Chapter 4

Stochastic Differential Equations

In this chapter we connect the *rough difference equations (RDE)* discussed in the previous chapter, see (3.18), with the classical *stochastic differential equations (SDE)* $\mathrm{d}Y_t = \sigma(Y_t)\,\mathrm{d}B_t$ driven by a Brownian motion B. Indeed, both RDE and SDE are ways to make sense of the ill-posed differential equation $\dot{Y}_t = \sigma(Y_t)\,\dot{B}_t$.

We fix a time horizon $T > 0$ and two dimensions $k, d \in \mathbb{N}$. Let $B = (B_t)_{t\in[0,T]}$ be a d-dimensional Brownian motion (with continuous paths) relative to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We fix a sufficiently regular function $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ and we consider a solution $Y = (Y_t)_{t\in[0,T]}$ of the SDE

$$\mathrm{d}Y_t = \sigma(Y_t)\,\mathrm{d}B_t \qquad \text{i.e.} \qquad Y_t = Y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s\,, \qquad t \geqslant 0, \tag{4.1}$$

where the stochastic integral is in the Ito sense. *We always fix a version of Y with continuous paths* (we recall that the Ito integral is a continuous local martingale).

We want to show that Y solves a *rough difference equation* driven by the *rough path* $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$ (see Definition 3.2) defined by

$$\mathbb{B}^1_{st} := B_t - B_s, \qquad \mathbb{B}^2_{st} := \int_s^t (B_r - B_s) \otimes \mathrm{d}B_r, \quad 0 \leqslant s \leqslant t \leqslant T, \tag{4.2}$$

where the stochastic integral is in the Ito sense. More explicitly, for $i, j \in \{1, \ldots, d\}$

$$(\mathbb{B}^1_{st})^i := B^i_t - B^i_s\,, \qquad (\mathbb{B}^2_{st})^{ij} := \int_s^t (B^i_r - B^i_s)\,\mathrm{d}B^j_r\,, \tag{4.3}$$

where we write $B_t = (B^1_t, \ldots, B^d_t)$, so that $\mathbb{B}^1: [0,T]^2_{\leqslant} \to \mathbb{R}^d$ and $\mathbb{B}^2: [0,T]^2_{\leqslant} \to \mathbb{R}^d \otimes \mathbb{R}^d$.

Our first main result is that $(\mathbb{B}^1, \mathbb{B}^2)$ is indeed a rough path over B.

Theorem 4.1. (Ito rough path) *Almost surely, $\mathbb{B} := (\mathbb{B}^1, \mathbb{B}^2)$ is an α-rough path over B (see Definition 3.2) for any $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right[$, namely it satisfies a.s.*

$$\begin{aligned} \delta\mathbb{B}^2_{sut} &:= \mathbb{B}^2_{st} - \mathbb{B}^2_{su} - \mathbb{B}^2_{ut} = \mathbb{B}^1_{su} \otimes \mathbb{B}^1_{ut}\,, \\ |\mathbb{B}^1_{st}| &\lesssim |t-s|^{\alpha}\,, \qquad |\mathbb{B}^2_{st}| \lesssim |t-s|^{2\alpha}\,. \end{aligned} \tag{4.4}$$

Our second main result is that, under suitable assumptions, the solution Y of the SDE (4.1) solves the RDE (3.18) driven by the Ito rough path $\mathbb{X} = \mathbb{B}$.

THEOREM 4.2. (SDE & RDE) *If $\sigma\colon \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is of class C^2, then almost surely any solution $Y = (Y_t)_{t\in[0,T]}$ of the SDE (4.1) is also a solution of the RDE*

$$\delta Y_{st} = \sigma(Y_s)\,\mathbb{B}^1_{st} + \sigma_2(Y_s)\,\mathbb{B}^2_{st} + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{4.5}$$

(We recall that $\sigma_2(\cdot) := \nabla\sigma(\cdot)\,\sigma(\cdot)$ is defined in (3.5).)

If $\sigma(\cdot)$ is of class C^3 and, furthermore, $\sigma(\cdot)$ and $\sigma_2(\cdot)$ are globally Lipschitz, i.e. $\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty < \infty$, then almost surely both the SDE (4.1) and the RDE (4.5) admit a unique solution $Y = (Y_t)_{t\in[0,T]}$ and these solutions coincide.

The key tool we exploit in this chapter is a *local expansion of stochastic integrals*, see Theorem 4.3 in the next Section 4.1. The proofs of Theorems 4.1 and 4.2 are direct consequences of this result, see Section 4.2.

In Sections 4.3 and 5.1 we discuss useful generalizations of the SDE (4.1), where we add a drift and we allow for stochastic integration in the Stratonovich sense, which leads to generalized versions of Theorems 4.1 and 4.2.

In Section 5.2 we present the celebrated result by Wong-Zakai on the limit of solutions of the SDE (4.1) with a regularized Brownian motion (via convolution).

Finally, Section 4.4 is devoted to a far-reaching generalization of Kolmogorov's continuity criterion, which leads to the proof of Theorem 4.3 in Section 4.5.

4.1. LOCAL EXPANSION OF STOCHASTIC INTEGRALS

We recall that $B = (B_t)_{t\in[0,T]}$ is a d-dimensional Brownian motion. Let $h = (h_t)_{t\in[0,T]}$ be a stochastic process with values in $\mathbb{R}^k \otimes (\mathbb{R}^d)^*$. We assume that h is adapted and has continuous paths, in particular $\int_0^T |h_s|^2\,\mathrm{d}s < \infty$, hence the Itô integral

$$I_t := I_0 + \int_0^t h_r\,\mathrm{d}B_r \tag{4.6}$$

is well-defined as a local martingale. It is a classical result that the stochastic process $I = (I_t)_{t\in[0,T]}$ admits a version with continuous paths, which we always fix.

We now state the main technical result of this chapter, proved in Section 4.5 below, which connects the regularity of h to the regularity of I.

THEOREM 4.3. (LOCAL EXPANSION OF STOCHASTIC INTEGRALS) *Let $h\colon [0,T] \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ be an adapted process with a.s. continuous paths. Fix any $\alpha \in \left]0, \frac{1}{2}\right[$ and recall $(\mathbb{B}^1, \mathbb{B}^2)$ from (4.2).*

1. *Almost surely I is of class $\mathcal{C}^\alpha$, i.e.*

$$|I_t - I_s| \lesssim (t-s)^\alpha, \qquad \forall 0 \leqslant s \leqslant t \leqslant T. \tag{4.7}$$

   (We recall that the implicit constant in the relation $\lesssim$ is random.)

