# Chapter 3

## Difference equations: the rough case

We have so far considered the difference equation (2.3), that is

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{3.1}$$

where $X$ is given, $Z$ is the unknown and $\sigma(\cdot)$ is sufficiently regular. This is a generalization of the differential equation $\dot{Z}_t = \sigma(Z_t)\,\dot{X}_t$ which is meaningful for non smooth $X$, as we showed in Chapter 2, where we proved *well-posedness* in the so-called *Young case*, i.e. assuming that $X \in \mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{2}, 1\right]$.

However, the restriction $\alpha > \frac{1}{2}$ is a substantial limitation: in particular, we cannot take $X = B$ as a typical path of Brownian motion, which is in $\mathcal{C}^\alpha$ only for $\alpha < \frac{1}{2}$. For this reason, we show in this chapter how to *enrich* the difference equation (3.1) and prove *well-posedness when $X \in \mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$*, called the *rough case*. This will be applied to Brownian motion in the next Chapter 4, in ordered to obtain a robust formulation of classical *stochastic differential equations*.

Notation. *Throughout this book we write $f_{st} \lesssim g_{st}$ to mean that $f_{st} \leqslant C\, g_{st}$ for all $0 \leqslant s \leqslant t \leqslant T$, where $C < \infty$ is a suitable random constant.*

**Remark 3.1.** (Young vs. Rough case) The restriction $\alpha > \frac{1}{2}$ for the study of the difference equation (3.1) has a substantial reason, namely *there is no solution to (3.1) for general $X \in \mathcal{C}^\alpha$ with $\alpha \leqslant \frac{1}{2}$*. Indeed, taking the "increment" $\delta$ of both sides of (3.1) and recalling (1.23) and (1.32), we obtain

$$(\sigma(Z_u) - \sigma(Z_s))\,(X_t - X_u) = o(t-s) \qquad \text{for } 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{3.2}$$

If $X \in \mathcal{C}^\alpha$, for any $\alpha \in (0,1]$, then we know from Lemma 2.6 that $Z \in \mathcal{C}^\alpha$, but not better in general (e.g. when $\sigma(\cdot) \equiv c$ is constant we have $Z = c\,X$), hence the LHS of (3.2) is $\lesssim (u-s)^\alpha\,(t-u)^\alpha \lesssim (t-s)^{2\alpha}$, but not better in general. This shows that the restriction $\alpha > \frac{1}{2}$ is generally necessary for (3.1) to have solutions.

### 3.1. Enhanced Taylor expansion

We fix $d, k \in \mathbb{N}$, a time horizon $T > 0$ and a sufficiently regular function $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$. Our goal is to give a meaning to the integral equation

$$Z_t = Z_0 + \int_0^t \sigma(Z_s)\,\dot{X}_s\,\mathrm{d}s, \qquad 0 \leqslant t \leqslant T, \tag{3.3}$$

where $Z: [0,T] \to \mathbb{R}^k$ is the unknown and $X: [0,T] \to \mathbb{R}^d$ is a non smooth path, more precisely $X \in \mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$.

The difference equation (3.1) is no longer enough, for the crucial reason that typically *it admits no solutions for* $\alpha \leqslant \frac{1}{2}$, see Remark 3.1. We are going to solve this problem by *enriching the RHS of (3.1)* in a suitable, but non canonical way: this leads to the key notion of *rough path* which is central in this book.

To provide motivation, suppose for the moment that $X$ is continuously differentiable, so that (3.3) is meaningful. As we saw in (1.3), an integration yields for $s<t$

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + \int_s^t (\sigma(Z_u) - \sigma(Z_s))\,\dot{X}_u\,\mathrm{d}u. \tag{3.4}$$

In Chapter 1 we observed that the integral is $o(t-s)$, which leads to the difference equation (3.1). More precisely, the integral is $O((t-s)^2)$ if $X \in C^1$ and $\sigma$ is locally Lipschitz (note that $Z \in C^1$). The idea is now to go further, expanding the integral to get a more accurate local description, with a better remainder $O((t-s)^3)$.

To this purpose, we assume that $\sigma$ is differentiable and we introduce the key function $\sigma_2 \colon \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^d)^*$ by

$$\sigma_2(z) := \nabla\sigma(z)\,\sigma(z), \qquad \text{i.e.} \qquad [\sigma_2(z)]^i_{j\ell} := \sum_{a=1}^k \frac{\partial \sigma^i_j}{\partial z_a}(z)\,\sigma^a_\ell(z)\,. \tag{3.5}$$

Since $\frac{\mathrm{d}}{\mathrm{d}r}\sigma(Z_r) = \nabla\sigma(Z_r)\,\dot{Z}_r = \sigma_2(Z_r)\,\dot{X}_r$ by (3.3), we can write for $s<u$

$$\begin{aligned}\sigma(Z_u) - \sigma(Z_s) &= \int_s^u \sigma_2(Z_r)\,\dot{X}_r\,\mathrm{d}r \\ &= \sigma_2(Z_s)\,(X_u - X_s) + \int_s^u (\sigma_2(Z_r) - \sigma_2(Z_s))\,\dot{X}_r\,\mathrm{d}r, \end{aligned}\tag{3.6}$$

where for $z \in \mathbb{R}^d$ and $a \in \mathbb{R}^d$ we define $\sigma_2(z)\,a \in \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ by

$$[\sigma_2(z)\,a]^i_j = \sum_{\ell=1}^d [\sigma_2(z)]^i_{j\ell}\,a^\ell.$$

If we assume that $\sigma_2$ is locally Lipschitz, then the last integral in (3.6) is $O((u-s)^2)$ (recall that $X \in C^1$). Plugging this into (3.4), we then obtain

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + \sigma_2(Z_s)\int_s^t (X_u - X_s) \otimes \dot{X}_u\,\mathrm{d}u + O((t-s)^3), \tag{3.7}$$

where now for $z \in \mathbb{R}^d$ and $B \in \mathbb{R}^d \otimes \mathbb{R}^d$ we define $\sigma_2(z)\,B \in \mathbb{R}^k$ by

$$[\sigma_2(z)\,B]^i = \sum_{\ell,m=1}^d [\sigma_2(z)]^i_{\ell m}\,B^{m\ell}. \tag{3.8}$$

Let us rewrite the integral in the right-hand side of (3.7) more conveniently. To this purpose we introduce the shorthands

$$\mathbb{X}^1_{st} := X_t - X_s\,, \qquad \mathbb{X}^2_{st} := \int_s^t (X_r - X_s) \otimes \dot{X}_r\,\mathrm{d}r, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{3.9}$$

so that $\mathbb{X}^1: [0,T]^2_\leqslant \to \mathbb{R}^d$ and $\mathbb{X}^2: [0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$, see (1.7). More explicitly:

$$(\mathbb{X}^2_{st})^{ij} := \int_s^t (X^i_r - X^i_s)\, \dot{X}^j_r\, \mathrm{d}r, \qquad i,j \in \{1,\ldots,d\}\,.$$

We can thus rewrite (3.7), replacing $O((t-s)^3)$ by $o(t-s)$, in the compact form

$$Z_t - Z_s = \sigma(Z_s)\, \mathbb{X}^1_{st} + \sigma_2(Z_s)\, \mathbb{X}^2_{st} + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{3.10}$$

where for the product $\sigma_2(Z_s)\, \mathbb{X}^2_{st}$ we use the contraction rule (3.8).

We have obtained an *enhanced Taylor expansion*: comparing with (3.1), we added a "second order term" containing $\mathbb{X}^2_{st}$. The idea is to take this new difference equation, that we call *rough difference equation*, as a generalized formulation of the differential equation (3.3), just as we did in Chapter 1 (see Section 1.2). However, there is a problem: the term $\mathbb{X}^2_{st}$ *depends on the derivative* $\dot{X}$, see (3.9), so it is not clearly defined for a non-differentiable $X$.

To overcome this problem, we will *assign* a suitable function $\mathbb{X}^2 = (\mathbb{X}^2_{st})_{0\leqslant s\leqslant t\leqslant T}$ playing the role of the integral (3.9) when $X$ is not differentiable: this leads to the notion of *rough paths*, defined in the next section and studied in depth in Chapter 8. We will show in this chapter that rough paths are the key to a robust solution theory of rough difference equations when $X$ of class $\mathcal{C}^\alpha$ with $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$.

## 3.2. Rough paths

Let us fix a path $X: [0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$ with $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$. Motivated by the previous section, we are going to reformulate the ill-posed integral equation (3.3) as the difference equation (3.10), which contains $\mathbb{X}^1$ and $\mathbb{X}^2$.

We can certainly define $\mathbb{X}^1_{st} := X_t - X_s$ as in (3.9), but there is no canonical definition of $\mathbb{X}^2_{st} = \int_s^t (X_r - X_s) \otimes \dot{X}_r\, \mathrm{d}r$, since $X$ may not be differentiable. We therefore *assign* a function $\mathbb{X}^2_{st}$ which satisfies *suitable properties*. Note that when $X$ is continuously differentiable the function $\mathbb{X}^2$ in (3.9) satisfies:

- an algebraic identity known as *Chen's relation*: for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

  $$\mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut} = (X_u - X_s) \otimes (X_t - X_u)\,, \tag{3.11}$$

  which follows from (3.9) noting that

  $$\mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \int_u^t (X_r - X_s) \otimes \dot{X}_r\, \mathrm{d}r = (X_u - X_s) \otimes (X_t - X_u)\,;$$

- the analytic bounds

  $$|\mathbb{X}^1_{st}| \lesssim |t-s|\,, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^2, \tag{3.12}$$

  which follow from the fact that $\dot{X}$ is bounded.

The algebraic relation (3.11) is still meaningful for non-differentiable $X$, while the analytic bounds (3.12) can naturally be adapted to the case of Hölder paths $X \in \mathcal{C}^\alpha$ by changing the exponents $1, 2$ to $\alpha, 2\alpha$. This leads to the following key definition.

DEFINITION 3.2. (ROUGH PATHS) *Fix* $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$, $d \in \mathbb{N}$ *and a path* $X: [0,T] \to \mathbb{R}^d$ *of class* $\mathcal{C}^\alpha$. *An* ***α-rough path over X*** *is a pair* $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ *where the functions* $\mathbb{X}^1: [0,T]^2_\leqslant \to \mathbb{R}^d$ *and* $\mathbb{X}^2: [0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$ *satisfy, for* $0 \leqslant s \leqslant u \leqslant t \leqslant T$:

- *the algebraic relations*

$$\mathbb{X}^1_{st} = X_t - X_s\,, \qquad \delta\mathbb{X}^2_{sut} := \mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}\,, \tag{3.13}$$

  *where the second identity is called Chen's relation;*

- *the analytic bounds*

$$|\mathbb{X}^1_{st}| \lesssim |t-s|^\alpha\,, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}\,. \tag{3.14}$$

*We call* $\mathcal{R}_{\alpha,d}(X)$ *the set of d-dimensional α-rough paths* $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ *over X and* $\mathcal{R}_{\alpha,d} = \bigcup_{X \in \mathcal{C}^\alpha} \mathcal{R}_{\alpha,d}(X)$ *the set of all d-dimensional α-rough paths.*

When $X$ is of class $C^1$, the choice (3.9) yields by (3.11)-(3.12) a $\alpha$-rough path for any $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$ which we call the *canonical rough path*, see Section 8.7 below.

