Chapter 2

Difference equations: the Young case

Fix a time horizon $T > 0$ and two dimensions $k, d \in \mathbb{N}$. We study the following *controlled difference equation* for an unknown path $Z: [0,T] \to \mathbb{R}^k$:

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + o(t-s)\,, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{2.1}$$

where the "driving path" $X: [0,T] \to \mathbb{R}^d$ and the function $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ are given, and $o(t-s)$ is *uniform* for $0 \leqslant s \leqslant t \leqslant T$ (see Remark 1.1).

The difference equation (2.1) is a natural generalized formulation of the *controlled differential equation*

$$\dot{Z}_t = \sigma(Z_t)\,\dot{X}_t\,, \qquad 0 \leqslant t \leqslant T. \tag{2.2}$$

Indeed, as we showed in Chapter 1 (see Section 1.2), equations (2.1) and (2.2) are *equivalent* when X is continuously differentiable and σ is continuous, but (2.1) is meaningful also when X is non differentiable.

In this chapter we prove *well-posedness for the difference equation (2.1)* when the driving path $X \in \mathcal{C}^\alpha$ is Hölder continuous in the regime $\alpha \in \left]\frac{1}{2}, 1\right]$, called the *Young case*. The more challenging regime $\alpha \leqslant \frac{1}{2}$, called the *rough case*, is the object of the next Chapter 3, where new ideas will be introduced.

2.1. Summary

Using the increment notation $\delta f_{st} := f_t - f_s$ from (1.11), we rewrite (2.1) as

$$\delta Z_{st} = \sigma(Z_s)\,\delta X_{st} + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{2.3}$$

so that a solution of (2.3) is any path $Z: [0,T] \to \mathbb{R}^k$ such that the *"remainder"*

$$Z_{st}^{[2]} := \delta Z_{st} - \sigma(Z_s)\,\delta X_{st} \qquad \text{satisfies} \qquad Z_{st}^{[2]} = o(t-s)\,. \tag{2.4}$$

We summarize the main results of this chapter stating *local and global existence, uniqueness of solutions and continuity of the solution map* for the difference equation (2.3) under natural assumptions on σ. We will actually prove more precise results, which yield quantitative estimates.

Theorem 2.1. (Well-posedness) *Let $X: [0,T] \to \mathbb{R}^d$ be of class $\mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2}, 1\right]$ and let $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$. Then we have:*

- ***local existence**: if σ is locally γ-Hölder with $\gamma \in \left(\frac{1}{\alpha} - 1, 1\right]$ (e.g. of class C^1), then for every $z_0 \in \mathbb{R}^k$ there is a possibly shorter time horizon $T' = T'_{\alpha,X,\sigma}(z_0) \in \left]0, T\right]$ and a path $Z: [0,T'] \to \mathbb{R}^k$ starting from $Z_0 = z_0$ which solves (2.3) for $0 \leqslant s \leqslant t \leqslant T'$;*

- ***global existence***: *if σ is globally γ-Hölder with $\gamma \in \left(\frac{1}{\alpha}-1, 1\right]$ (e.g. of class C^1 with $\|\nabla\sigma\|_\infty < \infty$), then we can take $T'_{\alpha,X,\sigma}(z_0) = T$ for any $z_0 \in \mathbb{R}^d$;*
- ***uniqueness***: *if σ is of class $\mathcal{C}^\gamma$ with $\gamma \in \left(\frac{1}{\alpha}, 2\right]$ (e.g. if σ is of class C^2), then there is exactly one solution Z of (2.3) with $Z_0 = z_0$;*
- ***continuity of the solution map***: *if σ is differentiable with bounded and globally $(\gamma-1)$-Hölder gradient with $\gamma \in \left(\frac{1}{\alpha}, 2\right]$ (i.e. $\|\nabla\sigma\|_\infty < \infty$, $[\nabla\sigma]_{\mathcal{C}^{\gamma-1}} < \infty$), then the solution Z of (2.3) is a continuous function of the starting point z_0 and driving path X: the map $(z_0, X) \mapsto Z$ is continuous from $\mathbb{R}^k \times \mathcal{C}^\alpha \to \mathcal{C}^\alpha$.*

In the first part of this chapter, we give for granted the existence of solutions and we focus on their properties: we prove *a priori estimates* in Section 2.3, *uniqueness of solutions* in Section 2.4 and *continuity of the solution map* in Section 2.5. A key role is played by the Sewing Bound from Chapter 1, see Theorems 1.9 and 1.17, and its discrete version, see Theorem 1.18.

The proof of local and global *existence of solutions of (2.3)* is given at the end of this chapter, see Section 2.6, exploiting a suitable Euler scheme.

2.2. Set-up

We collect here some notions and tools that will be used extensively.

We recall that C_1 denotes the space of continuous functions $f\colon [0,T] \to \mathbb{R}^k$. Similarly, C_2 and C_3 are the spaces of continuous functions of two and three ordered variables, i.e. defined on $[0,T]^2_\leqslant$ and $[0,T]^3_\leqslant$, see (1.7)-(1.8).

We are going to exploit the *weighted semi-norms* $\|\cdot\|_{\eta,\tau}$, see (1.33)-(1.34) (see also (1.9) for the original norm $\|\cdot\|_\eta$). These are useful to bound the *weighted supremum norm* $\|f\|_{\infty,\tau}$ of a function $f \in C_1$, see (1.37) and (1.40):

$$\|f\|_{\infty,\tau} \leqslant |f_0| + 3\,(\tau \wedge T)^\eta\, \|\delta f\|_{\eta,\tau}, \qquad \forall \eta, \tau > 0. \tag{2.5}$$

It follows directly from the definitions (1.33)-(1.34) that

$$\|\cdot\|_{\eta,\tau} \leqslant (\tau \wedge T)^{\eta'}\, \|\cdot\|_{\eta+\eta',\tau}\,, \qquad \forall \eta, \eta' > 0, \tag{2.6}$$

because $(t-s)^\eta \geqslant (t-s)^{\eta+\eta'}\,(\tau \wedge T)^{-\eta'}$ for $0 \leqslant s \leqslant t \leqslant T$ with $t - s \leqslant \tau$.

Remark 2.2. The factor $(\tau \wedge T)^{\eta'}$ in the RHS of (2.6) can be made small *by choosing τ small while keeping T fixed*. This is why we included the indicator function $\mathbb{1}_{\{0<t-s\leqslant\tau\}}$ in the definition (1.33)-(1.34) of the norms $\|\cdot\|_{\eta,\tau}$: without this indicator function, instead of $(\tau\wedge T)^{\eta'}$ we would have $T^{\eta'}$, which is small only when T is small.

We will often work with functions $F \in C_2$ or $F \in C_3$ that are *product of two factors*, like $F_{st} = g_s\, H_{st}$ or $F_{sut} = G_{su}\, H_{ut}$. We show in the next result that the semi-norm $\|F\|_{\eta,\tau}$ can be controlled by a product of suitable norms for each factor.

LEMMA 2.3. (WEIGHTED BOUNDS) *For any* $\eta, \eta' \in (0,\infty)$ *and* $\tau > 0$*, we have*

$$\text{if} \quad F_{st} = g_s H_{st} \quad \text{or} \quad F_{st} = g_t H_{st} \qquad \text{then} \qquad \|F\|_{\eta,\tau} \leqslant \|g\|_{\infty,\tau} \|H\|_{\eta}, \tag{2.7}$$

$$\text{if} \quad F_{sut} = G_{su} H_{ut} \qquad \text{then} \qquad \|F\|_{\eta+\eta',\tau} \leqslant \|G\|_{\eta,\tau} \|H\|_{\eta'}. \tag{2.8}$$

Proof. If $F_{st} = g_t H_{st}$, by (1.37) we can estimate $e^{-t/\tau} |g_t| \leqslant \|g\|_{\infty,\tau}$ to get (2.7). If $F_{st} = g_s H_{st}$, for $s \leqslant t$ we can bound $e^{-t/\tau} \leqslant e^{-s/\tau}$ in the definition (1.33)-(1.34) of $\|\cdot\|_{\eta,\tau}$, hence again by (1.37) we can estimate $e^{-s/\tau} |g_s| \leqslant \|g\|_{\infty,\tau}$ to get (2.7).

If $F_{sut} = G_{su} H_{ut}$, we can further bound $(t-s)^{\eta+\eta'} \geqslant (t-u)^{\eta} (u-s)^{\eta'}$ in (1.34) and then estimate $e^{-s/\tau} G_{su} / (u-s)^{\eta} \leqslant \|G\|_{\eta,\tau}$, which yields (2.8). □

We stress that in the RHS of (2.7) and (2.8) *only one factor gets the weighted norm or semi-norm*, while the other factor gets the non-weigthed norm $\|\cdot\|_{\eta}$. We will sometimes need an extra weight, which can be introduced as follows.