2. *Assume that, almost surely, $|\delta h_{sr}| \lesssim (r-s)^\beta$ for some $\beta \in]0,1]$ (i.e. h is of class $\mathcal{C}^\beta$). Then, almost surely,*

$$|\delta I_{st} - h_s\,\mathbb{B}^1_{st}| = \left|\int_s^t \delta h_{sr}\,\mathrm{d}B_r\right| \lesssim (t-s)^{\alpha+\beta}, \qquad \forall 0 \leqslant s \leqslant t \leqslant T. \tag{4.8}$$

3. *Assume that, almost surely,* $|\delta h_{sr} - h_s^1\,\mathbb{B}_{sr}^1| \lesssim (r-s)^{\eta+\alpha}$*, for some adapted process* $h^1 = (h_t^1)_{t\in[0,T]}$ *of class* $\mathcal{C}^\eta$ *with* $\eta \in]0,1]$*. Then, almost surely,*

$$\begin{aligned} |\delta I_{st} - h_s\,\mathbb{B}_{st}^1 - h_s^1\,\mathbb{B}_{st}^2| &= \left|\int_s^t (\delta h_{sr} - h_s^1\,\mathbb{B}_{sr}^1)\,\mathrm{d}B_r\right| \\ &\lesssim (t-s)^{\eta+2\alpha}, \qquad \forall 0\leqslant s\leqslant t\leqslant T. \end{aligned} \tag{4.9}$$

The proof of Theorem 4.3 is postponed to Section 4.5.

4.2. Brownian rough path and SDE

In this section we exploit Theorem 4.3 to prove Theorems 4.1 and 4.2.

Proof. (of Theorem 4.1) We need to verify that $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$ satisfies the Chen relation (3.13) and the analytic bounds (3.14).

The Chen relation $\delta\mathbb{B}_{sut}^2 = \mathbb{B}_{su}^1 \otimes \mathbb{B}_{ut}^1$ for $0\leqslant s\leqslant u\leqslant t\leqslant T$ holds by (4.3):

$$\begin{aligned} \delta(\mathbb{B}^2)_{sut}^{ij} &= (\mathbb{B}^2)_{st}^{ij} - (\mathbb{B}^2)_{su}^{ij} - (\mathbb{B}^2)_{ut}^{ij} \\ &= \int_s^t (B_r^i - B_s^i)\,\mathrm{d}B_r^j - \int_s^u (B_r^i - B_s^i)\,\mathrm{d}B_r^j - \int_u^t (B_r^i - B_u^i)\,\mathrm{d}B_r^j \\ &= \int_u^t (B_u^i - B_s^i)\,\mathrm{d}B_r^j = (B_u^i - B_s^i)\int_u^t 1\,\mathrm{d}B_r^j = (B_u^i - B_s^i)(B_t^j - B_u^j), \end{aligned}$$

by the properties of the Itô integral and the fact that the times $s\leqslant u\leqslant t$ are ordered.

The first analytic bound $|\mathbb{B}_{st}^1| \lesssim |t-s|^\alpha$ for $\alpha \in \left]0, \frac{1}{2}\right[$ is a well-known almost sure property of Brownian motion, which also follows from Theorem 4.3, applying (4.7) with $h\equiv 1$. Finally, the second analytic bound $|\mathbb{B}_{st}^2| \lesssim |t-s|^{2\alpha}$ is also a consequence of Theorem 4.3: it suffices to apply (4.8) with $h_s := B_s$ and $\beta = \alpha$. □

Proof. (Theorem 4.2) We first prove the second part of the statement.

- When σ is globally Lipschitz ($\|\nabla\sigma\|_\infty < +\infty$), it is a classical result that for the SDE (4.1) there is existence of strong solutions and pathwise uniqueness.
- When σ is of class C^3, by Theorem 3.10 there is uniqueness of solutions for the RDE (3.19), and if both σ and σ_2 are globally Lipschitz ($\|\nabla\sigma\|_\infty < +\infty$ and $\|\nabla\sigma_2\|_\infty < +\infty$) there is also existence of solutions, by Theorem 3.12.

Therefore we only need to prove the first part of the statement: we assume that σ is of class C^2 and we show that given a solution $Y = (Y_t)_{t\in[0,T]}$ of the SDE (4.1), almost surely Y is also a solution to the RDE (4.5).

Since Y is solution to (4.1), recalling (4.2) we can write

$$\begin{aligned} \delta Y_{st} - \sigma(Y_s)\,\mathbb{B}_{st}^1 - \sigma_2(Y_s)\,\mathbb{B}_{st}^2 &= \int_s^t (\sigma(Y_r) - \sigma(Y_s))\,\mathrm{d}B_r - \sigma_2(Y_s)\int_s^t (B_r - B_s)\,\mathrm{d}B_r \\ &= \int_s^t (\delta\sigma(Y)_{sr} - \sigma_2(Y_s)\,\mathbb{B}_{sr}^1)\,\mathrm{d}B_r\,. \end{aligned}$$

Let us fix $\alpha \in \left]0, \frac{1}{2}\right[$. We prove below that, almost surely,

$$|\delta\sigma(Y)_{st} - \sigma_2(Y_s)\mathbb{B}_{st}^1| \lesssim (t-s)^{2\alpha}, \qquad \forall 0\leqslant s\leqslant t\leqslant T. \tag{4.10}$$

This means that the assumptions of part 3 of Theorem 4.3 are satisfied by $h_r = \sigma(Y_r)$ and $h_r^1 = \sigma_2(Y_r)$ with $\eta = \alpha$: applying (4.9) we then obtain, almost surely,

$$|\delta Y_{st} - \sigma(Y_s)\, \mathbb{B}_{st}^1 - \sigma_2(Y_s)\, \mathbb{B}_{st}^2| \lesssim (t-s)^{3\alpha}.$$

If we fix $\alpha > \frac{1}{3}$, this shows that Y is indeed a solution of the RDE (4.5).