When $X = B$ is Brownian motion, the theory of stochastic integration provides a natural candidate for $\mathbb{X}^2$, in fact *multiple candidates* (think of Ito vs. Stratonovich integration), as we discuss in Chapter 4 below. Incidentally, this makes it clear that the construction of $\mathbb{X}^2$ is in general *non canonical*, i.e. there are multiple choices of $\mathbb{X}^2$ for a given path $X$. *This is a strength of the theory of rough paths*, since it allows to treat different non equivalent forms of integration.

**Remark 3.3.** The existence of rough paths over any given path $X$ (i.e. the fact that $\mathcal{R}_{\alpha,d}(X) \neq \emptyset$) is a non trivial fact, which will be proved in Chapter 8.

**Remark 3.4.** ($\mathbb{X}^2$ AS A "PATH") The two-parameters function $\mathbb{X}^2_{st}$ is determined by the one-parameter function

$$\mathbb{I}_t := \mathbb{X}^2_{0t} + X_0 \otimes (X_t - X_0)\,, \tag{3.15}$$

which intuitively describes the integral $\int_0^t X_r \otimes \dot{X}_r \,\mathrm{d}r$. Indeed, we can write

$$\mathbb{X}^2_{st} = \mathbb{I}_t - \mathbb{I}_s - X_s \otimes (X_t - X_s)\,, \tag{3.16}$$

since $\mathbb{X}^2_{st} = \mathbb{X}^2_{0t} - \mathbb{X}^2_{0s} - (X_s - X_0) \otimes (X_t - X_s)$ by Chen's relation (3.13).

Vice versa, given a function $\mathbb{I}: [0,T] \to \mathbb{R}^d$, if we *define* $\mathbb{X}^2$ by (3.16), then Chen's relation (3.13) is automatically satisfied (recall (1.32)). In order to satisfy the analytic bound in (3.14), we must require that

$$|\mathbb{I}_t - \mathbb{I}_s - X_s \otimes (X_t - X_s)| \lesssim (t-s)^{2\alpha}\,, \tag{3.17}$$

which is a natural estimate if $\mathbb{I}_t - \mathbb{I}_s$ should describe "$= \int_s^t X_r \otimes \dot{X}_r \,\mathrm{d}r$".

Summarizing: given any path $X: [0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$, *it is equivalent to assign* $\mathbb{X}^2: [0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$ *satisfying (3.13)-(3.14) or to assign* $\mathbb{I}: [0,T] \to \mathbb{R}^d$ *satisfying (3.17)*, the correspondence being given by (3.15)-(3.16).

## 3.3. Rough difference equations

Given a time horizon $T>0$ and two dimensions $d,k\in\mathbb{N}$, let us fix:

- a path $X\colon[0,T]\to\mathbb{R}^d$ of class $\mathcal{C}^\alpha$ with $\alpha\in\,]\frac{1}{3},\frac{1}{2}]$;
- an $\alpha$-rough path $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ over $X$, see Definition 3.2;
- a *differentiable* function $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$, which lets us define the function

$$\sigma_2\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*\otimes(\mathbb{R}^d)^* \qquad (\text{see } (3.5))\,.$$

Motivated by the previous discussions, see in particular (3.10), we study in this chapter the following *rough difference equation* for an unknown path $Z\colon[0,T]\to\mathbb{R}^k$:

$$\delta Z_{st}=\sigma(Z_s)\,\mathbb{X}^1_{st}+\sigma_2(Z_s)\,\mathbb{X}^2_{st}+o(t-s),\qquad 0\leqslant s\leqslant t\leqslant T, \tag{3.18}$$

where we recall the increment notation $\delta Z_{st}:=Z_t-Z_s$ and the contraction rule (3.8), and we stress that $o(t-s)$ is *uniform* for $0\leqslant s\leqslant t\leqslant T$, see Remark 1.1. In analogy with (2.3)-(2.4), a solution of (3.18) is a path $Z\colon[0,T]\to\mathbb{R}^k$ such that

$$Z^{[3]}_{st}:=\delta Z_{st}-\sigma(Z_s)\,\mathbb{X}^1_{st}-\sigma_2(Z_s)\,\mathbb{X}^2_{st}=o(t-s)\,. \tag{3.19}$$

We stress that the rough difference equation (3.18) is a generalization of the integral equation (3.3), as we show in the next result.

Proposition 3.5. *If $X$ and $\sigma$ are of class $C^1$ and $\sigma_2$ is locally Lipschitz (e.g. if $\sigma$ is of class $C^2$), then any solution $Z$ to the integral equation (3.3) satisfies the difference equation (3.18) for the canonical rough path $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ in (3.9).*

**Proof.** If $X\in C^1$, then $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ defined in (3.9) is an $\alpha$-rough path over $X$ for any $\alpha\in\,]\frac{1}{3},\frac{1}{2}]$, as we showed in (3.11)-(3.12). Given a solution $Z$ of (3.3), if $\sigma_2$ is locally Lipschitz we derived the Taylor expansion (3.10), hence (3.18) holds. ☐

We now state *local and global existence, uniqueness of solutions and continuity of the solution map* for the rough difference equation (3.18) under natural assumptions on $\sigma$ and $\sigma_2$, summarizing the main results of this chapter. We refer to the next sections for more precise and quantitative results.

**To be completed.**

Proposition 3.6. *Let $z_0\in\mathbb{R}^d$. We suppose that $\sigma$ and $\sigma_2$ are of class $C^1$ and globally Lipschitz, namely $\|\nabla\sigma\|_\infty+\|\nabla\sigma_2\|_\infty<+\infty$. Let $D:=\max\{1,\|\nabla\sigma\|_\infty,\|\nabla\sigma_2\|_\infty\}$ and $M>0$.*

*There exists $T_{M,D,\alpha}>0$ such that, for all $T\in(0,T_{M,D,\alpha})$ and $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)\in\mathcal{R}_{\alpha,d}$ such that $\|\mathbb{X}^1\|_\alpha+\|\mathbb{X}^2\|_{2\alpha}\leqslant M$, there exists a solution $Z$ to (3.19) on the interval $[0,T]$ such that $Z_0=z_0$ and*

$$\|Z\|_\alpha\leqslant 15\,M(|\sigma(z_0)|+|\sigma_2(z_0)|). \tag{3.20}$$

The proof of this Proposition, based on a discretization argument, is postponed to section 3.9 below.

We are going to use the Sewing Bound (1.26), its weighted version (1.41) and its discrete formulation (1.45).

## 3.4. SET-UP

We recall that the *weighted semi-norms* $\|\cdot\|_{\eta,\tau}$ are defined in (1.33)-(1.34). We are going to use the various properties that we recalled in Section 2.2, see in particular (2.5), (2.6) and (2.7)-(2.8), as well as the natural generalization

$$\text{if} \quad F_{sut} = G_{su} H_{ut} \quad \text{then} \qquad \|F\|_{3\eta,\tau} \begin{cases} \leqslant \|G\|_{2\eta,\tau} \|H\|_{\eta}, \\ \leqslant \|G\|_{\eta,\tau} \|H\|_{2\eta}. \end{cases} \tag{3.21}$$

In all these bounds, whenever there is a product, *only one factor gets the weighted semi-norm, while the other factor gets the ordinary semi-norm*. We sometimes need to introduce an additional weight, which is possible applying (2.9).

In Chapter 2 a key tool to study the Young difference equation (2.4) was the estimate on the “difference of increments” in Lemma 2.8. This tool is still crucial in this chapter, but we will need an additional ingredient that we now present.

LEMMA 3.7. (TAYLOR IDENTITY) *Let $z_1, z_2 \in \mathbb{R}^k$ and $x \in \mathbb{R}^d$. If $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is of class $C^1$, defining $\sigma_2: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^d)^*$ by (3.5) and setting $\delta z_{12} := z_2 - z_1$, we have the identities*

$$\begin{aligned} &\sigma(z_2) - \sigma(z_1) - \sigma_2(z_1)\, x \\ &\quad = \nabla\sigma(z_1)(\delta z_{12} - \sigma(z_1)\, x) + \int_0^1 [(\nabla\sigma(z_1 + r\,\delta z_{12}) - \nabla\sigma(z_1))\,\delta z_{12}]\,\mathrm{d}r, \end{aligned} \tag{3.22}$$

*and*

$$\begin{aligned} \sigma(z_2) - \sigma(z_1) - \sigma_2(z_1)\, x \;=\; & \int_0^1 [(\sigma_2(z_1 + r\,\delta z_{12}) - \sigma_2(z_1))\, x]\,\mathrm{d}r \\ & + \int_0^1 [\nabla\sigma(z_1 + r\,\delta z_{12})\,(\delta z_{12} - \sigma(z_1)\, x)]\,\mathrm{d}r \\ & - \int_0^1 \nabla\sigma(z_1 + r\delta z_{12}) \left( \int_0^r [\nabla\sigma(z_1 + v\,\delta z_{12})\,\delta z_{12}\, x]\,\mathrm{d}v \right) \mathrm{d}r. \end{aligned} \tag{3.23}$$

**Proof.** The first formula is based on elementary manipulations and on the fact that

$$\sigma(z_2) - \sigma(z_1) = \int_0^1 [\nabla\sigma(z_1 + r\,\delta z_{12})\,\delta z_{12}]\,\mathrm{d}r.$$

For the second formula, setting $\delta z := \delta z_{12}$ for short, we similarly write

$$\begin{aligned} \sigma(z_2) - \sigma(z_1) \;&=\; \int_0^1 [\nabla\sigma(z_1 + r\,\delta z)\,\delta z]\,\mathrm{d}r \\ &=\; \int_0^1 [\nabla\sigma(z_1 + r\,\delta z)\,(\delta z - \sigma(z_1)\, x)]\,\mathrm{d}r + \underbrace{\int_0^1 [\nabla\sigma(z_1 + r\,\delta z)\,\sigma(z_1)\, x]\,\mathrm{d}r}_{A} \end{aligned}$$

and then, recalling the definition (3.5) of $\sigma_2$,

$$A = \int_0^1 [\sigma_2(z_1 + r\,\delta z)\,x]\,\mathrm{d}r - \underbrace{\int_0^1 [\nabla\sigma(z_1 + r\,\delta z)\,(\sigma(z_1 + r\,\delta z) - \sigma(z_1))\,x]\,\mathrm{d}r}_{B}.$$

Finally

$$B \;=\; \int_0^1 \nabla\sigma(z_1 + r\,\delta z)\left(\int_0^r [\nabla\sigma(z_1 + v\,\delta z)\,\delta z\,x]\,\mathrm{d}v\right)\mathrm{d}r$$

from which (3.23) follows easily. □

We will see below that (3.22) is useful for the comparison between *two solutions*, as in the proofs of uniqueness (Theorem 3.10) and continuity of the solution map (Theorem 3.11), while (3.23) is well suited for a priori estimates on a *single solution* (Theorem 3.9) or on a discretization scheme (Lemma 3.13).