LEMMA 2.4. (EXTRA WEIGHT) *For any* $\eta, \bar{\tau} \in (0,\infty)$ *and* $0 < \tau \leqslant \bar{\tau}$*, we have*

$$\text{if} \quad F_{st} = g_s H_{st} \quad \text{or} \quad F_{st} = g_t H_{st} \qquad \text{then} \qquad \|F\|_{\eta,\tau} \leqslant \|g\|_{\infty,\tau} e^{\frac{T}{\bar{\tau}}} \|H\|_{\eta,\bar{\tau}}. \tag{2.9}$$

Proof. Recall the definition (1.33)-(1.34) of $\|\cdot\|_{\eta,\tau}$ and note that for $0 \leqslant s \leqslant t \leqslant T$ we have $e^{-t/\tau} |g_t| \leqslant \|g\|_{\infty,\tau}$ and $e^{-s/\tau} |g_s| \leqslant \|g\|_{\infty,\tau}$ (see the proof of Lemma 2.3). Finally, for $t - s \leqslant \tau \leqslant \bar{\tau}$ we can estimate $|H_{st}| \leqslant e^{T/\bar{\tau}} e^{-t/\bar{\tau}} |H_{st}| \leqslant e^{T/\bar{\tau}} \|H\|_{\eta,\bar{\tau}} (t-s)^{\eta}$. □

We recall that $\mathbb{R}^k \otimes (\mathbb{R}^d)^* \simeq \mathbb{R}^{k \times d}$ is the space of linear applications from $\mathbb{R}^d$ to $\mathbb{R}^k$ equipped with the Hilbert-Schmidt (Euclidean) norm $|\cdot|$. We say that a function is of class C^m if it is continuously differentiable m times. Given $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ of class C^2, that we represent by $\sigma^i_j(z)$ with $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, d\}$, we denote by $\nabla\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^k)^*$ its gradient and by $\nabla^2\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^k)^* \otimes (\mathbb{R}^k)^*$ its Hessian, represented for $i, a, b \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, d\}$ by

$$(\nabla\sigma(z))^i_{ja} = \frac{\partial \sigma^i_j}{\partial z_a}(z), \qquad (\nabla^2\sigma(z))^i_{jab} = \frac{\partial^2 \sigma^i_j}{\partial z_a \partial z_b}(z).$$

Remark 2.5. (NORM OF THE GRADIENT OF LIPSCHITZ FUNCTIONS) For a *locally Lipschitz function* $\psi: \mathbb{R}^k \to \mathbb{R}^\ell$ we can define the "norm of the gradient" at any point (even where ψ may not be differentiable):

$$|\nabla\psi(z)| := \limsup_{y \to z} \frac{|\psi(y) - \psi(z)|}{|y - z|} \in [0, \infty).$$

Similarly, $|\nabla^2\psi(z)|$ is well defined as soon as ψ is *differentiable with locally Lipschitz gradient* $\nabla\psi$ (which is slightly less than requiring $\psi \in C^2$).

2.3. A PRIORI ESTIMATES

In this section we prove *a priori estimates* for solutions of (2.3) assuming that σ is *globally Lipschitz*, that is $\|\nabla\sigma\|_\infty < \infty$ (recall Remark 2.5).

We first observe that if the driving path X is of class $\mathcal{C}^\alpha$, then any solution Z of (2.3) is also of class $\mathcal{C}^\alpha$, as soon as σ is continuous.

LEMMA 2.6. (HÖLDER REGULARITY) *Let X be of class $\mathcal{C}^\alpha$ with $\alpha \in]0,1]$ and let σ be continuous. Then any solution Z of (2.3) is of class $\mathcal{C}^\alpha$.*

Proof. We know by Lemma 1.2 that Z is continuous, more precisely by (1.6) we have $|\delta Z_{st}| \leqslant C\,|\delta X_{st}| + o(t-s)$ with $C < \infty$. Since $|\delta X_{st}| \leqslant \|\delta X\|_\alpha\,(t-s)^\alpha$ and $o(t-s) = o((t-s)^\alpha)$ for any $\alpha \leqslant 1$, it follows that $Z \in \mathcal{C}^\alpha$. □

We next formulate the announced a priori estimates. It is convenient to use the weighted semi-norms $\|\cdot\|_{\eta,\tau}$ in (1.33)-(1.34) (note that the usual norms $\|\cdot\|_\eta$ in (1.9) can be recovered by letting $\tau \to \infty$).

THEOREM 2.7. (A PRIORI ESTIMATES) *Let X be of class $\mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2},1\right]$ and let σ be globally γ-Hölder with $\gamma \in \left(\frac{1}{\alpha}-1,1\right]$. Then, for any solution $Z\colon [0,T] \to \mathbb{R}^k$ of (2.3), the remainder $Z^{[2]}_{st} := \delta Z_{st} - \sigma(Z_s)\,\delta X_{st}$ satisfies $Z^{[2]} \in C_2^{(\gamma+1)\alpha}$, more precisely for any $\tau > 0$*

$$\|Z^{[2]}\|_{(\gamma+1)\alpha,\tau} \leqslant C_{\alpha,\gamma,X,\sigma}\,\|\delta Z\|^\gamma_{\alpha,\tau} \qquad \text{with } C_{\alpha,\gamma,X,\sigma} := K_{(\gamma+1)\alpha}\,\|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}\,, \tag{2.10}$$

where $K_\eta = (1-2^{1-\eta})^{-1}$. Moreover, if either T or τ is small enough, we have

$$\|\delta Z\|_{\alpha,\tau} \leqslant 1 \vee (2\,\|\delta X\|_\alpha\,|\sigma(Z_0)|) \qquad \text{for } (\tau \wedge T)^{\alpha\gamma} \leqslant \varepsilon_{\alpha,\gamma,X,\sigma}, \tag{2.11}$$

where we define

$$\varepsilon_{\alpha,\gamma,X,\sigma} := \frac{1}{2\,(K_{(\gamma+1)\alpha}+3)\,\|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}}\,. \tag{2.12}$$

If σ is globally Lipschitz, namely if we can take $\gamma = 1$, we can improve (2.11) to

$$\|\delta Z\|_{\alpha,\tau} \leqslant 2\,\|\delta X\|_\alpha\,|\sigma(Z_0)| \qquad \text{for } (\tau \wedge T)^\alpha \leqslant \varepsilon_{\alpha,1,X,\sigma}\,. \tag{2.13}$$

Proof. We first prove (2.10). Since $Z^{[2]}_{st} = o(t-s)$ by definition of solution, see (2.4), *we can estimate $Z^{[2]}$ in terms of $\delta Z^{[2]}$*, by the weighted Sewing Bound (1.41). Let us compute $\delta Z^{[2]}_{sut} = Z^{[2]}_{st} - Z^{[2]}_{su} - Z^{[2]}_{ut}$: recalling (2.4) and (1.32), since $\delta \circ \delta = 0$, we have

$$\delta Z^{[2]}_{sut} = \delta\sigma(Z)_{su}\,\delta X_{ut} = (\sigma(Z_u) - \sigma(Z_s))\,(X_t - X_u)\,. \tag{2.14}$$

Since $|\sigma(z) - \sigma(\bar{z})| \leqslant [\sigma]_{\mathcal{C}^\gamma}\,|z - \bar{z}|^\gamma$ for all $z, \bar{z} \in \mathbb{R}^d$, we can bound

$$\|\delta\sigma(Z)\|_{\gamma\alpha,\tau} \leqslant [\sigma]_{\mathcal{C}^\gamma}\,\|\delta Z\|^\gamma_{\alpha,\tau}, \tag{2.15}$$

hence by (2.8) we obtain

$$\|\delta Z^{[2]}\|_{(\gamma+1)\alpha,\tau} \leqslant \|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}\,\|\delta Z\|^\gamma_{\alpha,\tau}.$$

Applying the weighted Sewing Bound (1.41), for $(\gamma+1)\alpha > 1$ we then obtain

$$\|Z^{[2]}\|_{(\gamma+1)\alpha,\tau} \leqslant K_{(\gamma+1)\alpha}\,\|\delta X\|_\alpha\,[\sigma]_{\mathcal{C}^\gamma}\,\|\delta Z\|^\gamma_{\alpha,\tau}\,, \tag{2.16}$$

which proves (2.10).