It remains to prove (4.10). By Itô's formula and (4.1) we have, for $0 \leqslant s < t \leqslant T$,

$$\begin{aligned}
\sigma(Y_t) &= \sigma(Y_s) + \int_s^t \sum_{a=1}^k \partial_a \sigma(Y_r)\, \mathrm{d}Y_r^a + \frac{1}{2} \int_s^t \sum_{a,b=1}^k \partial_{ab}\sigma(Y_r)\, \mathrm{d}\langle Y^a, Y^b \rangle_r \\
&= \sigma(Y_s) + \int_s^t \sum_{a=1}^k \partial_a \sigma(Y_r) \sum_{c=1}^d \sigma_c^a(Y_r)\, \mathrm{d}B_r^c + \\
&\quad + \int_s^t \underbrace{\frac{1}{2} \sum_{a,b=1}^k \sum_{c=1}^d \partial_{ab}\sigma\,(Y_r)\, \sigma_c^a(Y_r)\, \sigma_c^b(Y_r)}_{p(Y_r)}\, \mathrm{d}r \\
&= \sigma(Y_s) + \int_s^t \sigma_2(Y_r)\, \mathrm{d}B_r + \int_s^t p(Y_r)\, \mathrm{d}r, \qquad (4.11)
\end{aligned}$$

therefore

$$\delta\sigma(Y)_{st} - \sigma_2(Y_s)\, \mathbb{B}_{st}^1 = \int_s^t (\sigma_2(Y_r) - \sigma_2(Y_s))\, \mathrm{d}B_r + \int_s^t p(Y_r)\, \mathrm{d}r.$$

To prove (4.10), we show that both integrals in the RHS are $O((t-s)^{2\alpha})$.

- Since σ is of class C^2 and Y has continuous paths, the random function $r \mapsto p(Y_r)$ is continuous, hence bounded for $r \in [0, T]$, therefore

$$\left| \int_s^t p(Y_r)\, \mathrm{d}r \right| \lesssim (t-s) \lesssim (t-s)^{2\alpha}, \qquad \forall 0 \leqslant s \leqslant t \leqslant T.$$

- Almost surely Y is of class $\mathcal{C}^\alpha$, thanks to (4.7) from Theorem 4.3 and (4.1). Since σ_2 is of class C^1, hence locally Lipschitz, $r \mapsto \sigma_2(Y_r)$ is of class $\mathcal{C}^\alpha$ too. Applying (4.8) from Theorem 4.3 with $\beta = \alpha$ we then obtain, almost surely,

$$\left| \int_s^t (\sigma_2(Y_r) - \sigma_2(Y_s))\, \mathrm{d}B_r \right| \lesssim (t-s)^{2\alpha}, \qquad \forall 0 \leqslant s \leqslant t \leqslant T.$$

This completes the proof. □

4.3. SDE with a drift

It is natural to consider the SDE (4.1) with a non-zero drift term:

$$\mathrm{d}Y_t = b(Y_t)\, \mathrm{d}t + \sigma(Y_t)\, \mathrm{d}B_t \qquad \text{i.e.}$$

$$Y_t = Y_0 + \int_0^t b(Y_s)\, \mathrm{d}s + \int_0^t \sigma(Y_s)\, \mathrm{d}B_s, \qquad t \geqslant 0, \qquad (4.12)$$

where $b: \mathbb{R}^k \to \mathbb{R}^k$ and $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ are given and we recall that $B = (B_t)_{t \geqslant 0}$ is a d-dimensional Brownian motion. We can generalize Theorem 4.2 as follows.

THEOREM 4.4. (SDE & RDE WITH DRIFT) *If $\sigma(\cdot)$ is of class C^2 and $b(\cdot)$ is continuous, then almost surely any solution $Y = (Y_t)_{t \in [0,T]}$ of the SDE (4.12) is also a solution of the RDE*

$$\delta Y_{st} = b(Y_s)\,(t-s) + \sigma(Y_s)\,\mathbb{B}^1_{st} + \sigma_2(Y_s)\,\mathbb{B}^2_{st} + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{4.13}$$

If $\sigma(\cdot)$ and $b(\cdot)$ are of class C^3 and, furthermore, $\sigma(\cdot)$, $\sigma_2(\cdot)$ and $b(\cdot)$ are globally Lipschitz, i.e. $\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty + \|\nabla b\|_\infty < \infty$, then almost surely the SDE (4.12) and the RDE (4.13) have the same unique solution $Y = (Y_t)_{t \in [0,T]}$.

Proof. We cast the generalized SDE (4.12) in the "usual framework" by adding a component to the driving noise B, i.e. we define $\tilde{B}: [0,T] \to \mathbb{R}^d \times \mathbb{R}$ by

$$\tilde{B}_t := (B_t, t) = (B^1_t, \ldots, B^d_t, t), \qquad t \in [0,T],$$

and accordingly we define $\tilde{\sigma}: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^{d+1})^*$ by

$$\tilde{\sigma}(\cdot)\,\tilde{b} := \sigma(\cdot)\,b + b(\cdot)\,t \qquad \text{for} \qquad \tilde{b} = (b,t) \in \mathbb{R}^d \times \mathbb{R},$$

that is $\tilde{\sigma}(\cdot)^i_j = \sigma(\cdot)^i_j\,\mathbb{1}_{\{j \leqslant d\}} + b(\cdot)^i\,\mathbb{1}_{\{j = d+1\}}$. We can then rewrite the SDE (4.12) as

$$\mathrm{d}Y_t = \tilde{\sigma}(Y_t)\,\mathrm{d}\tilde{B}_t \qquad \text{i.e.} \qquad Y_t = Y_0 + \int_0^t \tilde{\sigma}(Y_s)\,\mathrm{d}\tilde{B}_s, \qquad t \geqslant 0. \tag{4.14}$$

We next extend the Ito rough path $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$ from (4.2), defining

$$\tilde{\mathbb{B}}^1_{st} \;:=\; \tilde{B}_t - \tilde{B}_s = \begin{pmatrix} \mathbb{B}^1_{st} \\ t-s \end{pmatrix}, \tag{4.15}$$

$$\tilde{\mathbb{B}}^2_{st} \;:=\; \int_s^t (\tilde{B}_r - \tilde{B}_s) \otimes \mathrm{d}\tilde{B}_r = \begin{pmatrix} \mathbb{B}^2_{st} & \int_s^t (B_r - B_s)\,\mathrm{d}r \\ \int_s^t (r-s)\,\mathrm{d}B_r & \int_s^t (r-s)\,\mathrm{d}r = \frac{(t-s)^2}{2} \end{pmatrix}. \tag{4.16}$$