## 3.5. A PRIORI ESTIMATES

In this section we prove *a priori estimates* for solutions of the rough difference equation (3.18) for *globally Lipschitz* $\sigma$ and $\sigma_2$, i.e. $\|\nabla\sigma\|_\infty < \infty$ and $\|\nabla\sigma_2\|_\infty < \infty$. A sufficient condition is that $\sigma$, $\nabla\sigma$, $\nabla^2\sigma$ are bounded, see (3.5), but it is interesting that *boundedness of $\sigma$ is not necessary* (think of the case of linear $\sigma$).

Given a solution $Z$ of (3.18), we define the "remainders" $Z^{[3]}$ and $Z^{[2]}$ by

$$Z^{[3]}_{st} = \delta Z_{st} - \sigma(Z_s)\,\mathbb{X}^1_{st} - \sigma_2(Z_s)\,\mathbb{X}^2_{st}\,, \qquad Z^{[2]}_{st} = \delta Z_{st} - \sigma(Z_s)\,\mathbb{X}^1_{st}\,. \tag{3.24}$$

Let us first show, by easy arguments, that any solution $Z$ of (3.18) has the same Hölder regularity $\mathcal{C}^\alpha$ of the driving path $X$ (in analogy with Lemmas 1.2 and 2.6), and that the "level 2 remainder" $Z^{[2]}_{st}$ is in $C^{2\alpha}_2$, that is $|Z^{[2]}_{st}| \lesssim (t-s)^{2\alpha}$.

LEMMA 3.8. (HÖLDER REGULARITY) *Let $\sigma$ be of class $C^1$ and let $Z$ be a solution of (3.18). There is a constant $C = C(Z) < \infty$ such that*

$$\begin{cases} |Z^{[2]}_{st}| \leqslant C\,|\mathbb{X}^2_{st}| + o(t-s), \\ |\delta Z_{st}| \leqslant C\,(|\mathbb{X}^1_{st}| + |\mathbb{X}^2_{st}|) + o(t-s), \end{cases} \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{3.25}$$

*In particular, if $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ is an $\alpha$-rough path, then $Z^{[2]} \in C^{2\alpha}_2$ and $Z$ is of class $\mathcal{C}^\alpha$.*

**Proof.** If $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ is an $\alpha$-rough path, then by the first bound in (3.25) we have $|Z^{[2]}_{st}| \lesssim (t-s)^{2\alpha} + o(t-s) \lesssim (t-s)^{2\alpha}$, that is $Z^{[2]} \in C^{2\alpha}_2$. Similarly, the second bound in (3.25) gives $|\delta Z_{st}| \lesssim (t-s)^\alpha + (t-s)^{2\alpha} + o(t-s) \lesssim (t-s)^\alpha$, that is $Z$ is of class $\mathcal{C}^\alpha$.

It remains to prove (3.25). This follows by (3.18) with $C := \sup_{0\leqslant s\leqslant T}\{|\sigma(Z_s)| + |\sigma_2(Z_s)|\}$, so we need to show that $C < \infty$. Since $\sigma$ and $\sigma_2$ are continuous (because $\sigma$ is of class $C^1$), it is enough to prove that $Z$ is bounded: $\sup_{0\leqslant t\leqslant T}|Z_t| < \infty$.

Arguing as in the proof of Lemma 1.2, we fix $\bar{\delta} > 0$ such that $|o(t-s)| \leqslant 1$ for all $0 \leqslant s \leqslant t \leqslant T$ with $|t-s| \leqslant \bar{\delta}$. Since $[0,T]$ is a finite union of intervals $[\bar{s}, \bar{t}]$ with $\bar{t} - \bar{s} \leqslant \bar{\delta}$, we may focus on one such interval: by (3.18) we can bound

$$\sup_{t\in[\bar{s},\bar{t}]} |Z_t| \leqslant |Z_{\bar{s}}| + |\sigma(Z_{\bar{s}})| \sup_{t\in[\bar{s},\bar{t}]} |\mathbb{X}^1_{st}| + |\sigma_2(Z_{\bar{s}})| \sup_{t\in[\bar{s},\bar{t}]} |\mathbb{X}^2_{st}| + 1 < \infty.$$

This completes the proof that $\sup_{0\leqslant t\leqslant T}|Z_t|<\infty$. □

We next get to our main a priori estimates, showing in particular that the "level 3 remainder" $Z_{st}^{[3]}$ is in $C_2^{3\alpha}$, that is $|Z_{st}^{[3]}|\lesssim|t-s|^{3\alpha}$. Let us first record a useful computation: recalling (1.23) and (1.32), by $\delta\circ\delta=0$ and (3.13), we have

$$\begin{aligned}\delta Z_{sut}^{[3]} &= Z_{st}^{[3]}-Z_{su}^{[3]}-Z_{ut}^{[3]}\\ &= \underbrace{(\sigma(Z_u)-\sigma(Z_s)-\sigma_2(Z_s)\,\mathbb{X}_{su}^1)}_{B_{su}}\mathbb{X}_{ut}^1+(\sigma_2(Z_u)-\sigma_2(Z_s))\,\mathbb{X}_{ut}^2\,. \end{aligned}\tag{3.26}$$

THEOREM 3.9. (ROUGH A PRIORI ESTIMATES) *Let $X$ be of class $\mathcal{C}^\alpha$ with $\alpha\in\left]\frac{1}{3},\frac{1}{2}\right]$ and let $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ be an $\alpha$-rough path over $X$. Let $\sigma$ and $\sigma_2$ be globally Lipschitz.*

*For any solution $Z$ of (3.18), recalling the "remainders" $Z^{[3]}$ and $Z^{[2]}$ from (3.24), we have $Z^{[3]}\in C_2^{3\alpha}$: more precisely, for any $\tau>0$,*

$$\|Z^{[3]}\|_{3\alpha,\tau}\leqslant K_{3\alpha}\,c'_{\alpha,\mathbb{X},\sigma}\,(\|\delta Z\|_{\alpha,\tau}+\|Z^{[2]}\|_{2\alpha,\tau})\,,\tag{3.27}$$

*where we recall that $K_{3\alpha}=(1-2^{1-3\alpha})^{-1}$ and we define the constant*

$$c'_{\alpha,\mathbb{X},\sigma}:=\|\nabla\sigma\|_\infty\,\|\mathbb{X}^1\|_\alpha+\|\nabla\sigma_2\|_\infty\,\|\mathbb{X}^2\|_{2\alpha}+(\|\nabla\sigma\|_\infty^2+\|\nabla\sigma_2\|_\infty)\,\|\mathbb{X}^1\|_\alpha^2\,.\tag{3.28}$$

*Moreover, if either $T$ or $\tau$ is small enough, we have*

$$\|\delta Z\|_{\alpha,\tau}+\|Z^{[2]}\|_{2\alpha,\tau}\leqslant 2\,(\sigma(Z_0)\,\|\mathbb{X}^1\|_\alpha+\sigma_2(Z_0)\,\|\mathbb{X}^2\|_{2\alpha})\quad \text{for } (T\wedge\tau)^\alpha\leqslant\varepsilon'_{\alpha,\mathbb{X},\sigma}\,,\tag{3.29}$$

*where we set*

$$\varepsilon'_{\alpha,\mathbb{X},\sigma}:=\frac{1}{4\,(K_{3\alpha}+3)\,(c'_{\alpha,\mathbb{X},\sigma}+1)}\,.\tag{3.30}$$

**Proof.** Let us prove (3.27). Since $3\alpha>1$ and $Z_{st}^{[3]}=o(t-s)$, see (3.19), we can apply the weighted Sewing Bound (1.41) which gives $\|Z^{[3]}\|_{3\alpha,\tau}\leqslant K_{3\alpha}\,\|\delta Z^{[3]}\|_{3\alpha,\tau}$. It remains to estimate $\delta Z^{[3]}$ from (3.26): applying (3.21) we can write

$$\|\delta Z^{[3]}\|_{3\alpha,\tau}\leqslant\|B\|_{2\alpha,\tau}\,\|\mathbb{X}^1\|_\alpha+\|\delta\sigma_2(Z)\|_{\alpha,\tau}\,\|\mathbb{X}^2\|_{2\alpha}\,.\tag{3.31}$$

We now focus on $B_{su}$ from (3.26): by (3.23) we have

$$\begin{aligned}B_{su} &= \int_0^1[(\sigma_2(Z_s+u\,\delta Z_{su})-\sigma_2(Z_s))\,\mathbb{X}_{su}^1]\,\mathrm{d}u+\int_0^1[\nabla\sigma(Z_s+u\,\delta Z_{su})\,Z_{su}^{[2]}]\,\mathrm{d}u\\ &\quad-\int_0^1\nabla\sigma(Z_s+u\,\delta Z_{su})\left(\int_0^u[\nabla\sigma(Z_s+v\,\delta Z_{su})\,\delta Z_{su}\,\mathbb{X}_{su}^1]\,\mathrm{d}v\right)\mathrm{d}u\,,\end{aligned}$$

so that, by (2.8),

$$\|B\|_{2\alpha,\tau}\leqslant(\|\nabla\sigma_2\|_\infty+\|\nabla\sigma\|_\infty^2)\,\|\mathbb{X}^1\|_\alpha\,\|\delta Z\|_{\alpha,\tau}+\|\nabla\sigma\|_\infty\,\|Z^{[2]}\|_{2\alpha,\tau}\,.\tag{3.32}$$

We can plug this estimate into (3.31), together with the elementary bound

$$\|\delta\sigma_2(Z)\|_{\alpha,\tau}\leqslant\|\nabla\sigma_2\|_\infty\,\|\delta Z\|_{\alpha,\tau}\,.\tag{3.33}$$

Recalling that $\|Z^{[3]}\|_{3\alpha,\tau} \leqslant K_{3\alpha} \|\delta Z^{[3]}\|_{3\alpha,\tau}$, we have proved (3.27)-(3.28).

We next prove (3.29), for which we need to estimate $Z^{[2]}$ and $\delta Z$. Writing $Z_{st}^{[2]} = \sigma_2(Z_s)\,\mathbb{X}_{st}^2 + Z_{st}^{[3]}$ and setting $\varepsilon := (\tau \wedge T)^\alpha$ for short, we can bound by (2.6) and (2.7)

$$\|Z^{[2]}\|_{2\alpha,\tau} \leqslant \|\sigma_2(Z)\|_{\infty,\tau}\,\|\mathbb{X}^2\|_{2\alpha} + \varepsilon\,\|Z^{[3]}\|_{3\alpha,\tau}\,.$$

By (2.5) we have $\|\sigma_2(Z)\|_{\infty,\tau} \leqslant \sigma_2(Z_0) + 3\varepsilon\,\|\delta\sigma_2(Z)\|_{\alpha,\tau}$ and we can bound $\|\delta\sigma_2(Z)\|_{\alpha,\tau}$ by (3.33). Applying (3.27) and recalling (3.28), we then obtain

$$\begin{aligned}\|Z^{[2]}\|_{2\alpha,\tau} &\leqslant \sigma_2(Z_0)\,\|\mathbb{X}^2\|_{2\alpha} + \varepsilon\,(K_{3\alpha}+3)\,c'_{\alpha,\mathbb{X},\sigma}\,(\|\delta Z\|_{\alpha,\tau} + \|Z^{[2]}\|_{2\alpha,\tau})\\ &\leqslant \sigma_2(Z_0)\,\|\mathbb{X}^2\|_{2\alpha} + \frac{1}{4}\,\frac{\varepsilon}{\varepsilon'_{\alpha,\mathbb{X},\sigma}}\,(\|\delta Z\|_{\alpha,\tau} + \|Z^{[2]}\|_{2\alpha,\tau})\,,\end{aligned} \tag{3.34}$$

where we recall that $\varepsilon'_{\alpha,\mathbb{X},\sigma}$ is defined in (3.30).