We next prove (2.11). To simplify notation, let us set $\varepsilon := (\tau \wedge T)^{\alpha}$. Recalling (2.7) and (2.6), we obtain by (2.4)

$$\begin{aligned} \|\delta Z\|_{\alpha,\tau} &\leqslant \|\sigma(Z)\,\delta X\|_{\alpha,\tau} + \|Z^{[2]}\|_{\alpha,\tau} \\ &\leqslant \|\sigma(Z)\|_{\infty,\tau}\,\|\delta X\|_{\alpha} + \varepsilon^{\gamma}\,\|Z^{[2]}\|_{(\gamma+1)\alpha,\tau}\,. \end{aligned} \tag{2.17}$$

We can estimate $\|\sigma(Z)\|_{\infty,\tau}$ by (2.5) and (2.15):

$$\|\sigma(Z)\|_{\infty,\tau} \leqslant |\sigma(Z_0)| + 3\,\varepsilon^{\gamma}\,[\sigma]_{\mathcal{C}^{\gamma}}\,\|\delta Z\|_{\alpha,\tau}^{\gamma}\,.$$

Plugging this and (2.16) into (2.17), we get

$$\begin{aligned} \|\delta Z\|_{\alpha,\tau} &\leqslant (|\sigma(Z_0)| + 3\,\varepsilon^{\gamma}\,[\sigma]_{\mathcal{C}^{\gamma}}\,\|\delta Z\|_{\alpha,\tau}^{\gamma})\,\|\delta X\|_{\alpha} + \\ &\qquad + \varepsilon^{\gamma}\,K_{(\gamma+1)\alpha}\,\|\delta X\|_{\alpha}\,[\sigma]_{\mathcal{C}^{\gamma}}\,\|\delta Z\|_{\alpha,\tau}^{\gamma} \\ &= \|\delta X\|_{\alpha}\,|\sigma(Z_0)| + \frac{1}{2}\,\frac{\varepsilon^{\gamma}}{\varepsilon_{\alpha,\gamma,X,\sigma}}\,\|\delta Z\|_{\alpha,\tau}^{\gamma}\,, \end{aligned}$$

where $\varepsilon_{\alpha,\gamma,X,\sigma}$ is defined in (2.12). For $\varepsilon^{\gamma} \leqslant \varepsilon_{\alpha,\gamma,X,\sigma}$ the last term is bounded by $\frac{1}{2}\|\delta Z\|_{\alpha,\tau}^{\gamma}$ *which is finite* by Lemma 2.6. If $\|\delta Z\|_{\alpha,\tau} \leqslant 1$ then (2.11) holds trivially; if not, $\frac{1}{2}\|\delta Z\|_{\alpha,\tau}^{\gamma} \leqslant \frac{1}{2}\|\delta Z\|_{\alpha,\tau}$. Bringing this term in the LHS we obtain (2.11).

To prove (2.13), we argue as for (2.11) and since $\gamma = 1$ we obtain

$$\|\delta Z\|_{\alpha,\tau} \leqslant \|\delta X\|_{\alpha}\,|\sigma(Z_0)| + \frac{1}{2}\,\frac{\varepsilon}{\varepsilon_{\alpha,1,X,\sigma}}\,\|\delta Z\|_{\alpha,\tau}.$$

For $\varepsilon \leqslant \varepsilon_{\alpha,1,X,\sigma}$ the last term is bounded by $\frac{1}{2}\|\delta Z\|_{\alpha,\tau}$ *which is finite* by Lemma 2.6. Bringing this term in the LHS we obtain (2.13), and this completes the proof. □

2.4. UNIQUENESS

In this section we prove uniqueness of solutions to (2.3) assuming that σ is of class C^1 *with locally Hölder gradient* (we stress that we make no boundedness assumption on σ). This improves on Theorem 1.7, both because we allow for non-linear σ and because we do not require that the time horizon $T > 0$ is small.

We first need an elementary but fundamental estimate on the difference of increments of a function. Given $\Psi: \mathbb{R}^k \to \mathbb{R}^{\ell}$, we use the notation

$$C_{\Psi,R} := \sup\,\{|\Psi(x)|:\; x \in \mathbb{R}^k,\; |x| \leqslant R\}\,. \tag{2.18}$$

LEMMA 2.8. (DIFFERENCE OF INCREMENTS) *Let $\psi: \mathbb{R}^k \to \mathbb{R}^{\ell}$ be of class $\mathcal{C}^{1+\rho}_{\mathrm{loc}}$ for some $0 < \rho \leqslant 1$ (i.e. ψ is differentiable with $\nabla\psi$ of class $\mathcal{C}^{\rho}_{\mathrm{loc}}$). Then for any $R > 0$ and for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}^k$ with $\max\{|x|, |y|, |\bar{x}|, |\bar{y}|\} \leqslant R$ we can estimate*

$$\begin{aligned} &|[\psi(x) - \psi(y)] - [\psi(\bar{x}) - \psi(\bar{y})]| \\ &\qquad \leqslant C_R'\,|(x-y) - (\bar{x}-\bar{y})| + C_R''\,\{|x-y|^{\rho} + |\bar{x}-\bar{y}|^{\rho}\}\,|y-\bar{y}|\,, \end{aligned} \tag{2.19}$$

where $C_R' := \sup\{|\nabla\psi(x)|:\; |x| \leqslant R\}$ and $C_R'' := \sup\left\{\frac{|\nabla\psi(x) - \nabla\psi(y)|}{|x-y|^{\rho}}:\; |x|, |y| \leqslant R\right\}$.

Proof. For $z, w \in \mathbb{R}^k$ we can write

$$\psi(z) - \psi(w) = \hat{\psi}(z, w)\,(z - w),$$

where $\hat{\psi}(z, w) := \int_0^1 \nabla\psi(u\,z + (1-u)\,w)\,\mathrm{d}u \in \mathbb{R}^\ell \otimes (\mathbb{R}^k)^*$, therefore

$$\begin{aligned}[\psi(x) - \psi(y)] - [\psi(\bar{x}) - \psi(\bar{y})] &= [\psi(x) - \psi(\bar{x})] - [\psi(y) - \psi(\bar{y})] \\ &= \hat{\psi}(x, \bar{x})\,(x - \bar{x}) - \hat{\psi}(y, \bar{y})\,(y - \bar{y}) \\ &= \hat{\psi}(x, \bar{x})\,[(x - \bar{x}) - (y - \bar{y})] \\ &\quad + [\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{y})]\,(y - \bar{y})\,.\end{aligned}$$

By definition of C_R' and C_R'' we have $|\hat{\psi}(x, \bar{x})| \leqslant C_R'$ and

$$\begin{aligned}|\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{y})| &\leqslant |\hat{\psi}(x, \bar{x}) - \hat{\psi}(y, \bar{x})| + |\hat{\psi}(y, \bar{x}) - \hat{\psi}(y, \bar{y})| \\ &\leqslant C_R''\{|x - y|^\rho + |\bar{x} - \bar{y}|^\rho\},\end{aligned}$$

hence (2.19) follows. □

We are now ready to state and prove the announced uniqueness result.

Theorem 2.9. (Uniqueness) *Let X be of class $\mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2}, 1\right]$ and let σ be of class $\mathcal{C}^\gamma$ for some $\gamma > \frac{1}{\alpha}$ (for instance, we can take $\sigma \in \mathcal{C}^2$). Then for every $z_0 \in \mathbb{R}^k$ there exists at most one solution Z to (2.3) with $Z_0 = z_0$.*

Proof. Let Z and $\bar{Z}$ be two solutions of (2.3), i.e. they satisfy (2.4), and set

$$Y := Z - \bar{Z}\,.$$

We want to show that, for $\tau > 0$ small enough, we have

$$\|Y\|_{\infty,\tau} \leqslant 2\,|Y_0|,$$

where the weighted norm $\|\cdot\|_{\infty,\tau}$ was defined in (1.37). In particular, if we assume that $Z_0 = \bar{Z}_0$, we obtain $\|Y\|_{\infty,\tau} = 0$ and hence $Z = \bar{Z}$.

We know by (2.5) that for any $\tau > 0$

$$\|Y\|_{\infty,\tau} \leqslant |Y_0| + 3\tau^\alpha\,\|\delta Y\|_{\alpha,\tau}\,, \tag{2.20}$$

where we recall that the weighted semi-norm $\|\cdot\|_{\alpha,\tau}$ was defined in (1.33). We now define $Y^{[2]}$ as the difference between the remainders $Z^{[2]}$ and $\bar{Z}^{[2]}$ of the solutions Z and $\bar{Z}$ as defined in (2.4), that is

$$Y_{st}^{[2]} := Z_{st}^{[2]} - \bar{Z}_{st}^{[2]} = \delta Y_{st} - (\sigma(Z_s) - \sigma(\bar{Z}_s))\,\delta X_{st}\,. \tag{2.21}$$

(We are slightly abusing notation, since $Y^{[2]}$ is not the remainder of Y when σ is not linear.) By assumption $\sigma \in \mathcal{C}^\gamma$ for some $\gamma > \frac{1}{\alpha}$: renaming γ as $\gamma \wedge 2$, we may assume that $\gamma \in \left]\frac{1}{\alpha}, 2\right]$. We are going to prove the following inequalities: for any $\tau > 0$

$$\|\delta Y\|_{\alpha,\tau} \leqslant c_1\,\|Y\|_{\infty,\tau} + \tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau}\,, \tag{2.22}$$

$$\|Y^{[2]}\|_{\gamma\alpha,\tau} \leqslant c_2\,\|Y\|_{\infty,\tau} + c_2'\,\tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau}\,, \tag{2.23}$$

for finite constants c_i, c_i' *that may depend on* $X, \sigma, Z, \bar{Z}$ *but not on* τ.