One can show that $\tilde{\mathbb{B}} = (\tilde{\mathbb{B}}^1, \tilde{\mathbb{B}}^2)$ *is a rough path over $\tilde{B}$*, following closely the proof of Theorem 4.1. Indeed, if we fix $\alpha \in \left]0, \frac{1}{2}\right[$, we have almost surely $B \in \mathcal{C}^\alpha$, hence

$$\left|\int_s^t (B_r - B_s)\,\mathrm{d}r\right| \lesssim (t-s)^{\alpha+1}, \qquad \left|\int_s^t (r-s)\,\mathrm{d}B_r\right| \lesssim (t-s)^{\alpha+1}. \tag{4.17}$$

We can now write the RDE which generalizes (4.5):

$$\delta Y_{st} = \tilde{\sigma}(Y_s)\,\tilde{\mathbb{B}}^1_{st} + \tilde{\sigma}_2(Y_s)\,\tilde{\mathbb{B}}^2_{st} + o(t-s)\,. \tag{4.18}$$

Interestingly, plugging the definitions of $\tilde{\mathbb{B}}$ and $\tilde{\sigma}$ into (4.18) *we do not obtain (4.13)*, because the components of $\tilde{\mathbb{B}}^2_{st}$ other than $\mathbb{B}^2_{st}$ are missing in (4.13), see (4.16). The point is that these components can be absorbed in the reminder $o(t-s)$, see (4.17), hence *the RDE (4.18) and (4.13) are fully equivalent*.

To complete the proof, we are left with comparing the SDE (4.14) with the RDE (4.18). This can be done following the very same arguments as in the proof of Theorem 4.2. The details are left to the reader. □

Remark 4.5. The strategy of adding the drift term as an additional component of the driving noise, as in the proof of Theorem 4.4, suffers from a technical limitation, namely *we are forced to use the same regularity exponent α for all components*, due to Definition 3.2 of rough paths. This prevents us from exploiting the additional regularity of the drift term: for instance, in the second part of Theorem 4.4, the assumption that $b(\cdot)$ is of class C^3 could be removed, because the "driving noise" t is smooth and the classical theory of ordinary differential equations applies.

A natural solution would be to generalize Definition 3.2, allowing rough paths to have a different regularity exponent for each component. The key results can be generalized to this setting, but for simplicity we refrain from pursuing this path.

4.4. A refined Kolmogorov criterion

In this section we prepare the ground for the proof of Lemmas 4.10 and 4.11 in Section 4.5 below, which are the main technical tools in the proof of Theorem 4.3. We suppose without loss of generality that $T=1$, namely our processes are defined on the interval $[0,1]$. Define the set $\mathbb{D}$ of dyadic points in $[0,1]$ by

$$\mathbb{D} := \bigcup_{k\geq 0} D_k, \qquad \text{where} \qquad D_k := \left\{d_i^k := \tfrac{i}{2^k}\right\}_{0\leqslant i\leqslant 2^k}. \tag{4.19}$$

Given $d,\tilde{d}\in\mathbb{D}$, we write $d\to\tilde{d}$ if and only if $\tilde{d}$ is consecutive to d in some layer D_k of $\mathbb{D}$, that is $d=d_i^k$ and $\tilde{d}=d_{i+1}^k$, for some $k\geq 0$ and $0\leqslant i\leqslant 2^k-1$.

Remarkably, in order to prove relation (4.35), *it is enough to have a suitable control on $R_{d,\tilde{d}}$ for consecutive points $d\to\tilde{d}$* (together with a global control on δR), as the next result shows. This turns out to be at the heart of the Kolmogorov continuity criterion, but we stress that it is a deterministic statement.

THEOREM 4.6. (KOLMOGOROV CRITERION: DETERMINISTIC PART) *Given a function $A\colon\mathbb{D}^2_<\to\mathbb{R}$, for $0<\rho<\gamma$ we define the constants*

$$Q_\gamma := \sup_{d,\tilde{d}\in\mathbb{D}: d\to\tilde{d}} \frac{|A_{d,\tilde{d}}|}{|\tilde{d}-d|^\gamma}, \tag{4.20}$$

$$K_{\rho,\gamma} := \sup_{\substack{0\leqslant s<u<t\leqslant 1\\ s,u,t\in\mathbb{D}}} \frac{|\delta A_{s,u,t}|}{\min(u-s,t-u)^\rho |t-s|^{\gamma-\rho}}. \tag{4.21}$$

Then there is a constant $C_{\rho,\gamma}<\infty$ such that

$$|A_{st}|\leqslant C_{\rho,\gamma}(Q_\gamma+K_{\rho,\gamma})|t-s|^\gamma, \quad \forall (s,t)\in\mathbb{D}^2_<. \tag{4.22}$$

A key tool for Theorem 4.6 is the next result, proved at this end of this section, which ensures the existence of suitable *short paths* in $\mathbb{D}$.

LEMMA 4.7. (DYADIC PATHS) *For any $s,t\in\mathbb{D}$ with $s<t$, there are integers $n,m\geq 1$ and a path of $(m+n+1)$ points in $\mathbb{D}$ which leads from s to t, labelled as follows:*

$$s=s_m<\ldots<s_1<s_0=t_0<t_1<\ldots<t_n=t, \tag{4.23}$$

with the property that for all $i\in\{0,\ldots,m-1\}$ and $j\in\{0,\ldots,n-1\}$

$$s_{i+1}\to s_i,\quad t_j\to t_{j+1};\quad |s_i-s_{i+1}|<\frac{|t-s|}{2^i},\quad |t_{j+1}-t_j|<\frac{|t-s|}{2^j}. \tag{4.24}$$

Proof of Theorem 4.6. Fix $s,t\in\mathbb{D}$ with $s<t$. We use Lemma 4.7 with the same notation. By the definition of δA, we write

$$A_{st}=A_{st_0}+A_{t_0t}+\delta A_{s,t_0,t}\,.$$

In the case $m\geq 2$, we can develop A_{st_0} as follows (recall that $s=s_m$ and $s_0=t_0$):

$$A_{st_0}\;=\;\sum_{i=0}^{m-1}A_{s_{i+1}s_i}+\sum_{i=0}^{m-2}\delta A_{s,s_{i+1},s_i}\,.$$

Similarly, when $n\geq 2$, we develop

$$A_{t_0t}=\sum_{j=0}^{n-1}A_{t_jt_{j+1}}+\sum_{j=0}^{n-2}\delta A_{t_j,t_{j+1},t},$$

so that

$$\begin{aligned}A_{st}\;=\;&\underbrace{\sum_{i=0}^{m-1}A_{s_{i+1}s_i}+\sum_{j=0}^{n-1}A_{t_jt_{j+1}}+}_{\Xi_1}\\&\underbrace{+\delta A_{s,t_0,t}+\sum_{i=0}^{m-2}\delta A_{s,s_{i+1},s_i}+\sum_{j=0}^{n-2}\delta A_{t_j,t_{j+1},t}.}_{\Xi_2}\end{aligned} \tag{4.25}$$