Similarly, writing $\delta Z_{st} = \sigma(Z_s)\,\mathbb{X}_{st}^1 + Z_{st}^{[2]}$ we can bound, by (2.6) and (2.7),

$$\|\delta Z\|_{\alpha,\tau} \leqslant \|\sigma(Z)\|_{\infty,\tau}\,\|\mathbb{X}^1\|_\alpha + \varepsilon\,\|Z^{[2]}\|_{2\alpha,\tau},$$

and since $\|\sigma(Z)\|_{\infty,\tau} \leqslant \sigma(Z_0) + 3\,\varepsilon\,\|\delta\sigma(Z)\|_{\alpha,\tau} \leqslant \sigma(Z_0) + 3\,\varepsilon\,\|\nabla\sigma\|_\infty\,\|\delta Z\|_{\alpha,\tau}$ we get, recalling (3.28),

$$\begin{aligned}\|\delta Z\|_{\alpha,\tau} &\leqslant \sigma(Z_0)\,\|\mathbb{X}^1\|_\alpha + 3\,\varepsilon\,c'_{\alpha,\mathbb{X},\sigma}\,\|\delta Z\|_{\alpha,\tau} + \varepsilon\,\|Z^{[2]}\|_{2\alpha,\tau}\\ &\leqslant \sigma(Z_0)\,\|\mathbb{X}^1\|_\alpha + \frac{1}{4}\,\frac{\varepsilon}{\varepsilon'_{\alpha,\mathbb{X},\sigma}}\,\|\delta Z\|_{\alpha,\tau} + \varepsilon\,\|Z^{[2]}\|_{2\alpha,\tau}\,.\end{aligned} \tag{3.35}$$

Finally, for $\varepsilon \leqslant \varepsilon'_{\alpha,\mathbb{X},\sigma}$ (hence $\varepsilon \leqslant \frac{1}{4}$, see (3.28)), by (3.34) and (3.35) we obtain

$$\|\delta Z\|_{\alpha,\tau} + \|Z^{[2]}\|_{2\alpha,\tau} \leqslant \sigma(Z_0)\,\|\mathbb{X}^1\|_\alpha + \sigma_2(Z_0)\,\|\mathbb{X}^2\|_{2\alpha} + \frac{1}{2}\,(\|\delta Z\|_{\alpha,\tau} + \|Z^{[2]}\|_{2\alpha,\tau})\,.$$

Since $\|\delta Z\|_{\alpha,\tau} + \|Z^{[2]}\|_{2\alpha,\tau} < \infty$ by Lemma 3.8, we have proved (3.29). □

## 3.6. UNIQUENESS

In this section we prove uniqueness of solutions of (3.18) under the assumption that $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is *of class $C^\gamma$ with $\gamma > \frac{1}{\alpha}$* (e.g. it suffices that $\sigma$ is of class $C^3$). This implies that $\sigma_2$ from (3.5) is of class $C^1$ with locally $(\gamma-2)$-Hölder gradient $\nabla\sigma_2$. We stress that $\sigma$ and $\sigma_2$ are *not* required to be bounded.

THEOREM 3.10. (UNIQUENESS) *Let $X$ be of class $\mathcal{C}^\alpha$ with $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$, let $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ be an $\alpha$-rough path over $X$, and let $\sigma$ be of class $C^\gamma$ with $\gamma > \frac{1}{\alpha}$ (e.g. if $\sigma$ is of class $C^3$). Then for every $z_0 \in \mathbb{R}^k$ there exists at most one solution $Z$ to (3.18) such that $Z_0 = z_0$.*

**Proof.** Let us fix two solutions $Z, \bar{Z}$ of (3.18) and define their difference

$$Y := Z - \bar{Z}.$$

Our goal is to show that, for $\tau > 0$ small, we have $\|Y\|_{\infty,\tau} \leqslant 2\,|Y_0|$. In particular, if $Z_0 = \bar{Z}_0$, then $Y_0 = 0$ and therefore $\|Y\|_{\infty,\tau} = 0$, i.e. $Z = \bar{Z}$, which completes the proof.

We know by (2.5) that

$$\|Y\|_{\infty,\tau} \leqslant |Y_0| + 3\,\tau^{\alpha}\,\|\delta Y\|_{\alpha,\tau}\,. \tag{3.36}$$

With some abuse of notation, we denote by $Y_{st}^{[2]} := Z_{st}^{[2]} - \bar{Z}_{st}^{[2]}$ and $Y_{st}^{[3]} := Z_{st}^{[3]} - \bar{Z}_{st}^{[3]}$ the "differences of remainders", recall (3.24), so that we can write

$$\delta Y_{st} = (\sigma(Z_s) - \sigma(\bar{Z}_s))\,\mathbb{X}_{st}^1 + Y_{st}^{[2]}, \tag{3.37}$$

$$Y_{st}^{[2]} = (\sigma_2(Z_s) - \sigma_2(\bar{Z}_s))\,\mathbb{X}_{st}^2 + Y_{st}^{[3]}. \tag{3.38}$$

We are going to show that, for $\tau > 0$ small enough, the following bounds hold:

$$\|\delta Y\|_{\alpha,\tau} \leqslant c_1\,\|Y\|_{\infty,\tau} + \tau^{\alpha}\,\|Y^{[2]}\|_{2\alpha,\tau}\,, \tag{3.39}$$

$$\|Y^{[2]}\|_{2\alpha,\tau} \leqslant c_2\,\|Y\|_{\infty,\tau} + \tau^{(\gamma-2)\alpha}\,\|Y^{[3]}\|_{\gamma\alpha,\tau}\,, \tag{3.40}$$

$$\|Y^{[3]}\|_{\gamma\alpha,\tau} \leqslant c_3\,\|Y\|_{\infty,\tau} + c_3'\,\tau^{(\gamma-2)\alpha}\,\|Y^{[3]}\|_{\gamma\alpha,\tau}\,, \tag{3.41}$$

for suitable constants $c_i, c_i'$ that may depend on $Z, \bar{Z}, \mathbb{X}^1, \mathbb{X}^2, \sigma$, *but not on* $\tau$.

We can easily complete the proof, assuming (3.39)-(3.41): if we fix $\tau > 0$ small enough so that $c_3'\,\tau^{(\gamma-2)\alpha} < \frac{1}{2}$, by (3.41) we have $\|Y^{[3]}\|_{\gamma\alpha,\tau} \leqslant 2\,c_3\,\|Y\|_{\infty,\tau}$; plugging this into (3.40) and taking $\tau > 0$ small, we obtain $\|Y^{[2]}\|_{2\alpha,\tau} \leqslant 2\,c_2\,\|Y\|_{\infty,\tau}$, which plugged into (3.39) yields $\|\delta Y\|_{\alpha,\tau} \leqslant 2\,c_1\,\|Y\|_{\infty,\tau}$, for $\tau > 0$ is small enough. Finally, by (3.36) we obtain, for $\tau > 0$ small, our goal $\|Y\|_{\infty,\tau} \leqslant 2\,|Y_0|$.

It remains to prove (3.39)-(3.41). Recalling (2.18), let us define the constants

$$C_1' := C_{\nabla\sigma, \|Z\|_\infty \vee \|\bar{Z}\|_\infty}, \qquad C_1'' := C_{\nabla^2\sigma, \|Z\|_\infty \vee \|\bar{Z}\|_\infty}, \qquad C_2' := C_{\nabla\sigma_2, \|Z\|_\infty \vee \|\bar{Z}\|_\infty},$$

$$C_1''' := \sup\left\{\frac{|\nabla^2\sigma(x) - \nabla^2\sigma(y)|}{|x-y|^{\gamma-2}}\colon\ |x|, |y| \leqslant \|Z\|_\infty \vee \|\bar{Z}\|_\infty\right\},$$

$$C_2'' := \sup\left\{\frac{|\nabla\sigma_2(x) - \nabla\sigma_2(y)|}{|x-y|^{\gamma-2}}\colon\ |x|, |y| \leqslant \|Z\|_\infty \vee \|\bar{Z}\|_\infty\right\}.$$

(Note that $\|Z\|_\infty, \|\bar{Z}\|_\infty < \infty$ because $Z, \bar{Z}$ are continuous, see Lemma 3.8.)

We can prove (3.39) and (3.40) arguing as in the proof of Theorem 2.9, see (2.24) and (2.25). Indeed, from (3.37) we can bound, by (2.6) and (2.7),

$$\begin{aligned} \|\delta Y\|_{\alpha,\tau} &\leqslant \|\sigma(Z) - \sigma(\bar{Z})\|_{\infty,\tau}\,\|\mathbb{X}^1\|_\alpha + \tau^\alpha\,\|Y^{[2]}\|_{2\alpha,\tau} \\ &\leqslant C_1'\,\|Y\|_{\infty,\tau}\,\|\mathbb{X}^1\|_\alpha + \tau^\alpha\,\|Y^{[2]}\|_{2\alpha,\tau}\,, \end{aligned} \tag{3.42}$$

because $|\sigma(Z_t) - \sigma(\bar{Z}_t)| \leqslant C_1'\,|Z_t - \bar{Z}_t|$, hence (3.39) holds with $c_1 = C_1'\,\|\mathbb{X}^1\|_\alpha$. Similarly, by (3.38) we can bound

$$\begin{aligned} \|Y^{[2]}\|_{2\alpha,\tau} &\leqslant \|\sigma_2(Z) - \sigma_2(\bar{Z})\|_{\infty,\tau}\,\|\mathbb{X}^2\|_{2\alpha} + \tau^{(\gamma-2)\alpha}\,\|Y^{[3]}\|_{\gamma\alpha,\tau} \\ &\leqslant C_2'\,\|Y\|_{\infty,\tau}\,\|\mathbb{X}^2\|_{2\alpha} + \tau^{(\gamma-2)\alpha}\,\|Y^{[3]}\|_{\gamma\alpha,\tau}\,, \end{aligned} \tag{3.43}$$

because $|\sigma_2(Z_t) - \sigma_2(\bar{Z}_t)| \leqslant C_2'\,|Z_t - \bar{Z}_t|$, hence also (3.40) holds with $c_2 = C_2'\,\|\mathbb{X}^2\|_{2\alpha}$.