Let us complete the proof assuming (2.22) and (2.23). Note that $(\gamma-1)\,\alpha>0$ by assumption. If we fix $\tau>0$ small, so that $c_2'\,\tau^{(\gamma-1)\alpha}<\frac{1}{2}$, from (2.23) we get $\|Y^{[2]}\|_{\gamma\alpha,\tau}\leqslant 2\,c_2\,\|Y\|_{\infty,\tau}$ which plugged into (2.22) yields $\|\delta Y\|_{\alpha,\tau}\leqslant 2\,c_1\,\|Y\|_{\infty,\tau}$ for $\tau>0$ small (it suffices that $2\,c_2\,\tau^{(\gamma-1)\alpha}<c_1$). Finally, plugging this into (2.20) and possibly choosing $\tau>0$ even smaller, we obtain our goal $\|Y\|_{\infty,\tau}\leqslant 2\,|Y_0|$ which completes the proof.

It remains to prove (2.22) and (2.23). Using the notation from Lemma 2.8 we set

$$\begin{aligned} C_1' &:= \sup\{|\nabla\sigma(x)|:\ |x|\leqslant\|Z\|_\infty\vee\|\bar{Z}\|_\infty\}, \\ C_1'' &:= \sup\left\{\frac{|\nabla\sigma(x)-\nabla\sigma(y)|}{|x-y|^\rho}:\ |x|,|y|\leqslant\|Z\|_\infty\vee\|\bar{Z}\|_\infty\right\}. \end{aligned}$$

so that $|\sigma(Z_t)-\sigma(\bar{Z}_t)|\leqslant C_1'\,|Z_t-\bar{Z}_t|$ and, therefore,

$$\|\sigma(Z)-\sigma(\bar{Z})\|_{\infty,\tau}\leqslant C_1'\,\|Y\|_{\infty,\tau}\,. \tag{2.24}$$

We now exploit (2.21) to estimate $\|\delta Y\|_{\alpha,\tau}$: applying (2.7) we obtain

$$\begin{aligned} \|\delta Y\|_{\alpha,\tau} &\leqslant \|\sigma(Z)-\sigma(\bar{Z})\|_{\infty,\tau}\,\|\delta X\|_\alpha+\|Y^{[2]}\|_{\alpha,\tau} \\ &\leqslant C_1'\,\|Y\|_{\infty,\tau}\,\|\delta X\|_\alpha+\tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau}\,, \end{aligned} \tag{2.25}$$

where we note that $\|Y^{[2]}\|_{\alpha,\tau}\leqslant\tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau}$ by (2.6). We have shown that (2.22) holds with $c_1=C_1'\,\|\delta X\|_\alpha$.

We finally prove (2.23). Since $Y_{st}^{[2]}=o(t-s)$, see (2.21) and (2.4), we bound $Z^{[2]}$ by its increment $\delta Z^{[2]}$ through the weighted Sewing Bound (1.41):

$$\|Y^{[2]}\|_{\gamma\alpha,\tau}\ \leqslant K_{\gamma\alpha}\,\|\delta Y^{[2]}\|_{\gamma\alpha,\tau}\,, \tag{2.26}$$

hence we focus on $\|\delta Y^{[2]}\|_{\gamma\alpha,\tau}$. By (2.21) and (1.32), since $\delta\circ\delta=0$, we have

$$\delta Y_{sut}^{[2]}=(\delta\sigma(Z)_{su}-\delta\sigma(\bar{Z})_{su})\,\delta X_{ut}\,. \tag{2.27}$$

Applying the estimate (2.19) for $x=Z_u$, $y=Z_s$, $\bar{x}=\bar{Z}_u$, $\bar{y}=\bar{Z}_s$, we can write

$$\begin{aligned} |\delta\sigma(Z)_{su}-\delta\sigma(\bar{Z})_{su}| &\leqslant C_1'\,|\delta Z_{su}-\delta\bar{Z}_{su}|+C_1''\,\{|\delta Z_{su}|^{\gamma-1}+|\delta\bar{Z}_{su}|^{\gamma-1}\}|Z_s-\bar{Z}_s| \\ &= C_1'\,|\delta Y_{su}|+C_1''\,\{|\delta Z_{su}|^{\gamma-1}+|\delta\bar{Z}_{su}|^{\gamma-1}\}\,|Y_s|. \end{aligned} \tag{2.28}$$

hence by (2.7) we get

$$\begin{aligned} \|\delta\sigma(Z)-\delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau} &\leqslant C_1'\,\|\delta Y\|_{(\gamma-1)\alpha,\tau}+ \\ &+\ C_1''\,\{\|\delta Z\|_\alpha^{\gamma-1}+\|\delta\bar{Z}\|_\alpha^{\gamma-1}\}\,\|Y\|_{\infty,\tau}. \end{aligned} \tag{2.29}$$

If we take $\tau\leqslant 1$ we can bound $\|\delta Y\|_{(\gamma-1)\alpha,\tau}\leqslant\|\delta Y\|_{\alpha,\tau}$ by (2.6) (recall that we are assuming $\gamma\leqslant 2$). Then by (2.27) we obtain, recalling (2.8),

$$\|\delta Y^{[2]}\|_{\gamma\alpha,\tau}\leqslant\|\delta X\|_\alpha\,\|\delta\sigma(Z)-\delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau}\leqslant\tilde{c}_1\,(\|\delta Y\|_{\alpha,\tau}+\|Y\|_{\infty,\tau})\,,$$

for a suitable (explicit) constant $\tilde{c}_1=\tilde{c}_1(\sigma,Z,\bar{Z},X)$. Applying (2.22), we obtain

$$\|\delta Y^{[2]}\|_{\gamma\alpha,\tau}\leqslant(c_1+1)\,\tilde{c}_1\,\|Y\|_{\infty,\tau}+\tilde{c}_1\,\tau^{(\gamma-1)\alpha}\,\|Y^{[2]}\|_{\gamma\alpha,\tau},$$

which plugged into (2.26) shows that (2.23) holds. The proof is complete. □

We conclude with an example of (2.19).

Example 2.10. If $\sigma\colon\mathbb{R}\to\mathbb{R}$ is $\sigma(x)=x^2$, then we have

$$\begin{aligned}&(\sigma(x)-\sigma(y))-(\sigma(\bar{x})-\sigma(\bar{y}))\\&\quad=(x^2-y^2)-(\bar{x}^2-\bar{y}^2)=(x^2-\bar{x}^2)-(y^2-\bar{y}^2)\\&\quad=(x-\bar{x})\,(x+\bar{x})-(y-\bar{y})\,(y+\bar{y})\\&\quad=[(x-\bar{x})-(y-\bar{y})]\,(y+\bar{y})+(x-\bar{x})\,[(x+\bar{x})-(y+\bar{y})]\\&\quad=[(x-\bar{x})-(y-\bar{y})]\,(y+\bar{y})+(x-\bar{x})\,[(x-y)+(\bar{x}-\bar{y})],\end{aligned}$$

where in the second last equality we have summed and subtracted $(y-\bar{y})\,(x+\bar{x})$. If we use this formula for $x=Z_t$, $y=Z_s$ and $\bar{x}=\bar{Z}_t$, $\bar{y}=\bar{Z}_s$, then we obtain

$$\delta(Z^2-\bar{Z}^2)_{st}=\delta(Z-\bar{Z})_{st}\,(Z_s+\bar{Z}_s)+(Z_t-\bar{Z}_t)\,[\delta Z_{st}+\delta Z_{st}],$$

which is in the spirit of (2.19) with $\rho=1$. It follows that

$$\|\delta(Z^2-\bar{Z}^2)\|_\alpha\leqslant 2\,\|\bar{Z}\|_\infty\,\|\delta(Z-\bar{Z})\|_\alpha+\|Z-\bar{Z}\|_\infty\,[\|\delta Z\|_\alpha+\|\delta\bar{Z}\|_\alpha],$$

which is the form that (2.29) takes in this particular case.