By the definition of Q_γ, for any $d\to\tilde{d}$ we can bound

$$|A_{d\tilde{d}}|\leqslant Q_\gamma|\tilde{d}-d|^\gamma.$$

By Lemma 4.7, this bound applies to any couple (s_{i+1},s_i) and (t_j,t_{j+1}). Then we can estimate Ξ_1 in (4.25) as follows, exploiting the bounds in (4.24):

$$\begin{aligned}&Q_\gamma\left\{\sum_{i=0}^{m-1}|s_i-s_{i+1}|^\gamma+\sum_{j=0}^{n-1}|t_{j+1}-t_j|^\gamma\right\}\leqslant\\&\leqslant Q_\gamma\left\{\sum_{i=0}^{\infty}(2^{-i})^\gamma+\sum_{j=0}^{\infty}(2^{-j})^\gamma\right\}|t-s|^\gamma=\\&=Q_\gamma\left\{\frac{2}{1-2^{-\gamma}}\right\}|t-s|^\gamma,\end{aligned}$$

which agrees with (4.22). On the other hand, thanks to (4.21) and (4.24),

$$|\delta A_{s,s_{i+1},s_i}| \leqslant K_{\rho,\gamma}\left(\frac{|t-s|}{2^i}\right)^{\rho} |t-s|^{\gamma-\rho} = K_{\rho,\gamma}\, 2^{-i\rho}\, |t-s|^{\gamma}$$

and similarly for $\delta A_{t_j,t_{j+1},t}$, so that the term Ξ_2 can be bounded above by

$$K_{\rho,\gamma}|t-s|^{\gamma}\left(1+\sum_{i=0}^{m-2} 2^{-i\rho}+\sum_{j=0}^{n-2} 2^{-j\rho}\right) \leqslant K_{\rho,\gamma}|t-s|^{\gamma}\left(1+\frac{2}{1-2^{-\rho}}\right).$$

This completes the proof of (4.22). □

As a simple consequence of Theorem 4.6, we show that suitable moment conditions ensure the finiteness of the constant Q_γ in (4.20), as in the classical Kolmogorov criterion.

PROPOSITION 4.8. (KOLMOGOROV CRITERION: PROBABILISTIC PART) *Let* $A = (A_{st})_{(s,t)\in\mathbb{D}^2_<}$ *be a stochastic process which satisfies the following bound, for some* γ_0, $p, c \in (0,\infty)$*:*

$$\mathbb{E}[|A_{st}|^p] \leqslant c|t-s|^{p\gamma_0}, \qquad \forall (s,t)\in\mathbb{D}^2_<.$$

Then, for any value of γ *such that*

$$\gamma < \gamma_0 - \tfrac{1}{p}, \tag{4.26}$$

the random variable $Q_\gamma = Q_\gamma(A)$ *defined in (4.20) is in* L^p*:*

$$\mathbb{E}[|Q_\gamma|^p] \leqslant \frac{c}{1-2^{1-p(\gamma_0-\gamma)}} < \infty. \tag{4.27}$$

In particular, a.s. $Q_\gamma < \infty$*.*

Proof. By definition of Q_γ in (4.20), bounding the supremum with a sum we can write

$$|Q_\gamma|^p \leqslant \sum_{d,\tilde d\in\mathbb{D}: d\to\tilde d} \left(\frac{|A_{d,\tilde d}|}{|\tilde d - d|^{\gamma}}\right)^p = \sum_{k\geq 0}\sum_{i=0}^{2^k-1} \frac{|A_{d_i^k d_{i+1}^k}|^p}{|d_{i+1}^k - d_i^k|^{p\gamma}}.$$

Let us write $\gamma = \gamma_0 - \frac{1+\epsilon}{p}$, for some $\epsilon > 0$. Since $d_{i+1}^k - d_i^k = \frac{1}{2^k}$ we have

$$\begin{aligned}
\mathbb{E}[|Q_\gamma|^p] &\leqslant \sum_{k\geq 0}\sum_{i=0}^{2^k-1} c|d_{i+1}^k - d_i^k|^{p(\gamma_0-\gamma)} \\
&\leqslant \sum_{k\geq 0}\sum_{i=0}^{2^k-1} \frac{c}{2^{(1+\epsilon)k}} = \sum_{k\geq 0}\frac{c}{2^{\epsilon k}} = \frac{c}{1-2^{-\epsilon}} < \infty.
\end{aligned}$$

The proof is complete. □

Remark 4.9. Given a stochastic process $(X_t)_{t\in\mathbb{D}}$ defined on dyadic times, if we apply Theorem 4.6 and Proposition 4.8 to $(A_{st} := \delta X_{st} = X_t - X_s)_{(s,t)\in\mathbb{D}^2_<}$ we obtain the classical Kolmogorov continuity criterion. Note that in this case $K_{\rho,\gamma} = 0$ because $\delta A = 0$.

Proof of Lemma 4.7. We refer to Figure 4.1 for a graphical representation. Given $s, t \in \mathbb{D}$ with $s < t$, since $0 < t - s \leqslant 1$, we can define $\ell \geq 1$ as the unique integer such that

$$\frac{1}{2^\ell} < t - s \leqslant \frac{1}{2^{\ell-1}}. \tag{4.28}$$

We now take the smallest $k \in \{0, \ldots, 2^\ell - 1\}$ for which $d_k^\ell > s$ and define

$$s_0 := t_0 := d_k^\ell.$$

The definition of k guarantees that $d_k^\ell < t$, because if $d_k^\ell \geqslant t$ then $\frac{k}{2^\ell} - s \geqslant t - s > \frac{1}{2^\ell}$ and this would violate the minimality of k.