We finally prove (3.41). Since $Y_{st}^{[3]} = Z_{st}^{[3]} - \bar{Z}_{st}^{[3]} = o(t-s)$, see (3.19), we can bound $Z^{[3]}$ by its increment $\delta Z^{[3]}$ through the weighted Sewing Bound (1.41):

$$\|Y^{[3]}\|_{\gamma\alpha,\tau} \leqslant K_{\gamma\alpha} \|\delta Y^{[3]}\|_{\gamma\alpha,\tau} . \tag{3.44}$$

We are going to prove the following estimate:

$$\|\delta Y^{[3]}\|_{\gamma\alpha,\tau} \leqslant \tilde{c}_3 \|Y\|_{\infty,\tau} + \tilde{c}_3' \|\delta Y\|_{\alpha,\tau} + \tilde{c}_3'' \|Y^{[2]}\|_{2\alpha,\tau} , \tag{3.45}$$

for suitable constants $\tilde{c}_3, \tilde{c}_3'', \tilde{c}_3''$ that depend on $Z, \bar{Z}, \mathbb{X}^1, \mathbb{X}^2, \sigma$, *but not on* $\tau$. Plugging the estimates (3.39) and (3.40) (that we already proved) for $\|\delta Y\|_{\alpha,\tau}$ and $\|Y^{[2]}\|_{2\alpha,\tau}$, we obtain (3.41) for suitable (explicit) constants $c_3, c_3'$.

Let us then prove (3.45). Recalling (3.26), for $0 \leqslant s \leqslant u \leqslant t \leqslant T$ we can write

$$\delta Y_{sut}^{[3]} = (B_{su} - \bar{B}_{su})\, \mathbb{X}_{ut}^1 + (\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z}))_{su}\, \mathbb{X}_{ut}^2 ,$$

where $B_{su} := \sigma(Z_u) - \sigma(Z_s) - \sigma_2(Z_s)\, \mathbb{X}_{su}^1$ and similarly for $\bar{B}_{su}$, hence by (3.21)

$$\|\delta Y^{[3]}\|_{\gamma\alpha,\tau} \leqslant \|B - \bar{B}\|_{(\gamma-1)\alpha,\tau} \|\mathbb{X}\|_\alpha + \|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{(\gamma-2)\alpha,\tau} \|\mathbb{X}^2\|_{2\alpha} . \tag{3.46}$$

To obtain (3.45) we need to show that $\|B - \bar{B}\|_{(\gamma-1)\alpha,\tau}$ and $\|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{(\gamma-2)\alpha,\tau}$ can be bounded by *linear combinations of* $\|Y\|_{\infty,\tau}$, $\|\delta Y\|_{\alpha,\tau}$ *and* $\|Y^{[2]}\|_{2\alpha,\tau}$.

We start from $\|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{(\gamma-2)\alpha,\tau}$, which can be bounded as in (2.29):

$$\|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{(\gamma-2)\alpha,\tau} \leqslant C_2' \|\delta Y\|_{\alpha,\tau} + C_2'' \{\|\delta Z\|_\alpha^{\gamma-1} + \|\delta\bar{Z}\|_\alpha^{\gamma-1}\} \|Y\|_{\infty,\tau}.$$

We next focus on $\|B - \bar{B}\|_{(\gamma-1)\alpha,\tau}$, which we are going to estimate by the following explicit linear combination of $\|Y\|_{\infty,\tau}$, $\|\delta Y\|_{\alpha,\tau}$ and $\|Y^{[2]}\|_{2\alpha,\tau}$:

$$\begin{aligned} \|B - \bar{B}\|_{(\gamma-1)\alpha,\tau} \leqslant\ & C_1'' \|Y\|_{\infty,\tau} \|Z^{[2]}\|_{2\alpha} + C_1' \|Y^{[2]}\|_{2\alpha,\tau} \\ & + C_1'' \|\delta Y\|_{\alpha,\tau} \|\delta Z\|_\alpha + 2\, C_1''' \|Y\|_{\infty,\tau} \|\delta Z\|_\alpha^2 \\ & + C_1'' \|\delta\bar{Z}\|_\alpha \|\delta Y\|_{\alpha,\tau}, \end{aligned} \tag{3.47}$$

which completes the proof of (3.45) when plugged into (3.46).

It only remains to prove (3.47). Recalling (3.24), it follows by (3.22) that

$$\begin{aligned} B_{su} &:= \sigma(Z_u) - \sigma(Z_s) - \sigma_2(Z_s)\, \mathbb{X}_{su}^1 \\ &= \nabla\sigma(Z_s)\, Z_{su}^{[2]} + \int_0^1 \underbrace{(\nabla\sigma(Z_u + r\,\delta Z_{su}) - \nabla\sigma(Z_u))}_{F_{su}} \delta Z_{su}\, \mathrm{d}r, \end{aligned}$$

and likewise for $\bar{B}_{su}$ (with $\bar{F}_{su}$ defined similarly), therefore

$$|B_{su} - \bar{B}_{su}| \leqslant |\nabla\sigma(Z_s)\, Z_{su}^{[2]} - \nabla\sigma(\bar{Z}_s)\, \bar{Z}_{su}^{[2]}| + \int_0^1 |F_{su}\, \delta Z_{su} - \bar{F}_{su}\, \delta\bar{Z}_{su}|\, \mathrm{d}r . \tag{3.48}$$

By the elementary estimate $|a\,b - \bar{a}\,\bar{b}| = |a\,b - \bar{a}\,b + \bar{a}\,b - \bar{a}\,\bar{b}| \leqslant |a - \bar{a}|\,|b| + |\bar{a}|\ |b - \bar{b}|$, that we apply repeatedly, we can bound

$$\begin{aligned} |\nabla\sigma(Z_s)\, Z_{su}^{[2]} - \nabla\sigma(\bar{Z}_s)\, \bar{Z}_{su}^{[2]}| &\leqslant |\nabla\sigma(Z_s) - \nabla\sigma(\bar{Z}_s)|\, |Z_{su}^{[2]}| + |\nabla\sigma(\bar{Z}_s)|\, |Z_{su}^{[2]} - \bar{Z}_{su}^{[2]}| \\ &\leqslant C_1'' |Y_s|\, |Z_{su}^{[2]}| + C_1' |Y_{su}^{[2]}|, \end{aligned}$$

and note that by (2.7) we obtain the first line in the RHS of (3.47).

To complete the proof of (3.47), we look at the second term in the RHS of (3.48):

$$\begin{aligned} |F_{su}\,\delta Z_{su} - \bar{F}_{su}\,\delta \bar{Z}_{su}| &\leqslant |F_{su} - \bar{F}_{su}|\,|\delta Z_{su}| + |\bar{F}_{su}|\,|\delta Z_{su} - \delta \bar{Z}_{su}| \\ &\leqslant |F_{su} - \bar{F}_{su}|\,|\delta Z_{su}| + C_1''\,r\,|\delta \bar{Z}_{su}|\,|\delta Y_{su}|, \end{aligned} \tag{3.49}$$

because $|\bar{F}_{su}| \leqslant C_1''\,r\,|\delta \bar{Z}_{su}|$. We then see, applying (2.8), that the last term in (3.49) produces the third line in (3.47). Finally, by (2.19) we estimate

$$\begin{aligned} |F_{su} - \bar{F}_{su}| &= |(\nabla\sigma(Z_u + r\,\delta Z_{su}) - \nabla\sigma(Z_u)) - (\nabla\sigma(\bar{Z}_u + r\,\delta \bar{Z}_{su}) - \nabla\sigma(\bar{Z}_u))| \\ &\leqslant C_1''\,r\,|\delta Y_{su}| + C_1'''\,\{|r\,\delta Z_{su}|^{\gamma-2} + |r\,\delta Z_{su}|^{\gamma-2}\}\,|Y_s|\,. \end{aligned}$$

We obtain by (2.7) for $0 \leqslant r \leqslant 1$

$$\|F - \bar{F}\|_{(\gamma-2)\alpha,\tau} \leqslant C_1''\,\|\delta Y\|_{\alpha,\tau} + 2\,C_1'''\,\|Y\|_{\infty,\tau}\,\|\delta Z\|_{\alpha}^{\gamma-2}\,.$$

Applying again (2.8), we finally see that the first term in (3.49) yields the second line in (3.47), which completes the proof. □

## 3.7. Continuity of the solution map

In this section we assume that $\sigma$ has bounded first, second and third derivatives, while $\sigma_2$ has bounded first and second derivatives:

$$\|\nabla\sigma\|_\infty, \|\nabla^2\sigma\|_\infty, \|\nabla^3\sigma\|_\infty < \infty, \qquad \|\nabla\sigma_2\|_\infty, \|\nabla^2\sigma_2\|_\infty < \infty. \tag{3.50}$$

(We stress that no boundedness assumption is made on $\sigma$ and $\sigma_2$.) Under these assumptions, given any time horizon $T > 0$, any starting point $Z_0 \in \mathbb{R}^k$ and any $\alpha$-rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ with $\frac{1}{3} < \alpha \leqslant \frac{1}{2}$, we have *global existence and uniqueness* of solutions $Z\colon [0,T] \to \mathbb{R}^k$ to (3.18) (as we will prove in Theorem 3.12).

Denoting by $\mathcal{R}_{\alpha,d}$ the space of $d$-dimensional $\alpha$-rough paths $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$, that we endow with the norm $\|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha}$ we can thus consider the *solution map*:

$$\begin{aligned} \Phi\colon\ \mathbb{R}^k \times \mathcal{R}_{\alpha,d} &\longrightarrow \mathcal{C}^\alpha \\ (Z_0, \mathbb{X}) &\longmapsto Z := \begin{cases} \text{unique solution of (3.18) for } t \in [0,T] \\ \text{starting from } Z_0 \end{cases} . \end{aligned} \tag{3.51}$$

We prove the highly non-trivial result that this map is *locally Lipschitz*. In the space $\mathcal{C}^\alpha$ of Hölder functions we work with the weighted norm $\|f\|_{\infty,\tau} + \|\delta f\|_{\alpha,\tau}$, which is equivalent to the usual norm $\|f\|_{\mathcal{C}^\alpha} := \|f\|_\infty + \|\delta f\|_\alpha$, see Remark 1.15.