2.5. CONTINUITY OF THE SOLUTION MAP

In this section we assume that σ is *globally Lipschitz* and of class C^1 with a *globally γ-Hölder gradient*, i.e. $\|\nabla\sigma\|_\infty<\infty$ and $[\nabla\sigma]_{\mathcal{C}^\gamma}<\infty$, with $\gamma>\frac{1}{\alpha}$. Under these assumptions, we have *global existence and uniqueness* of solutions $Z\colon[0,T]\to\mathbb{R}^k$ to (2.3) for any time horizon $T>0$, for any starting point $Z_0\in\mathbb{R}^k$ and for any driving path X of class $\mathcal{C}^\alpha$ with $\frac{1}{2}<\alpha\leqslant 1$ (as we will prove in Section 2.6).

We can thus consider the *solution map*:

$$\begin{aligned}\Phi\colon\ \mathbb{R}^k\times\mathcal{C}^\alpha &\longrightarrow \mathcal{C}^\alpha\\ (Z_0,X)&\longmapsto Z:=\begin{cases}\text{unique solution of (2.3) for } t\in[0,T]\\ \text{starting from } Z_0\end{cases}.\end{aligned}\tag{2.30}$$

We prove in this section that this map is *continuous*, in fact *locally Lipschitz*.

Remark 2.11. The continuity of the solution map is a highly non-trivial property. Indeed, when X is of class C^1, note that Z solves the equation

$$Z_t=Z_0+\int_0^t\sigma(Z_s)\,\dot{X}_s\,\mathrm{d}s\,,\tag{2.31}$$

which is based on the derivative $\dot{X}$ of X. We instead consider driving paths $X\in\mathcal{C}^\alpha$ with $\alpha\in\left]\frac{1}{2},1\right]$ which are continuous but may be non-differentiable.

We shall see in the next chapters that the continuity of the solution map holds also in more complex situations such as $X\in\mathcal{C}^\alpha$ with $\alpha\leqslant\frac{1}{2}$, which cover the case when X is a Brownian motion and Z is the solution to a SDE.

Before stating the continuity of the solution map, we recall that the space $\mathcal{C}^\alpha$ is equipped with the norm $\|f\|_{\mathcal{C}^\alpha} := \|f\|_\infty + \|\delta f\|_\alpha$, see Remark 1.4, but *an equivalent norm is* $\|f\|_{\infty,\tau} + \|\delta f\|_{\alpha,\tau}$ *for any choice of the weight* $\tau > 0$, see Remark 1.15.

Theorem 2.12. (Continuity of the solution map) *Let* σ *be globally Lipschitz with a globally* $(\gamma-1)$*-Hölder gradient:* $\|\nabla\sigma\|_\infty < \infty$ *and* $[\nabla\sigma]_{\mathcal{C}^{\gamma-1}} < \infty$*, with* $\gamma \in \left(\frac{1}{\alpha}, 2\right]$*. Then, for any* $T > 0$ *and* $\alpha \in \left]\frac{1}{2}, 1\right]$*, the solution map* $(Z_0, X) \mapsto Z$ *in (2.30) is locally Lipschitz.*

More explicitly, given $M_0, M, D < \infty$*, if we assume that*

$$\max\{\|\nabla\sigma\|_\infty, [\nabla\sigma]_{\mathcal{C}^{\gamma-1}}\} \leqslant D,$$

and we consider starting points $Z_0, \bar{Z}_0 \in \mathbb{R}^d$ *and driving paths* $X, \bar{X} \in \mathcal{C}^\alpha$ *with*

$$\max\{|\sigma(Z_0)|, |\sigma(\bar{Z}_0)|\} \leqslant M_0, \qquad \max\{\|\delta X\|_\alpha, \|\delta\bar{X}\|_\alpha\} \leqslant M, \tag{2.32}$$

then the corresponding solutions $Z = (Z_s)_{s\in[0,T]}$*,* $\bar{Z} = (\bar{Z}_s)_{s\in[0,T]}$ *of (2.3) satisfy*

$$\|Z - \bar{Z}\|_{\infty,\tau} + \|\delta Z - \delta\bar{Z}\|_{\alpha,\tau} \leqslant \mathfrak{C}_M\,|Z_0 - \bar{Z}_0| + 6\,M_0\,\|\delta X - \delta\bar{X}\|_\alpha, \tag{2.33}$$

provided $0 < \tau \wedge T \leqslant \hat{\tau}$ *for a suitable* $\hat{\tau} = \hat{\tau}_{\alpha,\gamma,T,D,M_0,M} > 0$*, where we set*

$$\mathfrak{C}_M := 2\,(\|\nabla\sigma\|_\infty M + 1) \leqslant 2\,(D\,M + 1)\,.$$

Proof. Let us define the constant

$$\mathfrak{c}_M := \|\nabla\sigma\|_\infty M \leqslant D\,M\,. \tag{2.34}$$

We fix two solutions Z and $\bar{Z}$ of (2.3) with respective driving paths X and $\bar{X}$. If we define $Y := Z - \bar{Z}$, we can rewrite our goal (2.33) as

$$\|Y\|_{\infty,\tau} + \|\delta Y\|_{\alpha,\tau} \leqslant 6\,M_0\,\|\delta X - \delta\bar{X}\|_\alpha + 2\,(\mathfrak{c}_M + 1)\,|Y_0|\,. \tag{2.35}$$

Let us introduce the shorthand

$$\varepsilon := (\tau \wedge T)^\alpha$$

and let us agree that, whenever we write *for* ε *small enough* we mean *for* $0 < \varepsilon \leqslant \varepsilon_0$ *for a suitable* $\varepsilon_0 > 0$ *which depends on* α, T, M_0, M, D. By (2.5), *for* ε *small enough*,

$$\|Y\|_{\infty,\tau} \leqslant |Y_0| + \varepsilon\,\|\delta Y\|_{\alpha,\tau} \leqslant |Y_0| + \frac{1}{5}\,\|\delta Y\|_{\alpha,\tau}, \tag{2.36}$$

hence to prove (2.35) we can focus on $\|\delta Y\|_{\alpha,\tau}$.

Recalling (2.4), let us define $Y^{[2]} := Z^{[2]} - \bar{Z}^{[2]}$. We are going to establish the following two relations, *for* ε *small enough*:

$$\frac{4}{5}\,\|\delta Y\|_{\alpha,\tau} \leqslant 2\,M_0\,\|\delta X - \delta\bar{X}\|_\alpha + \mathfrak{c}_M\,|Y_0| + \|Y^{[2]}\|_{\alpha,\tau}\,, \tag{2.37}$$

$$\|Y^{[2]}\|_{\alpha,\tau} \leqslant M_0\,\|\delta X - \delta\bar{X}\|_\alpha + \frac{1}{2}|Y_0| + \frac{1}{5}\,\|\delta Y\|_{\alpha,\tau}\,. \tag{2.38}$$

Plugging (2.38) into (2.37) and applying (2.36), we obtain (2.35).

It remains to prove (2.37) and (2.38). We record some useful bounds. Let us set

$$\bar{\varepsilon} = \bar{\varepsilon}_{\alpha,D,M} := \frac{1}{2\,(K_{2\alpha}+3)\,D\,M}. \tag{2.39}$$

We exploit the a priori estimate (2.13) from Theorem 2.7: by (2.32), we have

$$\text{for } \varepsilon = (\tau \wedge T)^{\alpha} \leqslant \bar{\varepsilon}: \qquad \max\{\|\delta Z\|_{\alpha,\tau}, \|\delta \bar{Z}\|_{\alpha,\tau}\} \leqslant 2\,M_0\,M, \tag{2.40}$$

therefore

$$\|\delta\sigma(Z)\|_{\alpha,\tau} \leqslant \|\nabla\sigma\|_{\infty}\,\|\delta Z\|_{\alpha,\tau} \leqslant 2\,\|\nabla\sigma\|_{\infty}\,M_0\,M = 2\,M_0\,\mathfrak{c}_M, \tag{2.41}$$

and applying (2.5) and (2.32) we get, *for* ε *small enough*,

$$\|\sigma(Z)\|_{\infty,\tau} \leqslant |\sigma(Z_0)| + 3\varepsilon\,\|\delta\sigma(Z)\|_{\alpha,\tau} \leqslant M_0\,(1 + 6\,\mathfrak{c}_M\,\varepsilon) \leqslant 2\,M_0\ . \tag{2.42}$$