Note that $0 < d_k^\ell - s \leqslant d_k^\ell - d_{k-1}^\ell = \frac{1}{2^\ell}$ and $0 < t - d_k^\ell < t - s$, by (4.28), therefore

$$0 < s_0 - s < \frac{1}{2^{\ell-1}}, \qquad 0 < t - t_0 < \frac{1}{2^{\ell-1}}. \tag{4.29}$$

Since both $s_0 - s \in \mathbb{D}$ and $t - t_0 \in \mathbb{D}$, for suitable integers $m \geq 1$ and $n \geq 1$ we have

$$s_0 - s = \frac{1}{2^{q_1}} + \frac{1}{2^{q_2}} + \ldots + \frac{1}{2^{q_m}}, \qquad t - t_0 = \frac{1}{2^{r_1}} + \frac{1}{2^{r_2}} + \ldots + \frac{1}{2^{r_n}},$$

where $q_m > q_{m-1} > \ldots > q_1 \geq \ell$ and $r_n > \ldots > r_1 \geq \ell$. We can thus write

$$\begin{aligned} s &= s_0 - \frac{1}{2^{q_1}} - \frac{1}{2^{q_2}} - \ldots - \frac{1}{2^{q_m}}, \\ t &= t_0 + \frac{1}{2^{r_1}} + \frac{1}{2^{r_2}} + \ldots + \frac{1}{2^{r_n}}. \end{aligned}$$

We can finally define

$$\begin{aligned} s_i &:= s_0 - \frac{1}{2^{q_1}} - \frac{1}{2^{q_2}} - \ldots - \frac{1}{2^{q_i}} \quad \text{for } i = 1, \ldots, m, \\ t_j &:= t_0 + \frac{1}{2^{r_1}} + \frac{1}{2^{r_2}} + \ldots + \frac{1}{2^{r_j}} \quad \text{for } j = 1, \ldots, n. \end{aligned}$$

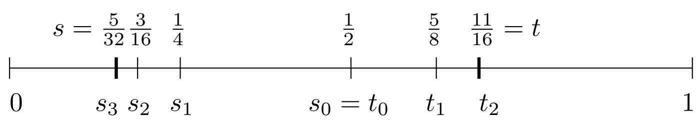

Figure 4.1. An instance of Lemma 4.7 with $s = \frac{5}{32}$ and $t = \frac{11}{16}$. Note that $\ell = 1$ (because $\frac{1}{2^1} < |t - s| = \frac{17}{32} \leq \frac{1}{2^0}$, cf. (4.28)) and $s_0 = t_0 = \frac{1}{2}$. The points $t_1, \ldots, t_n$ are built iteratively: first take the largest $\frac{1}{2^{r_1}}$ (i.e. the smallest r_1) such that $t_1 := t_0 + \frac{1}{2^{r_1}} \leq t$; if $t_1 < t$, then take the largest $\frac{1}{2^{r_2}}$ such that $t_2 := t_1 + \frac{1}{2^{r_2}} \leq t$; and so on, until $t_n = t$. Similarly for $s_1, \ldots, s_m$.

Since q_i and r_j are strictly increasing integers with $q_1 \geq \ell$ and $r_1 \geq \ell$, we have the bounds $q_i \geq \ell + (i - 1)$ and $r_j \geq \ell + (j - 1)$, for all $i \in \{0, \ldots, m - 1\}$ and $j \in \{0, \ldots, n - 1\}$, hence

$$\begin{aligned} |s_i - s_{i+1}| &= \frac{1}{2^{q_{i+1}}} \leqslant \frac{1}{2^i} \frac{1}{2^\ell} < \frac{|t - s|}{2^i}, \\ |t_{j+1} - t_j| &= \frac{1}{2^{q_{j+1}}} \leqslant \frac{1}{2^j} \frac{1}{2^\ell} < \frac{|t - s|}{2^j}. \end{aligned}$$

having used (4.28). This proves the bounds in (4.24).

We note that, for any integer $r \geq \ell$, we have the inclusion $D_\ell \subseteq D_r$. Then, given any $x \in D_\ell$, we have that $x \in D_r$, hence $x \to x + 2^{-r}$. Since $t_0 = d_k^\ell \in D_\ell$ and $r_1 \geq \ell$, this shows that $t_0 \to t_1 = t_0 + 2^{-r_1}$. Proceeding inductively, we have $t_j \to t_{j+1} = t_j + 2^{-r_{j+1}}$. A similar argument applies to the points s_i and completes the proof of (4.24). □

4.5. Proof of Theorem 4.3

In this section we prove the three assertions of Theorem 4.3.

Proof of the first assertion of Theorem 4.3. We want to prove that for any $\alpha \in \left(0, \frac{1}{2}\right)$, a.s. I is α-Hölder continuous, namely there is an a.s. finite random constant C such that

$$|\delta I_{st}| \leqslant C|t-s|^\alpha, \qquad \forall 0 \leqslant s \leqslant t \leqslant T. \tag{4.30}$$

First observation: if the claim holds under the stronger assumption $|h| \leqslant c$ almost surely, for some deterministic $c < \infty$, then we can deduce the general result by localization. Indeed, if we only assume that $\sup_{[0,T]} |h| < \infty$ a.s., we can define for $n \in \mathbb{N}$ the stopping times

$$\tau_n := \inf\{t \in [0,T]: |h_t| > n\}.$$

Let us define

$$h_s^{(n)} := h_{s \wedge \tau_n}, \qquad I_t^{(n)} := \int_0^t h_s^{(n)} \mathrm{d}B_s.$$

Note that $\sup_{[0,T]} |h^{(n)}| \leqslant n$ by the definition of τ_n. Then

$$|\delta I_{st}^{(n)}| \leqslant C^{(n)} |t-s|^\alpha, \qquad \forall 0 \leqslant s < t \leqslant T, \tag{4.31}$$

for a suitable a.s. finite random constant $C^{(n)}$. Let us define the events

$$A_n := \{\tau_n = \infty\} = \{\sup_{[0,T]} |h| \leqslant n\}$$

and note that $h = h^{(n)}$ on A_n. By the locality property of the stochastic integral, $I = I^{(n)}$ a.s. on A_n[4.1].

Note that $A := \bigcup_{n \in \mathbb{N}} A_n = \{\sup_{[0,T]} |h| < \infty\}$, hence $\mathbb{P}(A) = 1$. If we define $C := C^{(n)}$ on $A_n \setminus A_{n-1}$ (with $A_0 := \emptyset$) and $C := \infty$ on A^c, we have $C < \infty$ a.s. and relation (4.7) holds.

Second observation: if relation (4.30) holds for all s, t in a (deterministic) dense subset $\mathbb{D} \subseteq [0,T]$, then it holds for all $s, t \in [0,T]$, because δI_{st} is a continuous function of (s,t).