Theorem 3.11. (Continuity of the solution map) *Let $\sigma$ and $\sigma_2$ satisfy (3.50) (with no boundedness assumption on the functions $\sigma$ and $\sigma_2$). Then, for any $T > 0$ and $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$, the solution map $(Z_0, \mathbb{X}) \mapsto Z$ in (3.51) is locally Lipschitz.*

*More explicitly, given any $M_0, M, D < \infty$, if we assume that*

$$\max\{\|\nabla\sigma\|_\infty, \|\nabla^2\sigma\|_\infty, \|\nabla^3\sigma\|_\infty, \|\nabla\sigma_2\|_\infty, \|\nabla^2\sigma_2\|_\infty\} \leqslant D, \tag{3.52}$$

*and we consider starting points $Z_0, \bar{Z}_0 \in \mathbb{R}^d$ and rough paths $\mathbb{X}$, $\bar{\mathbb{X}} \in \mathcal{C}^\alpha$ with*

$$\max\{|\sigma(Z_0)|, |\sigma_2(Z_0)|, |\sigma(\bar{Z}_0)|, |\sigma_2(\bar{Z}_0)|\} \leqslant M_0, \tag{3.53}$$

$$\max\{\|\mathbb{X}^1\|_\alpha, \|\mathbb{X}^2\|_{2\alpha}, \|\bar{\mathbb{X}}^1\|_\alpha, \|\bar{\mathbb{X}}^2\|_{2\alpha}\} \leqslant M, \tag{3.54}$$

*then the corresponding solutions* $Z = (Z_s)_{s\in[0,T]}$*,* $\bar{Z} = (\bar{Z}_s)_{s\in[0,T]}$ *of (3.18) satisfy*

$$\begin{aligned}\|Z - \bar{Z}\|_{\infty,\tau} + \|\delta Z - \delta\bar{Z}\|_{\alpha,\tau} + \|Z^{[2]} - \bar{Z}^{[2]}\|_{2\alpha,\tau} \\ \leqslant \mathfrak{C}'_M\,|Z_0 - \bar{Z}_0| + 30\,M_0\,(\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}).\end{aligned} \tag{3.55}$$

*provided* $\tau$ *satisfies* $0 < \tau \wedge T \leqslant \hat{\tau}'$ *for a suitable* $\hat{\tau}' = \hat{\tau}'_{\alpha,T,D,M_0,M} > 0$*, where we set*

$$\mathfrak{C}'_M := 16\,\{(\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty)\,M + 1\} \leqslant 32\,(D\,M + 1).$$

**Proof.** It is convenient to define the constant

$$\mathfrak{c}'_M := (\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty)\,M \leqslant 2\,D\,M\,. \tag{3.56}$$

Let $Z$ and $\bar{Z}$ be two solutions of (3.18) with respective routh paths $\mathbb{X}$ and $\bar{\mathbb{X}}$. Defining $Y := Z - \bar{Z}$ and $Y^{[2]} := Z^{[2]} - \bar{Z}^{[2]}$, see (3.24), we rewrite our goal (3.55) as

$$\begin{aligned}\|Y\|_{\infty,\tau} + \|\delta Y\|_{\alpha,\tau} + \|Y^{[2]}\|_{2\alpha,\tau} \;\leqslant\; & 16\,(\mathfrak{c}'_M + 1)\,|Y_0| \\ & + 30\,M_0\,(\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha})\,.\end{aligned} \tag{3.57}$$

Throughout the proof we use the shorthand

$$\varepsilon := (\tau \wedge T)^\alpha \tag{3.58}$$

and we write *for* $\varepsilon$ *small enough* to mean *for all* $0 < \varepsilon < \varepsilon_0$*, for a suitable* $\varepsilon_0$ *depending on* $\alpha, T, M_0, M, D$. We claim that the following estimates hold for $\delta Y$ and $Y^{[2]}$:

$$\|\delta Y\|_{\alpha,\tau} \leqslant \mathfrak{c}'_M\,\|Y\|_{\infty,\tau} + 2\,M_0\,\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \varepsilon\,\|Y^{[2]}\|_{2\alpha,\tau}\,, \tag{3.59}$$

$$\|Y^{[2]}\|_{2\alpha,\tau} \leqslant \mathfrak{c}'_M\,\|Y\|_{\infty,\tau} + 2\,M_0\,\|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha} + \varepsilon\,\|Y^{[3]}\|_{3\alpha,\tau}, \tag{3.60}$$

and, moreover, *for* $\varepsilon$ *small enough* the following estimate holds for $Y^{[3]} := Z^{[3]} - \bar{Z}^{[3]}$:

$$\varepsilon\,\|Y^{[3]}\|_{3\alpha,\tau} \leqslant \|Y\|_{\infty,\tau} + M_0\,(\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}) + \|\delta Y\|_{\alpha,\tau} + \frac{1}{4}\,\|Y^{[2]}\|_{\alpha,\tau}. \tag{3.61}$$

It is now elementary (but tedious) to deduce our goal (3.57). Plugging (3.61) into (3.60) we obtain $\|Y^{[2]}\|_{2\alpha,\tau} \leqslant (\cdots) + \frac{1}{4}\,\|Y^{[2]}\|_{2\alpha,\tau}$ which yields $\|Y^{[2]}\|_{2\alpha,\tau} \leqslant \frac{4}{3}\,(\ldots)$ (since $\|Y^{[2]}\|_{2\alpha,\tau} < \infty$ by Lemma 3.8). Making $(\ldots)$ explicit, we get

$$\|Y^{[2]}\|_{2\alpha,\tau} \leqslant 2\,(\mathfrak{c}'_M + 1)\,\|Y\|_{\infty,\tau} + 4\,M_0\,(\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}) + 2\,\|\delta Y\|_{\alpha,\tau} \tag{3.62}$$

which plugged into (3.59) yields, *for* $\varepsilon$ *small enough* (it suffices that $\varepsilon \leqslant \frac{1}{4}$),

$$\|\delta Y\|_{\alpha,\tau} \leqslant 3\,(\mathfrak{c}'_M + 1)\,\|Y\|_{\infty,\tau} + 6\,M_0\,(\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha})\,, \tag{3.63}$$

and looking back at (3.62) we obtain

$$\|Y^{[2]}\|_{2\alpha,\tau} \leqslant 8\,(\mathfrak{c}'_M + 1)\,\|Y\|_{\infty,\tau} + 16\,M_0\,(\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}), \tag{3.64}$$

so that, overall,

$$\begin{aligned}\|Y\|_{\infty,\tau} + \|\delta Y\|_{\alpha,\tau} + \|Y^{[2]}\|_{2\alpha,\tau} \;\leqslant\; & 12\,(\mathfrak{c}'_M + 1)\,\|Y\|_{\infty,\tau} \\ & + 22\,M_0\,(\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha})\,.\end{aligned} \tag{3.65}$$

It only remains to make $\|Y\|_{\infty,\tau}$ explicit. Since $\|Y\|_{\infty,\tau} \leqslant |Y_0| + 3\,\varepsilon\,\|\delta Y\|_{\alpha,\tau}$ by (2.5), *for* $\varepsilon$ *small enough* (more precisely for $\varepsilon \leqslant \frac{1}{36\,(\mathfrak{c}'_M+1)}$) we can bound

$$(\mathfrak{c}'_M+1)\,\|Y\|_{\infty,\tau} \leqslant (\mathfrak{c}'_M+1)\,|Y_0| + \frac{1}{12}\,\|\delta Y\|_{\alpha,\tau}, \tag{3.66}$$

which inserted into (3.63) yields

$$\|\delta Y\|_{\alpha,\tau} \leqslant 4\,(\mathfrak{c}'_M+1)\,|Y_0| + 8\,M_0\,(\|\mathbb{X}^1-\bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2-\bar{\mathbb{X}}^2\|_{2\alpha}).$$

Plugging this into (3.66), and then (3.66) into (3.65), we obtain our goal (3.57).

It remains to prove (3.59), (3.60) and (3.61). We first state some useful bounds that will be used repeatedly. Recalling (3.52) and (3.28)-(3.30), let us define

$$\bar{\tau} = \bar{\tau}_{\alpha,D,M} := \frac{1}{\{4\,(K_{3\alpha}+3)\,(2\,(D^2+D)\,(M^2+M)+1)\}^{1/\alpha}}\,, \tag{3.67}$$

By the a priori estimate (3.29) we can then bound

$$\text{for } \varepsilon = (\tau \wedge T)^\alpha \leqslant \bar{\tau}^\alpha: \qquad \|\delta Z\|_{\alpha,\tau} + \|Z^{[2]}\|_{2\alpha,\tau} \leqslant 4\,M_0\,M, \tag{3.68}$$

hence

$$\max\{\|\delta\sigma(Z)\|_{\alpha,\tau}\,, \|\delta\sigma_2(Z)\|_{\alpha,\tau}\} \leqslant \max\{\|\nabla\sigma\|_\infty\,, \|\nabla\sigma_2\|_\infty\}\,\|\delta Z\|_{\alpha,\tau} \leqslant 4\,M_0\,\mathfrak{c}'_M, \tag{3.69}$$

which implies that, by (2.5) and *for* $\varepsilon$ *small enough*,

$$\max\{\|\sigma(Z)\|_{\infty,\tau}\,, \|\sigma_2(Z)\|_{\infty,\tau}\} \leqslant M_0 + 3\,\varepsilon\,4\,M_0\,\mathfrak{c}'_M \leqslant 2\,M_0.$$

We record the following simple bound, for any Lipschitz function $f$,

$$\|f(Z) - f(\bar{Z})\|_{\infty,\tau} \leqslant \|\nabla f\|_\infty\,\|Z-\bar{Z}\|_{\infty,\tau} = \|\nabla f\|_\infty\,\|Y\|_{\infty,\tau}. \tag{3.70}$$

We will also use a number of times the elementary estimate, for $a, b, \bar{a}, \bar{b} \in \mathbb{R}$,

$$|a\,b - \bar{a}\,\bar{b}| = |a\,b - a\,\bar{b} + a\,\bar{b} - \bar{a}\,\bar{b}| \leqslant |a|\,|b-\bar{b}| + |\bar{b}|\,|a-\bar{a}|\,. \tag{3.71}$$

We can now prove (3.59). Since $\delta Y_{st} = \delta Z_{st} - \delta\bar{Z}_{st} = \sigma(Z_s)\,\mathbb{X}^1_{st} - \sigma(\bar{Z}_s)\,\bar{\mathbb{X}}^1_{st} + Y^{[2]}_{st}$, see (3.24) for $Z$ and $\bar{Z}$, by (2.7) and (3.53)-(3.54) we get, applying (3.71),

$$\begin{aligned}\|\delta Y\|_{\alpha,\tau} &\leqslant \|\sigma(Z)\|_{\infty,\tau}\,\|\mathbb{X}^1-\bar{\mathbb{X}}^1\|_\alpha + \|\sigma(Z)-\sigma(\bar{Z})\|_{\infty,\tau}\,\|\bar{\mathbb{X}}^1\|_\alpha + \|Y^{[2]}\|_{\alpha,\tau}\\ &\leqslant 2\,M_0\,\|\mathbb{X}^1-\bar{\mathbb{X}}^1\|_\alpha + \|\sigma(Z)-\sigma(\bar{Z})\|_{\infty,\tau}\,M + \varepsilon\,\|Y^{[2]}\|_{2\alpha,\tau}\,,\end{aligned}$$

because $\|Y^{[2]}\|_{\alpha,\tau} \leqslant \varepsilon\,\|Y^{[2]}\|_{2\alpha,\tau}$ by (2.6) (recall the definition (3.58) of $\varepsilon$). Applying (3.70) with $f=\sigma$ and recalling $\mathfrak{c}'_M$ from (3.56), we obtain (3.59).