We can now prove (2.37). Defining $Y^{[2]} := Z^{[2]} - \bar{Z}^{[2]}$, we obtain from (2.4)

$$\begin{aligned} \delta Y_{st} &= \delta Z_{st} - \delta\bar{Z}_{st} = \sigma(Z_s)\,\delta X_{st} - \sigma(\bar{Z}_s)\,\delta\bar{X}_{st} + Y^{[2]}_{st} \\ &= \sigma(Z_s)\,(\delta X - \delta\bar{X})_{st} + (\sigma(Z_s) - \sigma(\bar{Z}_s))\,\delta\bar{X}_{st} + Y^{[2]}_{st}, \end{aligned}$$

hence by (2.7) we can bound

$$\begin{aligned} \|\delta Y\|_{\alpha,\tau} \leqslant\ & \|\sigma(Z)\|_{\infty,\tau}\,\|\delta X - \delta\bar{X}\|_{\alpha} \\ & + \|\delta\bar{X}\|_{\alpha}\,\|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau} + \|Y^{[2]}\|_{\alpha,\tau}\,. \end{aligned} \tag{2.43}$$

Let us look at the second term in the RHS of (2.43): by (2.5)

$$\begin{aligned} \|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau} &\leqslant \|\nabla\sigma\|_{\infty}\,\|Z - \bar{Z}\|_{\infty,\tau} \\ &\leqslant \|\nabla\sigma\|_{\infty}\,(|Y_0| + 3\varepsilon\,\|\delta Y\|_{\alpha,\tau}). \end{aligned} \tag{2.44}$$

Hence by (2.32) and (2.34) we get, *for* ε *small enough*,

$$\|\delta\bar{X}\|_{\alpha}\,\|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau} \leqslant \mathfrak{c}_M\,|Y_0| + \frac{1}{5}\,\|\delta Y\|_{\alpha,\tau}\,. \tag{2.45}$$

Plugging this into (2.43) we then obtain, by (2.42),

$$\frac{4}{5}\,\|\delta Y\|_{\alpha,\tau} \leqslant 2\,M_0\,\|\delta X - \delta\bar{X}\|_{\alpha} + \mathfrak{c}_M\,|Y_0| + \|Y^{[2]}\|_{\alpha,\tau}, \tag{2.46}$$

which proves (2.37).

We finally prove (2.38). Since $Y^{[2]}_{st} = Z^{[2]}_{st} - \bar{Z}^{[2]}_{st} = o(t-s)$, see (2.4), the weighted Sewing Bound (1.41) and (2.6) give

$$\|Y^{[2]}\|_{\alpha,\tau} \leqslant \varepsilon^{\gamma-1}\,\|Y^{[2]}\|_{\gamma\alpha,\tau} \leqslant K_{\gamma\alpha}\,\varepsilon^{\gamma-1}\,\|\delta Y^{[2]}\|_{\gamma\alpha,\tau}\ . \tag{2.47}$$

To estimate $\delta Y^{[2]} = \delta Z^{[2]} - \delta\,\bar{Z}^{[2]}$, note that by (2.4) and (1.32) we can write

$$\delta Y^{[2]}_{sut} = \delta\sigma(Z)_{su}\,(\delta X - \delta\bar{X})_{ut} + (\delta\sigma(Z) - \delta\sigma(\bar{Z}))_{su}\,\delta\bar{X}_{ut}\,, \tag{2.48}$$

hence by (2.8)

$$\|\delta Y^{[2]}\|_{\gamma\alpha,\tau} \leqslant \|\delta\sigma(Z)\|_{(\gamma-1)\alpha,\tau}\,\|\delta X - \delta\bar{X}\|_{\alpha} + \|\delta\bar{X}\|_{\alpha}\,\|\delta\sigma(Z) - \delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau}. \tag{2.49}$$

The first term is easy to control: by (2.41), *for ε small enough*,

$$K_{\gamma\alpha}\,\varepsilon^{\gamma-1}\,\|\delta\sigma(Z)\|_{(\gamma-1)\alpha,\tau}\,\|\delta X-\delta\bar{X}\|_{\alpha}\leqslant M_0\,\|\delta X-\delta\bar{X}\|_{\alpha}\,. \tag{2.50}$$

Let us now focus on the second term. By (2.19) we have, see also (2.28),

$$|\delta\sigma(Z)_{su}-\delta\sigma(\bar{Z})_{su}|\leqslant\|\nabla\sigma\|_{\infty}\,|\delta Y_{su}|+[\nabla\sigma]_{\mathcal{C}^{\gamma-1}}\,\{|\delta Z_{su}|^{\gamma-1}+|\delta\bar{Z}_{su}|^{\gamma-1}\}\,|Y_s|\,.$$

We apply (2.9) for $H=\delta Z$, $g=Y$ and $\bar{\tau}=(\bar{\varepsilon})^{1/\alpha}$ from (2.39):

$$\begin{aligned}\|\delta\sigma(Z)-\delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau}\;&\leqslant\;\|\nabla\sigma\|_{\infty}\,\|\delta Y\|_{(\gamma-1)\alpha,\tau}+\\&\qquad+[\nabla\sigma]_{\mathcal{C}^{\gamma-1}}\,\mathrm{e}^{\frac{T}{\bar{\tau}}}\,(\|\delta Z\|_{\alpha,\bar{\tau}}^{\gamma-1}+\|\delta\bar{Z}\|_{\alpha,\bar{\tau}}^{\gamma-1})\|Y\|_{\infty,\tau}\\&\leqslant\;D\,\|\delta Y\|_{\alpha,\tau}+2\,(2M_0\,M)^{\gamma-1}\,\mathrm{e}^{\frac{T}{\bar{\tau}}}D\,\|Y\|_{\infty,\tau},\end{aligned} \tag{2.51}$$

where we applied (2.40). Hence by (2.51), recalling (2.32), *for ε small enough* we obtain

$$K_{\gamma\alpha}\,\varepsilon^{\gamma-1}\,\|\delta\bar{X}\|_{\alpha}\,\|\delta\sigma(Z)-\delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau}\leqslant\frac{1}{10}\,\|\delta Y\|_{\alpha,\tau}+\frac{1}{2}\|Y\|_{\infty,\tau}, \tag{2.52}$$

and since $\|Y\|_{\infty,\tau}\leqslant|Y_0|+\frac{1}{5}\,\|\delta Y\|_{\alpha,\tau}$, see (2.36), we obtain

$$K_{\gamma\alpha}\,\varepsilon^{\gamma-1}\,\|\delta\bar{X}\|_{\alpha}\,\|\delta\sigma(Z)-\delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau}\leqslant\frac{1}{2}|Y_0|+\frac{1}{5}\,\|\delta Y\|_{\alpha,\tau}.$$

Finally, plugging this bound and (2.50) into (2.49) and (2.47), we obtain

$$\|Y^{[2]}\|_{\alpha,\tau}\leqslant M_0\,\|\delta X-\delta\bar{X}\|_{\alpha}+\frac{1}{2}|Y_0|+\frac{1}{5}\|\delta Y\|_{\alpha,\tau},$$

which proves (2.38) and completes the proof. ☐

Remark 2.13. An explicit choice for $\hat{\tau}$ in Theorem 2.12 is

$$\hat{\tau}^{\alpha}:=\frac{\mathrm{e}^{-\frac{T}{\bar{\tau}}}}{10\,(K_{2\alpha}+3)\,(1+M_0)\,(1+D\,(M+M^2))}, \tag{2.53}$$

with $\bar{\tau}=\bar{\tau}_{\alpha,D,M}$ defined in (2.39). This is obtained by tracking all the points in the proof of Theorem 2.12 where $\varepsilon=(\tau\wedge T)^{\alpha}$ was assumed to be *small enough*: see Section 2.8 for the details.

2.6. Euler scheme and local/global existence

In this section we discuss *global existence of solutions*, under the assumption that σ is globally γ-Hölder with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$, i.e. $[\sigma]_{\mathcal{C}^{\gamma}}<\infty$ (again with no boundedness assumption on σ). We also state a result of *local existence of solutions* for equation (2.3), where we only assume that σ is *locally γ-Hölder* with $\gamma\in\left(\frac{1}{\alpha}-1,1\right]$ (with no boundedness assumption on σ).

We fix $X\colon[0,T]\to\mathbb{R}^d$ of class $\mathcal{C}^{\alpha}$ with $\alpha\in\left]\frac{1}{2},1\right]$ and a starting point $z_0\in\mathbb{R}^k$. We split the proof in two parts: we first assume that $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$ is globally γ-Hölder, then we consider the case when σ is locally γ-Hölder.

First part: globally Hölder case.