In conclusion, the proof is reduced to showing (4.30) only for $s, t \in \mathbb{D}$, under the assumption that $\sup_{[0,T]} |h| \leqslant c < \infty$ almost surely. Suppose that this is the case and set $A_{st} := \delta I_{st}$, $0 \leqslant s \leqslant t \leqslant T$. Here $\delta A = 0$ and therefore the constant $K_{\rho,\gamma}$ in (4.21) is equal to zero for any $0 < \rho < \gamma$. It remains to estimate Q_α using Proposition 4.8.

4.1. We mean that $I^{(n)}$ and I are indistinguishable on A_n: for a.e. $\omega \in A_n$ one has $I_t^{(n)}(\omega) = I_t(\omega)$ for all $t \in [0,1]$ (we recall that we always fix continuous versions of the stochastic integrals).

By the BDG inequality of Proposition 4.12, for any $p \geqslant 2$

$$\mathbb{E}[|\delta I_{st}|^p] \leqslant c_p \,\mathbb{E}\bigg[\bigg(\int_s^t h_u^2 \,\mathrm{d}u\bigg)^{\frac{p}{2}}\bigg] \leqslant C_p |t-s|^{\frac{p}{2}}.$$

Then Proposition 4.8 applies with $\gamma_0 = \frac{1}{2}$ and any $\alpha = \gamma_0 - \frac{1}{p} \in \left(0, \frac{1}{2}\right)$ for p sufficiently large. By Theorem 4.6, we obtain (4.30) and the proof is complete. □

For $0 \leqslant s \leqslant t \leqslant T$ we define the (random) continuous function

$$R_{st} := I_t - I_s - h_s\,(B_t - B_s) = \int_s^t \delta h_{sr} \,\mathrm{d}B_r. \tag{4.32}$$

We recall that a.s. $B \in \mathcal{C}^\alpha$ for every $\alpha < \frac{1}{2}$.

Proof of the second assertion of Theorem 4.3. Let $\alpha < \frac{1}{2}$. We want to show that, if a.s. $h \in \mathcal{C}^\beta$, for some $\beta \in (0,1]$, then there is an a.s. finite random constant C such that

$$|R_{st}| \leqslant C\,|t-s|^{\alpha+\beta}, \qquad \forall 0 \leqslant s \leqslant t \leqslant T. \tag{4.33}$$

First observation: if the claim holds under the stronger assumption $\|\delta h\|_\beta \leqslant c$ almost surely, for some deterministic $c < \infty$, then we can deduce the general result by localization. Indeed, if we only assume that $\|\delta h\|_\beta < \infty$ a.s., we can define for $n \in \mathbb{N}$ the stopping times

$$\tau_n := \inf\{t \in [0,1] \colon \|\delta h\|_{\beta,[0,t]} > n\},$$

where $\|\delta h\|_{\alpha,[0,t]}$ is the Hölder semi-norm of h restricted to $[0,t]$ (equivalently, the Hölder semi-norm of $s \mapsto h_{s\wedge t}$ on the whole interval $s \in [0,1]$). Let us define

$$h_s^{(n)} := h_{s\wedge\tau_n}, \qquad I_t^{(n)} := \int_0^t h_s^{(n)} \mathrm{d}B_s, \qquad R_{st}^{(n)} := I_t^{(n)} - I_s^{(n)} - h_s^{(n)}(B_t - B_s).$$

Note that $\|\delta h^{(n)}\|_\beta \leqslant n$, by definition of τ_n. (Indeed, $\|\delta h\|_{\beta,[0,t]} \leqslant n$ for all $t < \tau_n$, which means that $|h(r) - h(s)| \leqslant n|r-s|^\beta$ for all $r, s \in [0, \tau_n)$; then, by continuity, $|h(r) - h(s)| \leqslant n|r-s|^\beta$ for all $r, s \in [0, \tau_n]$, which means that $\|\delta h\|_{\beta,[0,\tau_n]} = \|\delta h^{(n)}\|_\beta \leqslant n$). Then

$$|R_{st}^{(n)}| \leqslant C^{(n)} |t-s|^{\alpha+\beta}, \qquad \forall 0 \leqslant s < t \leqslant T, \tag{4.34}$$

for a suitable a.s. finite random constant $C^{(n)}$. Let us define the events

$$A_n := \{\tau_n = \infty\} = \{\|\delta h\|_\alpha \leqslant n\}$$

and note that $h = h^{(n)}$ on A_n. By the locality property of the stochastic integral, $I = I^{(n)}$ a.s. on A_n,[4.2] hence also $R = R^{(n)}$ a.s. on A_n. Redefining $C^{(n)} = \infty$ on the exceptional set $\{R = R^{(n)}\}^c$, we get by (4.34)

$$\text{on the event } A_n\colon \qquad |R_{st}| \leqslant C^{(n)} |t-s|^{\alpha+\beta}, \qquad \forall 0 \leqslant s < t \leqslant T.$$

4.2. We mean that $I^{(n)}$ and I are indistinguishable on A_n: for a.e. $\omega \in A_n$ one has $I_t^{(n)}(\omega) = I_t(\omega)$ for all $t \in [0,1]$ (we recall that we always fix continuous versions of the stochastic integrals).

Note that $A := \bigcup_{n\in\mathbb{N}} A_n = \{\|\delta h\|_\beta < \infty\}$, hence $\mathbb{P}(A) = 1$. If we define $C := C^{(n)}$ on $A_n \setminus A_{n-1}$ (with $A_0 := \emptyset$) and $C := \infty$ on A^c, we have $C < \infty$ a.s. and relation (4.8) holds.

Second observation: if relation (4.33) holds for all s,t in a (deterministic) dense subset $\mathbb{D} \subseteq [0,1]$, then it holds for all $s,t \in [0,1]$, because R_{st} is a continuous function of (s,t).

In conclusion, the proof is reduced to showing (4.33) only for $s,t \in \mathbb{D}$, under the assumption that $\|\delta h\|_\beta \leqslant c < \infty$. This technical result is formulated in the separate Lemma 4.10. □

LEMMA 4.10. *Let $0 < \alpha < \frac{1}{2}$ and $0 < \beta \leqslant 1$. Assume that $\mathbb{E}[\|\delta h\|_\beta^p] < \infty$ for all $p > 0$. Then there is an a.s. finite random constant C such that*

$$|R_{st}| \leqslant C|t-s|^{\alpha+\beta}, \qquad \forall s,t \in \mathbb{D} \quad \text{with } s \leqslant t. \tag{4.35}$$

Equivalently, a.s. $R \in C_2^{\alpha+\beta}$.