The proof of (3.60) is similar. Since $Z^{[3]}_{st} = Z^{[2]}_{st} - \sigma_2(Z_s)\,\mathbb{X}^2_{st}$ and similarly for $\bar{Z}^{[3]}$, see (3.24), we can write $Y^{[2]}_{st} = Z^{[2]} - \bar{Z}^{[2]} = \sigma_2(Z_s)\,\mathbb{X}^2_{st} - \sigma_2(\bar{Z}_s)\,\bar{\mathbb{X}}^2_{st} + Y^{[3]}_{st}$, therefore

$$\begin{aligned}\|Y^{[2]}\|_{2\alpha,\tau} &\leqslant \|\sigma_2(Z)\|_{\infty,\tau}\,\|\mathbb{X}^2-\bar{\mathbb{X}}^2\|_{2\alpha} + \|\sigma_2(Z)-\sigma_2(\bar{Z})\|_{\infty,\tau}\,\|\bar{\mathbb{X}}^2\|_{2\alpha} + \|Y^{[3]}\|_{2\alpha,\tau}\\ &\leqslant 2\,M_0\,\|\mathbb{X}^2-\bar{\mathbb{X}}^2\|_{2\alpha} + \|\sigma_2(Z)-\sigma_2(\bar{Z})\|_{\infty,\tau}\,M + \varepsilon\,\|Y^{[3]}\|_{3\alpha,\tau}\,,\end{aligned}$$

since $\|Y^{[3]}\|_{2\alpha,\tau} \leqslant \varepsilon\,\|Y^{[3]}\|_{3\alpha,\tau}$ by (2.6). Applying (3.70) for $f=\sigma_2$ we obtain (3.60).

We finally prove (3.61). Since $Y_{st}^{[3]} = Z_{st}^{[3]} - \bar{Z}_{st}^{[3]} = o(t-s)$, see (3.19), the weighted Sewing Bound (1.41) yields

$$\|Y^{[3]}\|_{3\alpha,\tau} \leqslant K_{3\alpha} \, \|\delta Y^{[3]}\|_{3\alpha,\tau} \, , \tag{3.72}$$

hence we can focus on $\delta Y^{[3]} = \delta Z^{[3]} - \delta \bar{Z}^{[3]}$. Let us recall (3.26): for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$\delta Z_{sut}^{[3]} = \underbrace{(\sigma(Z_u) - \sigma(Z_s) - \sigma_2(Z_s)\, \mathbb{X}_{su}^1)}_{B_{su}} \mathbb{X}_{ut}^1 + \delta\sigma_2(Z)_{su}\, \mathbb{X}_{ut}^2 \, ,$$

and analogously for $\delta \bar{Z}^{[3]}$ and $\bar{B}_{su}$, therefore by (3.71) and (3.21) we obtain

$$\begin{aligned} \|\delta Y^{[3]}\|_{3\alpha,\tau} \;\leqslant\; & \|B\|_{2\alpha,\tau}\, \|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|B - \bar{B}\|_{2\alpha,\tau}\, \|\bar{\mathbb{X}}^1\|_{\alpha,\tau} \\ & + \|\delta\sigma_2(Z)\|_{\alpha,\tau}\, \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha} + \|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{\alpha,\tau}\, \|\bar{\mathbb{X}}^2\|_{2\alpha} \, . \end{aligned} \tag{3.73}$$

It remains to estimate the four terms in the RHS: in view of (3.72), relation (3.61) is proved if we show that, *for $\varepsilon$ small enough*,

$$\varepsilon\, K_{3\alpha}\, \|B\|_{2\alpha,\tau}\, \|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha \;\leqslant\; M_0\, \|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha \, , \tag{3.74}$$

$$\varepsilon\, K_{3\alpha}\, \|B - \bar{B}\|_{2\alpha,\tau}\, \|\bar{\mathbb{X}}^1\|_{\alpha,\tau} \;\leqslant\; \frac{1}{2}(\|Y\|_{\infty,\tau} + \|\delta Y\|_{\alpha,\tau}) + \frac{1}{4}\|Y^{[2]}\|_{2\alpha,\tau}, \tag{3.75}$$

$$\varepsilon\, K_{3\alpha}\, \|\delta\sigma_2(Z)\|_{\alpha,\tau}\, \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha} \;\leqslant\; M_0\, \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha} \, , \tag{3.76}$$

$$\varepsilon\, K_{3\alpha}\, \|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{\alpha,\tau}\, \|\bar{\mathbb{X}}^2\|_{2\alpha} \;\leqslant\; \frac{1}{2}(\|Y\|_{\infty,\tau} + \|\delta Y\|_{\alpha,\tau}) \, . \tag{3.77}$$

We first deal with (3.76) and (3.77), then we focus on (3.74) and (3.75).

Proving (3.76) is very simple: since $\|\delta\sigma_2(Z)\|_{\alpha,\tau} \leqslant 4\, M_0\, \mathfrak{c}_M'$ by (3.69), we see that (3.76) holds *for $\varepsilon$ small enough*. To prove (3.77), note that by (2.51) we have

$$\|\delta\sigma(Z) - \delta\sigma(\bar{Z})\|_{(\gamma-1)\alpha,\tau} \;\leqslant\; \|\nabla\sigma\|_\infty\, \|\delta Y\|_{\alpha,\tau} + 4\, M_0\, M\, [\sigma]_{\mathcal{C}^{\gamma-1}}\, \|Y\|_{\infty,\tau} \, .$$

Applying (3.54) and (3.68) we obtain

$$\|\delta\sigma_2(Z) - \delta\sigma_2(\bar{Z})\|_{\alpha,\tau}\, \|\bar{\mathbb{X}}^2\|_{2\alpha} \leqslant \|\nabla\sigma_2\|_\infty\, M\, \|\delta Y\|_{\alpha,\tau} + \mathrm{e}^{\frac{T}{\tau}}\, \|\nabla^2\sigma_2\|_\infty\, 8\, M_0\, M^2\, \|Y\|_{\infty,\tau},$$

which shows that (3.77) holds *for $\varepsilon$ small enough*.

Let us now prove (3.74). By (3.22) we have, for $0 \leqslant s \leqslant t \leqslant T$,

$$B_{st} = \underbrace{\nabla\sigma(Z_s)\, Z_{st}^{[2]}}_{E_{st}} + \underbrace{\int_0^1 [(\nabla\sigma(Z_s + r\,\delta Z_{st}) - \nabla\sigma(Z_s))\, \delta Z_{st}]\, \mathrm{d}r}_{F_{st}} \tag{3.78}$$

and similarly for $\bar{E}_{st}$ and $\bar{F}_{st}$. In particular, recalling (3.68), we get

$$\begin{aligned} \|B\|_{2\alpha,\tau} \;\leqslant\; & \|\nabla\sigma\|_\infty\, \|Z^{[2]}\|_{2\alpha,\tau} + \|\nabla^2\sigma\|_\infty\, \|\delta Z\|_{\alpha,\tau}^2 \\ \leqslant\; & \|\nabla\sigma\|_\infty\, 4\, M_0\, M + \|\nabla^2\sigma\|_\infty\, (4\, M_0\, M)^2, \end{aligned}$$

hence we see that (3.74) holds *for $\varepsilon$ small enough*.

We finally prove (3.75), which is a bit tedious. In view of (3.78), we first consider

$$E_{st} - \bar{E}_{st} = (\nabla\sigma(Z_s) - \nabla\sigma(\bar{Z}_s))\, Z_{st}^{[2]} + \nabla\sigma(\bar{Z}_s)\, (Z_{st}^{[2]} - \bar{Z}_{st}^{[2]}) \, .$$

Applying (2.9) with $H = Z^{[2]}$ and $\bar{\tau}$ from (3.67), we obtain

$$\|E - \bar{E}\|_{2\alpha,\tau} \leqslant \|\nabla\sigma(Z) - \nabla\sigma(\bar{Z})\|_{\infty,\tau}\, e^{\frac{T}{\bar{\tau}}}\, \|Z^{[2]}\|_{2\alpha,\bar{\tau}} + \|\nabla\sigma\|_\infty \|Y^{[2]}\|_{2\alpha,\tau} .$$

By (3.70) with $f = \nabla\sigma$ and the a priori estimate (3.68) we obtain

$$\|E - \bar{E}\|_{2\alpha,\tau} \leqslant \|\nabla^2\sigma\|_\infty \|Y\|_{\infty,\tau}\, e^{\frac{T}{\bar{\tau}}}\, 4\, M_0\, M + \|\nabla\sigma\|_\infty \|Y^{[2]}\|_{2\alpha,\tau} . \tag{3.79}$$

We then consider $F_{st} - \bar{F}_{st}$. By (2.19), for $0 \leqslant r \leqslant 1$ we can estimate

$$\begin{aligned} &|(\nabla\sigma(Z_s + r\,\delta Z_{st}) - \nabla\sigma(Z_s)) - (\nabla\sigma(\bar{Z}_s + r\delta\bar{Z}_{st}) - \nabla\sigma(\bar{Z}_s))|\, |\delta Z_{st}| \\ &\quad\leqslant \|\nabla^2\sigma\|_\infty |\delta Y_{st}|\, |\delta Z_{st}| + \|\nabla^3\sigma\|_\infty \max_{0\leqslant u\leqslant 1} \{(1-u)\,|Y_s| + u\,|Y_t|\}\, |\delta Z_{st}|^2, \end{aligned}$$

as well as

$$|\nabla\sigma(Z_s + r\,\delta Z_{st}) - \nabla\sigma(Z_s)|\, |\delta Z_{st} - \delta\bar{Z}_{st}| \leqslant \|\nabla^2\sigma\|_\infty |\delta Z_{st}|\, |\delta Y_{st}| .$$

We can then estimate $F_{st} - \bar{F}_{st}$ from (3.78) as in (3.71): applying (2.9) twice with $H = \delta Z$ and $H = (\delta Z)^2$, always with $\bar{\tau}$ from (3.67), and recalling (3.68), we obtain

$$\begin{aligned} \|F - \bar{F}\|_{2\alpha,\tau} &\leqslant 2\, \|\nabla^2\sigma\|_\infty \|\delta Y\|_{\alpha,\tau}\, e^{\frac{T}{\bar{\tau}}}\, \|\delta Z\|_{\alpha,\bar{\tau}} + \|\nabla^3\sigma\|_\infty \|Y\|_{\infty,\tau}\, e^{\frac{T}{\bar{\tau}}}\, \|\delta Z\|^2_{\alpha,\bar{\tau}} \\ &\leqslant e^{\frac{T}{\bar{\tau}}} \{8\, M_0\, M\, \|\nabla^2\sigma\|_\infty \|\delta Y\|_{\alpha,\tau} + (4\, M_0\, M)^2 \|\nabla^3\sigma\|_\infty \|Y\|_{\infty,\tau}\} . \end{aligned} \tag{3.80}$$

Since $\|B - \bar{B}\|_{2\alpha,\tau} \leqslant \|E - \bar{E}\|_{2\alpha,\tau} + \|F - \bar{F}\|_{2\alpha,\tau}$ in view of (3.78), we see by (3.79) and (3.80) that (3.75) holds *for $\varepsilon$ small enough*. The proof is complete. □

## 3.8. GLOBAL EXISTENCE AND UNIQUENESS

Let us suppose that $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is of class $C^3$ with $\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty < +\infty$.

