ &\leqslant 6C + 20\varepsilon C(1 + D)(1 + \|\sigma\|_\infty)(1 + M)\\ &\leqslant 6C + 3C = 9C,\end{aligned}$$

where we have used first (11.14) and then (11.10)-(11.11). Similarly

$$\varepsilon(1 + M)c_1 \leqslant 9\varepsilon C(1 + M) = 9\varepsilon D\|\sigma\|_\infty(1 + M)^2 \leqslant 2,$$

and

$$c_2 = D + \varepsilon DL = D + 2\varepsilon C \leqslant D + \|\sigma\|_\infty.$$

Therefore

$$\varepsilon(1 + M)c_1 + c_2 \leqslant 2 + D + \|\sigma\|_\infty.$$

The proof is finished. □

11.4. Contraction

In this section we prove local existence by means of a fixed point argument, assuming σ to be of class C^3 and bounded with its first, second and third derivatives, namely $\|\sigma\|_\infty + \|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty + \|\nabla^3\sigma\|_\infty < +\infty$. Therefore the assumptions are stronger than for the discrete approximation of Section 3.9. However this method has the advantage of not requiring compactness of the image of Γ and therefore this approach works also for rough equations with values in infinite-dimensional spaces.

Let us fix $D > 0$ such that

$$D \geqslant \max\{\|\nabla\sigma\|_\infty, \|\nabla^2\sigma\|_\infty, \|\nabla^3\sigma\|_\infty\}.$$

Recalling that $\mathcal{B}$ was defined in (11.9), we can now show the following

LEMMA 11.6. *If* $T^\alpha \in\,]0, \varepsilon_0]$ *where* ε_0 *is as in (11.11), then* $\Gamma : \mathcal{B} \to \mathcal{B}$ *is a contraction for* $\|\cdot\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}}$.

Proof. Let $\boldsymbol{f} = (f, f^1)$ and $\bar{\boldsymbol{f}} = (\bar{f}, \bar{f}^1)$ be in $\mathcal{B}$. Since $f_0 = \bar{f}_0$ and $f_0^1 = \bar{f}_0^1$, by the definitions, see in particular (10.7),

$$\|\Gamma(\boldsymbol{f}) - \Gamma(\bar{\boldsymbol{f}})\|_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \;=\; [\Gamma(\boldsymbol{f}) - \Gamma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)}.$$

We set $\varepsilon := T^\alpha$. By (10.9)

$$[\Gamma(\boldsymbol{f}) - \Gamma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \;\leqslant\; \varepsilon 2(1+M)(1+K_{3\alpha})\,[\sigma(\boldsymbol{f}) - \sigma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)}.$$

Now by Lemma 11.5

$$[\sigma(\boldsymbol{f}) - \sigma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k \otimes (\mathbb{R}^d)^*)} \;\leqslant\; (2 + D + \|\sigma\|_\infty)\,[\boldsymbol{f} - \bar{\boldsymbol{f}}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)}.$$

Now $2 + D + \|\sigma\|_\infty \leqslant 2(1+D)(1+\|\sigma\|_\infty)$. Therefore

$$[\Gamma(\boldsymbol{f}) - \Gamma(\bar{\boldsymbol{f}})]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)} \;\leqslant c_4\,[\boldsymbol{f} - \bar{\boldsymbol{f}}]_{\mathcal{D}_{\mathbb{X}}^{2\alpha}(\mathbb{R}^k)},$$

with

$$c_4 = \varepsilon 2(1+M)(1+K_{3\alpha}) 2(1+D)(1+\|\sigma\|_\infty) \leqslant \frac{1}{2}$$

by (11.11). This concludes the proof. □