We consider a finite set $\mathbb{T} = \{0 = t_1 < \cdots < t_{\#\mathbb{T}}\} \subset \mathbb{R}_+$ and we define an approximate solution $Z = Z^{\mathbb{T}} = (Z_t)_{t \in \mathbb{T}}$ through the *Euler scheme*

$$Z_0 := z_0, \qquad Z_{t_{i+1}} := Z_{t_i} + \sigma(Z_{t_i})\, \delta X_{t_i, t_{i+1}} \qquad \text{for } 1 \leqslant i \leqslant \#\mathbb{T} - 1. \tag{2.54}$$

Let us define the “remainder”

$$R_{st} := \delta Z_{st} - \sigma(Z_s)\, \delta X_{st} \qquad \text{for } s < t \in \mathbb{T}. \tag{2.55}$$

We assume that σ is *globally* γ*-Hölder*, namely $[\sigma]_{\mathcal{C}^\gamma} < \infty$, with $\gamma \in \left(\frac{1}{\alpha} - 1, 1\right]$. We set

$$\hat{\varepsilon}_{\alpha,\gamma,X,\sigma} := \frac{1}{2\,(C_{(\gamma+1)\alpha} + 5)\, \|\delta X\|_\alpha\, [\sigma]_{\mathcal{C}^\gamma}}, \tag{2.56}$$

where the constant C_η is defined in (1.45). We prove the following *a priori estimates* on the Euler scheme (2.54), which are analogous to those in Theorem 2.7.

LEMMA 2.14. *If σ is globally γ-Hölder, namely $[\sigma]_{\mathcal{C}^\gamma} < \infty$, with $\gamma \in \left(\frac{1}{\alpha} - 1, 1\right]$, then*

$$\|R\|_{(\gamma+1)\alpha}^{\mathbb{T}} \leqslant C_{(\gamma+1)\alpha}\, [\sigma]_{\mathcal{C}^\gamma}\, (\|\delta Z\|_\alpha^{\mathbb{T}})^\gamma\, \|\delta X\|_\alpha, \tag{2.57}$$

$$\text{and for } \tau^{\gamma\alpha} \leqslant \hat{\varepsilon}_{\alpha,\gamma,X,\sigma}: \qquad \|\delta Z\|_\alpha^{\mathbb{T}} \leqslant 1 \vee (2\, |\sigma(z_0)|\, \|\delta X\|_\alpha). \tag{2.58}$$

Proof. Since $\delta R_{sut} = (\sigma(Z_s) - \sigma(Z_u))\, \delta X_{ut}$, recall (1.32), and since $R_{t_i t_{i+1}} = 0$ by (2.54), we can apply the discrete Sewing Bound (1.45) with $\eta = (\gamma + 1)\alpha > 1$ to get

$$\|R\|_{(\gamma+1)\alpha,\tau}^{\mathbb{T}} \leqslant C_{(\gamma+1)\alpha}\, \|\delta R\|_{(\gamma+1)\alpha,\tau}^{\mathbb{T}} \leqslant C_{(\gamma+1)\alpha}\, [\sigma]_{\mathcal{C}^\gamma}\, (\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}})^\gamma\, \|\delta X\|_\alpha. \tag{2.59}$$

We have proved (2.57).

We next prove (2.58). Recalling (2.55) we can bound, by (2.6) for $\|\cdot\|_{\gamma\alpha,\mathbb{T}_n}$,

$$\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}} \leqslant \|\sigma(Z)\|_{\infty,\tau}^{\mathbb{T}}\, \|\delta X\|_\alpha + \tau^{\gamma\alpha}\, \|R\|_{(\gamma+1)\alpha,\tau}^{\mathbb{T}}.$$

By (1.47)

$$\|\sigma(Z)\|_{\infty,\tau}^{\mathbb{T}} \leqslant |\sigma(z_0)| + 5\,\tau^{\gamma\alpha} \|\delta\sigma(Z)\|_{\gamma\alpha,\tau}^{\mathbb{T}} \leqslant |\sigma(z_0)| + 5\,\tau^{\gamma\alpha}\, [\sigma]_{\mathcal{C}^\gamma}\, (\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}})^\gamma.$$

We thus obtain, combining the previous bounds,

$$\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}} \leqslant |\sigma(z_0)|\, \|\delta X\|_\alpha + \{\tau^{\gamma\alpha}\, (C_{\gamma\alpha} + 5)\, [\sigma]_{\mathcal{C}^\gamma}\, \|\delta X\|_\alpha\}\, (\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}})^\gamma.$$

Now if $\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}} \leqslant 1$ then (2.58) is proved, otherwise $(\|\delta Z\|_{\alpha,\tau}^{\mathbb{T}})^\gamma \leqslant \|\delta Z\|_{\alpha,\tau}^{\mathbb{T}}$ and then for τ as in (2.56) the term in brackets is less than $\frac{1}{2}$ and we obtain (2.58). □

We can now prove the following

THEOREM 2.15. (GLOBAL EXISTENCE) *Let X be of class $\mathcal{C}^\alpha$, with $\alpha \in \left]\frac{1}{2}, 1\right]$, and let σ be globally γ-Hölder with $\gamma \in \left(\frac{1}{\alpha} - 1, 1\right]$, i.e. $[\sigma]_{\mathcal{C}^\gamma} < \infty$. For every $z_0 \in \mathbb{R}^k$, with no restriction on $T > 0$, there exists a solution $(Z_t)_{t \in [0,T]}$ of (2.3) with $Z_0 = z_0$.*

Proof. Given $n \in \mathbb{N}$, we construct an approximate solution $Z^n = (Z^n_t)_{t \in \mathbb{T}_n}$ of (2.3) defined in the discrete set of times $\mathbb{T}_n := (\{i2^{-n}\colon\ i = 0, 1, \ldots\} \cap [0, T]) \cup \{T\}$ through the *Euler scheme* (2.54).

$$Z^n_0 := z_0, \qquad Z^n_{t_{i+1}} := Z^n_{t_i} + \sigma(Z^n_{t_i})\, \delta X_{t_i, t_{i+1}} \qquad \text{for } t_i, t_{i+1} \in \mathbb{T}_n\,. \tag{2.60}$$

Let us define the "remainder"

$$R^n_{st} := \delta Z^n_{st} - \sigma(Z^n_s)\, \delta X_{st} \qquad \text{for } s < t \in \mathbb{T}_n\,. \tag{2.61}$$

We fix $T > 0$ such that

We extend Z^n by linear interpolation to a continuous function defined on $[0, T]$, still denoted by Z^n. Given two points $t_i \leqslant s < t \leqslant t_{i+1}$ inside the same interval $[t_i, t_{i+1}]$ of the partition $\mathbb{T}_n$, since $\delta Z^n_{st} = \frac{t-s}{t_{i+1} - t_i}\, \delta Z^n_{t_i t_{i+1}}$, we can bound for $\alpha \in (0, 1]$

$$\frac{|\delta Z^n_{st}|}{(t-s)^\alpha} = \left(\frac{t-s}{t_{i+1} - t_i} \right)^{1-\alpha} \frac{|\delta Z^n_{t_i t_{i+1}}|}{(t_{i+1} - t_i)^\alpha} \leqslant \frac{|\delta Z^n_{t_i t_{i+1}}|}{(t_{i+1} - t_i)^\alpha}\,.$$

Given two points $s < t$ in different intervals, say $t_i \leqslant s \leqslant t_{i+1} \leqslant t_j \leqslant t \leqslant t_{j+1}$ for some $i < j$, by the triangle inequality we can bound $|\delta Z^n_{st}| \leqslant |\delta Z^n_{s t_{i+1}}| + |\delta Z^n_{t_{i+1} t_j}| + |\delta Z^n_{t_j t}|$. Recalling (1.9) and (1.43), we then obtain $\|\cdot\|_\alpha \leqslant 3\, \|\cdot\|^{\mathbb{T}_n}_\alpha$, hence by (2.58) we get

$$\|\delta Z^n\|_{\alpha, \tau} \leqslant 3 \vee (6\, |\sigma(z_0)|\, \|\delta X\|_\alpha)\,. \tag{2.62}$$