THEOREM 3.12. *Let $\alpha > \frac{1}{3}$. If $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is of class $C^3$ with $\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty < +\infty$ then for every $z_0 \in \mathbb{R}^k$ and $T > 0$ there is a unique solution $(Z_t)_{t\in[0,T]}$ to (3.19) such that $Z_0 = z_0$.*

**Proof.** By Theorem 3.10 we have at most one solution. We now construct a solution on an arbitrary finite interval $[0, T]$, arguing as in the proof of Theorem 2.15. We define $\Lambda \subseteq [0, T]$ as the set of all $s$ such that there is a solution $(Z_t)_{t\in[0,s]}$ to (3.19). By Proposition 3.6, $\Lambda$ is an open subset of $[0, T]$ and contains 0. By the a priori estimates of Theorem 3.9, $\Lambda$ is a closed subset of $[0, T]$. Therefore $\Lambda = [0, T]$. □

## 3.9. MILSTEIN SCHEME AND LOCAL EXISTENCE

In this section we prove the local existence result of Proposition 3.6, under the assumption that $\sigma, \sigma_2$ are of class $C^1$ and uniformly Lipschitz. To construct a solution to (3.10), we set $t_i := \frac{i}{n}$, $i \geqslant 0$, and for a given $y_0 \in \mathbb{R}^k$

$$y_{t_{i+1}} = y_{t_i} + \sigma(y_{t_i})\, \mathbb{X}^1_{t_i t_{i+1}} + \sigma_2(y_{t_i})\, \mathbb{X}^2_{t_i t_{i+1}}, \qquad i \geqslant 0.$$

We set $D := \max\{1, \|\nabla\sigma\|_\infty, \|\nabla\sigma_2\|_\infty\}$, $\mathbb{T} := \{t_i : t_i \leqslant T\}$ and

$$\begin{aligned} \delta y_{t_i t_j} &:= y_{t_j} - y_{t_i}, \\ \|\delta y\|_\alpha^{\mathbb{T}} &:= \sup_{0<i<j\leqslant nT} \frac{|y_{t_j} - y_{t_i}|}{|t_j - t_i|^\alpha}, \\ A_{t_i t_j} &:= \sigma(y_{t_i})\,\mathbb{X}^1_{t_i t_j} + \sigma_2(y_{t_i})\,\mathbb{X}^2_{t_i t_j}. \end{aligned}$$

The main technical estimate is the following

LEMMA 3.13. *Let $M>0$. There exists $T_{M,D,\alpha}>0$ such that, for all $T\in(0,T_{M,D,\alpha})$ and $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)\in\mathcal{R}_{\alpha,d}$ such that $\|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha} \leqslant M$, we have*

$$\begin{aligned} \|\delta y\|_\alpha^{\mathbb{T}} &\leqslant 5M(|\sigma(y_0)| + |\sigma_2(y_0)|), \\ \|\delta y - A\|_{3\alpha}^{\mathbb{T}} &\lesssim_{M,D,\alpha} (|\sigma(y_0)| + |\sigma_2(y_0)|). \end{aligned}$$

**Proof.** Let us set $R_{t_i t_j} := \delta y_{t_i t_j} - A_{t_i t_j}$. By the definitions, $R_{t_i t_{i+1}} = 0$. Then we can apply the discrete Sewing bound (Theorem 1.18) to $R$ on $\mathbb{T} := \left\{\frac{i}{n} : i \leqslant nT\right\}$ and we obtain

$$\|R\|_{3\alpha}^{\mathbb{T}} \leqslant C_{3\alpha}\|\delta R\|_{3\alpha}^{\mathbb{T}}, \qquad C_{3\alpha} = 2^{3\alpha}\sum_{n\geqslant 1}\frac{1}{n^{3\alpha}}.$$

Now, analogously to (3.26), since $\delta R = -\delta A$,

$$\delta R_{t_i t_j t_k} = -\underbrace{(\sigma(y_{t_j}) - \sigma(y_{t_i}) - \sigma_2(y_{t_i})\,\mathbb{X}^1_{t_i t_j})}_{B_{ij}}\mathbb{X}^1_{t_j t_k} - \underbrace{(\sigma_2(y_{t_i}) - \sigma_2(y_{t_j}))}_{C_{ij}}\,\mathbb{X}^2_{t_j t_k},$$

so that

$$\|\delta R\|_{3\alpha}^{\mathbb{T}} \leqslant M(\|B\|_{2\alpha}^{\mathbb{T}} + \|C\|_\alpha^{\mathbb{T}}).$$

We set

$$H_{t_i t_j} := \delta y_{t_i t_j} - \sigma(y_{t_i})\,\mathbb{X}^1_{t_i t_j},$$

and by (3.23) we obtain

$$\begin{aligned} &B_{t_i t_j} = \sigma(y_{t_j}) - \sigma(y_{t_i}) - \sigma_2(y_{t_i})\,\mathbb{X}^1_{t_i t_j} = \\ &= \underbrace{\int_0^1 (\sigma_2(y_{t_i} + u\,\delta y_{t_i t_j}) - \sigma_2(y_{t_i}))\,\mathbb{X}^1_{t_i t_j}\mathrm{d}u}_{E_{ij}} + \underbrace{\int_0^1 \nabla\sigma(y_{t_i} + u\,\delta y_{t_i t_j})\,\mathrm{d}u\,H_{t_i t_j}}_{F_{ij}} \\ &\quad \underbrace{-\int_0^1 \nabla\sigma(y_{t_i} + u\,\delta y_{t_i t_j})(\sigma(y_{t_i} + u\,\delta y_{t_i t_j}) - \sigma(y_{t_i}))\mathbb{X}^1_{t_i t_j}\mathrm{d}u}_{G_{ij}}. \end{aligned}$$

First

$$\|E\|_{2\alpha}^{\mathbb{T}} \leqslant \|\nabla\sigma_2\|_\infty \|\delta y\|_\alpha^{\mathbb{T}}\,\|\mathbb{X}^1\|_\alpha \leqslant DM\,\|\delta y\|_\alpha^{\mathbb{T}}.$$

Similarly

$$\|G\|_{2\alpha}^{\mathbb{T}} \leqslant \|\nabla\sigma\|_\infty^2 \|\delta y\|_\alpha^{\mathbb{T}}\,\|\mathbb{X}^1\|_\alpha \leqslant D^2 M\,\|\delta y\|_\alpha^{\mathbb{T}}.$$

By the definition of $R_{t_it_j}$

$$\begin{aligned} |H_{t_it_j}| &\leqslant |R_{t_it_j}| + |\sigma_2(y_{t_i})\,\mathbb{X}^2_{t_it_j}| \\ &\leqslant [T^\alpha \|R\|^{\mathbb{T}}_{3\alpha} + (|\sigma_2(y_0)| + T^\alpha \|\nabla\sigma_2\|_\infty \|\delta y\|^{\mathbb{T}}_\alpha)\|\mathbb{X}^2\|_{2\alpha}]\,|t_j - t_i|^{2\alpha} \\ &\leqslant (T^\alpha \|R\|^{\mathbb{T}}_{3\alpha} + M|\sigma_2(y_0)| + T^\alpha D M \|\delta y\|^{\mathbb{T}}_\alpha)|t_j - t_i|^{2\alpha}. \end{aligned}$$

Therefore

$$\begin{aligned} \|F\|^{\mathbb{T}}_{2\alpha} &\leqslant D\|H\|^{\mathbb{T}}_{2\alpha} \\ &\leqslant D(T^\alpha \|R\|^{\mathbb{T}}_{3\alpha} + M|\sigma_2(y_0)| + T^\alpha D M \|\delta y\|^{\mathbb{T}}_\alpha). \end{aligned}$$

Finally

$$\begin{aligned} \|B\|^{\mathbb{T}}_{2\alpha} &\leqslant \|E\|^{\mathbb{T}}_{2\alpha} + \|F\|^{\mathbb{T}}_{2\alpha} + \|G\|^{\mathbb{T}}_{2\alpha} \\ &\leqslant D\,[M|\sigma_2(y_0)| + T^\alpha \|R\|^{\mathbb{T}}_{3\alpha} + DM(2 + T^\alpha)\|\delta y\|^{\mathbb{T}}_\alpha]\,. \end{aligned}$$

Analogously

$$\|C\|^{\mathbb{T}}_{2\alpha} \leqslant D\|\delta y\|^{\mathbb{T}}_\alpha.$$

Therefore

$$\|R\|^{\mathbb{T}}_{3\alpha} \leqslant C_{3\alpha} DM(M|\sigma_2(y_0)| + T^\alpha \|R\|^{\mathbb{T}}_{3\alpha} + [1 + DM(2 + T^\alpha)]\|\delta y\|^{\mathbb{T}}_\alpha).$$

If $T^\alpha C_{3\alpha} DM \leqslant \frac{1}{2}$ then

$$\|R\|^{\mathbb{T}}_{3\alpha} \leqslant 2C_{3\alpha} DM(M|\sigma_2(y_0)| + [1 + DM(2 + T^\alpha)]\|\delta y\|^{\mathbb{T}}_\alpha). \tag{3.81}$$

We set

$$L(y) := 2C_{3\alpha} DM(M|\sigma_2(y_0)| + [1 + DM(2 + T^\alpha)]\|\delta y\|^{\mathbb{T}}_\alpha)$$

Now we obtain by (3.81)

$$\begin{aligned} \|\delta y\|^{\mathbb{T}}_\alpha &\leqslant \|R\|^{\mathbb{T}}_\alpha + \|A\|^{\mathbb{T}}_\alpha \\ &\leqslant T^{2\alpha} L(y) + (|\sigma(y_0)| + |\sigma_2(y_0)| + 2DT^\alpha \|\delta y\|^{\mathbb{T}}_\alpha)M. \end{aligned}$$

If we assume also that $2DMT^\alpha \leqslant \frac{1}{2}$, we obtain

$$\|\delta y\|_\alpha \leqslant 2T^{2\alpha} L(y) + 2M(|\sigma(y_0)| + |\sigma_2(y_0)|).$$

By the definition of $L(y)$, if furthermore $2C_{3\alpha} DM[1 + DM(2 + T^\alpha)]\,T^{2\alpha} \leqslant \frac{1}{2}$, we obtain finally

$$\begin{aligned} \|\delta y\|^{\mathbb{T}}_\alpha &\leqslant 5M(|\sigma(y_0)| + |\sigma_2(y_0)|)\,, \\ L(y) &\leqslant 12C_{3\alpha} DM^2[1 + DM(2 + T^\alpha)](|\sigma(y_0)| + |\sigma_2(y_0)|) =: K, \end{aligned}$$

and by (3.81)

$$\|\delta y - A\|^{\mathbb{T}}_{3\alpha} \leqslant K.$$

The proof is complete. □

**Proof of Proposition 3.6.** Arguing as in Theorem 2.16 we obtain the result of local existence for equation (3.19) of Proposition 3.6. □