The family $(Z^n)_{n \in \mathbb{N}}$ is *equi-continuous* by (2.62) and *equi-bounded*, since $Z^n_0 = z_0$ for all $n \in \mathbb{N}$, hence by the Arzelà-Ascoli Theorem it is *compact* in the space $C([0, T], \mathbb{R}^k)$. Let us denote by $Z\colon [0, T] \to \mathbb{R}^k$ any limit point. Plugging (2.58) into (2.57), by (2.61) we can write

$$\text{if } T^\alpha \leqslant \hat{\varepsilon}_{\alpha, X, \sigma}\colon \qquad |\delta Z^n_{st} - \sigma(Z^n_s)\, \delta X_{st}| \leqslant c(z_0)\, (t-s)^{2\alpha} \qquad \forall s < t \in \mathbb{T}_n\,, \tag{2.63}$$

where $c(z_0) := C_{(\gamma+1)\alpha}\, [\sigma]_{C^\gamma}\, (3 \vee (6\, |\sigma(z_0)|\, \|\delta X\|_\alpha))^\gamma\, \|\delta X\|_\alpha$. Letting $n \to \infty$ and observing that $\mathbb{T}_n \subseteq \mathbb{T}_{n+1}$, we see that (2.63) still holds with Z^n replaced by Z and $\mathbb{T}_n$ replaced by the set $\mathbb{T} := \bigcup_{\ell \in \mathbb{N}} \mathbb{T}_{2^\ell} = \left(\left\{ \frac{i}{2^n}\colon\ i, n \in \mathbb{N} \right\} \cap [0, T] \right) \cup \{T\}$ of dyadic rationals:

$$\text{if } T^\alpha \leqslant \hat{\varepsilon}_{\alpha, X, \sigma}\colon \qquad |\delta Z_{st} - \sigma(Z_s)\, \delta X_{st}| \leqslant c(z_0)\, (t-s)^{2\alpha} \qquad \forall s < t \in \mathbb{T}.$$

Since $\mathbb{T}$ is dense in $[0, T]$ and Z is continuous, this bound extends to all $0 \leqslant s < t \leqslant T$, which shows that Z is a solution of (2.3). This completes the proof. □

Second part: locally Lipschitz case.

We now assume that σ is *locally* γ*-Hölder* and we fix $z_0 \in \mathbb{R}^k$. We also fix $T > 0$ such that $T \leqslant \tilde{\varepsilon}_{\alpha, X, \sigma}(z_0)$, see (2.64), and we prove that there exists a solution $Z\colon [0, T] \to \mathbb{R}^k$ of (2.3) with $Z_0 = z_0$.

THEOREM 2.16. (LOCAL EXISTENCE) *Let X be of class $\mathcal{C}^\alpha$, with $\alpha \in \left] \frac{1}{2}, 1 \right]$, and let σ be locally Lipschitz (e.g. of class C^1). For any $z_0 \in \mathbb{R}^k$ and for $T > 0$ small enough, i.e.*

$$T^\alpha \leqslant \tilde{\varepsilon}_{\alpha, X, \sigma}(z_0) := \frac{1}{2}\, \frac{1}{(C_{2\alpha} + 3)\, \|\delta X\|_\alpha\, \{1 + \sup_{|z - z_0| \leqslant |\sigma(z_0)|} |\nabla \sigma(z)|\}}\,, \tag{2.64}$$

there exists a solution $(Z_t)_{t\in[0,T]}$ *of (2.3) with* $Z_0 = z_0$.

Let $\tilde{\sigma}$ be a *globally* γ*-Hölder* function (depending on z_0) such that

$$\tilde{\sigma}(z) = \sigma(z) \quad \forall |z - z_0| \leqslant \sigma(z_0) \qquad \text{and} \qquad [\tilde{\sigma}]_{\mathcal{C}^\gamma} = \sup_{|z - z_0| \leqslant \sigma(z_0)} |\nabla\sigma(z)|. \tag{2.65}$$

Since $T \leqslant \tilde{\varepsilon}_{\alpha,X,\sigma}(z_0) \leqslant \hat{\varepsilon}_{\alpha,X,\sigma}$, see (2.64) and (2.56), by the first part of the proof *there exists a solution* Z *of (2.3) with* $\tilde{\sigma}$ *in place of* σ and $Z_0 = z_0$. We will prove that

$$|Z_t - z_0| \leqslant \sigma(z_0) \ \text{ for all } \ t \in [0, T], \tag{2.66}$$

therefore $\tilde{\sigma}(Z_t) = \sigma(Z_t)$ for all $t \in [0, T]$, see (2.65). This means that Z is a solution of the original (2.3) with σ, which completes the proof of Theorem 2.16.

To prove (2.66), we apply the a priori estimate (2.13) with $\tau = \infty$: we note that $T \leqslant \tilde{\varepsilon}_{\alpha,X,\sigma}(z_0) \leqslant \varepsilon_{\alpha,X,\sigma}$ (see (2.64) and (2.12), and note that $C_{2\alpha} \geqslant K_{2\alpha}$), therefore

$$\|\delta Z\|_\alpha \leqslant 2\, \|\delta X\|_\alpha\, |\sigma(z_0)|,$$

because $\tilde{\sigma}(z_0) = \sigma(z_0)$. Then for every $t \in [0, T]$ we can bound

$$|Z_t - z_0| \leqslant T^\alpha\, \|\delta Z\|_\alpha \leqslant 2\, T^\alpha\, \|\delta X\|_\alpha\, |\sigma(z_0)| \leqslant |\sigma(z_0)|,$$

where the last inequality holds because $T^\alpha \leqslant \tilde{\varepsilon}_{\alpha,X,\sigma}(z_0) \leqslant (2\, \|\delta X\|_\alpha)^{-1}$, see (2.64). This completes the proof of (2.66).

2.7. ERROR ESTIMATE IN THE EULER SCHEME

We suppose in this section that σ is of class C^2 with $\|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty < +\infty$.

THEOREM 2.17. *The Euler scheme converges at speed* $n^{2\alpha-1}$.

Proof. Let us set $z_i := \partial y_i / \partial y_0$, where $(y_i)_{i\geqslant 0}$ is defined by (2.60). Then

$$z_{i+1} = z_i + \nabla\sigma(y_i)\, z_i\, \delta X_{t_i t_{i+1}}, \qquad i \geqslant 0.$$

This shows that the pair $(y_i, z_i)_{i\geqslant 0}$ satisfies a recurrence which is similar to (2.60) with a map Σ of class C^1 and therefore we can apply the above results to obtain that $|z_i| \leqslant \text{const}$. In particular the map $y_0 \to y_k$ is Lipschitz-continuous, uniformly over $k \geqslant 0$.

Let us call, for $k \geqslant 0$, $(z_\ell^{(k)})_{\ell \geqslant k}$ as the sequence which satisfies (2.60) but has initial value $z_k^{(k)} = y(t_k)$. Since $(y(t))_{t\geqslant 0}$ is a solution to (2.4), we have

$$|z_{k+1}^{(k)} - y(t_{k+1})| \lesssim n^{-2\alpha}.$$

Since the map $y_0 \to y_k$ is Lipschitz-continuous uniformly over $k \geqslant 0$, we have

$$|z_\ell^{(k)} - z_\ell^{(k+1)}| \lesssim |z_{k+1}^{(k)} - y(t_{k+1})| \lesssim n^{-2\alpha}, \qquad \ell \geqslant k + 1.$$

Therefore

$$|y_\ell - y(t_\ell)| = |z_\ell^{(0)} - z_\ell^{(\ell)}| \leqslant \sum_{k=0}^{\ell-1} |z_\ell^{(k)} - z_\ell^{(k+1)}| \lesssim \frac{\ell}{n^{2\alpha}} = \frac{t_\ell}{n^{2\alpha-1}} \to 0$$

as t_ℓ is bounded and $n \to \infty$. □

2.8. Extra: a value for $\hat{\tau}$

We can give an explicit expression for $\hat{\tau} = \hat{\tau}_{M_0,M,T}$ in Theorem 2.12, by tracking all the points in the proof where τ *is small enough*, namely:

- for (2.36) we need $\tau^\alpha \leqslant \frac{1}{15}$;
- for (2.40) we need $\tau^\alpha \leqslant (\hat{\rho}_M)^\alpha := (2\,(K_{2\alpha} + 3)\,\mathfrak{c}_M)^{-1}$;
- for (2.42) we need $\tau^\alpha \leqslant (6\,\mathfrak{c}_M)^{-1}$, for (2.45) we need $\tau^\alpha \leqslant (15\,\mathfrak{c}_M)^{-1}$;
- for (2.50) we need $\tau^{(\gamma-1)\alpha} \leqslant (2\,K_{\gamma\alpha}\,\mathfrak{c}_M)^{-1}$;
- for (2.52) we need $\tau^{(\gamma-1)\alpha} \leqslant (10\,K_{\gamma\alpha}\,\mathfrak{c}_M)^{-1}$ (first term in the RHS) and also $\tau^{(\gamma-1)\alpha} \leqslant \left(K_{\gamma\alpha}\,\mathrm{e}^{\frac{T}{\hat{\rho}_M}}\,M_0\,M^2\,\|\nabla^2\sigma\|_\infty\right)^{-1}$ (second term in the RHS).

Since $\mathfrak{c}_M = M\,\|\nabla\sigma\|_\infty$, see (2.34), it is easy to check that all these constraints are satisfied for $0 < \tau \leqslant \hat{\tau}$ given by formula (2.53) in Remark 2.13.