Proof. We apply Theorem 4.6 to the (random) function $A(s,t) = R_{st}$, with $\gamma = \alpha + \beta$ and $\rho = \alpha \wedge \beta$. Then relation (4.22) yields (4.35). It remains to show that a.s. $Q_{\alpha+\beta} < \infty$ and $K_{\rho,\alpha+\beta} < \infty$.

We recall that R_{st} is defined in (4.32). In particular, for $s < u < t$

$$\delta R_{sut} = R_{st} - R_{su} - R_{ut} = (h_u - h_s)(B_t - B_u).$$

Then by (4.21), a.s.

$$K_{\rho,\alpha+\beta}(R) \leqslant \|\delta h\|_\beta \|\delta B\|_\alpha \sup_{0\leqslant s<u<t\leqslant 1} \frac{|u-s|^\beta |t-u|^\alpha}{\min(u-s, t-u)^{\alpha\wedge\beta} |t-s|^{\alpha\vee\beta}}.$$

By our assumption that $\|\delta h\|_\beta \in L^p$ and by the fact that B is a Brownian motion, it follows that $\|\delta h\|_\beta \|\delta B\|_\alpha < \infty$ a.s., hence it only remains to show that the constant defined by the supremum is bounded above by 1. However, this constant equals

$$\sup_{a,b>0,\ a+b=1} \frac{a^\alpha b^\beta}{(a\wedge b)^{\alpha\wedge\beta}} = \sup_{a,b>0,\ a+b=1} \left(\frac{ab}{(a\wedge b)}\right)^{\alpha\wedge\beta} a^{\alpha-\alpha\wedge\beta} b^{\beta-\alpha\wedge\beta} \leqslant 1.$$

We want now to estimate $Q_{\alpha+\beta}(R)$. We note that, for fixed $s < t$, we have a.s. $R_{st} = \int_s^t (h_u - h_s)\,\mathrm{d}B_u$. By the Burkholder-Davies-Gundy inequality, see Proposition 4.12, for any $p > 2$ there is a universal constant c_p such that

$$\begin{aligned}
\mathbb{E}[|R_{st}|^p] &\leqslant c_p \mathbb{E}\left[\left(\int_s^t (h_u - h_s)^2 \,\mathrm{d}u\right)^{\frac{p}{2}}\right] \\
&\leqslant c_p \mathbb{E}\left[\|\delta h\|_\beta^p \left(\int_s^t (u-s)^{2\beta} \,\mathrm{d}u\right)^{\frac{p}{2}}\right] \\
&\leqslant c_p \mathbb{E}[\|\delta h\|_\beta^p]\,(t-s)^{p\left(\beta+\frac{1}{2}\right)}.
\end{aligned}$$

By Proposition 4.8, we have $Q_\gamma < \infty$ a.s. for any $\gamma < \beta + \frac{1}{2} - \frac{1}{p}$. Plugging $\gamma = \alpha + \beta$ we get $\alpha < \frac{1}{2} - \frac{1}{p}$, which is satisfied for p large enough, since $\alpha < \frac{1}{2}$. □

Next, we suppose that there exists another adapted process $h^1 = (h^1_t)_{t\in[0,T]}$ with values in $\mathbb{R}^k \otimes (\mathbb{R}^d)^*$ such that a.s.

$$|\delta h_{st} - h^1_s\, \mathbb{B}^1_{st}| \lesssim |t-s|^{\eta+\alpha}.$$

Then we define

$$\begin{aligned}\hat{R}_{st} &:= R_{st} - h^1_s\, \mathbb{B}^2_{st} = \delta I_{st} - h_s\, \mathbb{B}^1_{st} - h^1_s\, \mathbb{B}^2_{st} \\ &= \int_s^t (\delta h_{sr} - h^1_s\, \mathbb{B}^1_{sr})\, \mathrm{d}B_r, \end{aligned} \tag{4.36}$$

where $\mathbb{B}^2$ is defined in (4.2). Then the third assertion of Theorem 4.3 follows with the same localisation argument as for the second one and from the following

LEMMA 4.11. *Assume that* $\mathbb{E}[\|\delta h^1\|^p_\eta + \|\delta h - h^1\mathbb{B}^1\|^p_{\eta+\alpha}] < \infty$, *for some* $\alpha \in (0, \frac{1}{2})$ *and for all* $p > 0$. *Then there is an a.s. finite random constant* C *such that*

$$|\hat{R}_{st}| \leqslant C|t-s|^{\eta+2\alpha}, \qquad \forall s,t \in \mathbb{D} \quad \text{with } s \leqslant t. \tag{4.37}$$

Equivalently, a.s. $\hat{R} \in C_2^{\eta+2\alpha}$.

Proof. We apply Theorem 4.6 to the (random) function $A(s,t) = R_{st}$, with $\gamma = \alpha + \beta$ and $\rho = \alpha \wedge \eta$. Then

$$\delta \hat{R}_{sut} = (\delta h_{su} - h^1_s\, \mathbb{B}^1_{su})\, \mathbb{B}^1_{ut} + \delta h^1_{su}\, \mathbb{B}^2_{ut}.$$

Now

$$\begin{aligned} K_{\rho,\eta+2\alpha}(\hat{R}) &\leqslant \|\delta h - h^1\mathbb{B}^1\|_{\eta+\alpha} \|\mathbb{B}^1\|_\alpha \sup_{0\leqslant s<u<t\leqslant 1} \frac{|u-s|^{\eta+\alpha}|t-u|^\alpha}{\min(u-s,t-u)^\rho |t-s|^{\eta+2\alpha-\rho}} \\ &\quad + \|\delta h^1\|_\eta \|\mathbb{B}^2\|_{2\alpha} \sup_{0\leqslant s<u<t\leqslant 1} \frac{|u-s|^\eta |t-u|^{2\alpha}}{\min(u-s,t-u)^\rho |t-s|^{\eta+2\alpha-\rho}}. \end{aligned}$$

We note that the first supremum is equal to

$$\sup_{a,b>0,a+b=1} \frac{a^{\eta+\alpha}\, b^\alpha}{(a\wedge b)^\rho} \leqslant \sup_{a,b>0,a+b=1} \left(\frac{a\,b}{a\wedge b}\right)^{\alpha\wedge\eta} a^{\alpha\vee\eta}\, b^{\alpha-\alpha\wedge\eta} \leqslant 1,$$

while the second supremum is equal to

$$\sup_{a,b>0,a+b=1} \frac{a^\eta\, b^{2\alpha}}{(a\wedge b)^\rho} \leqslant \sup_{a,b>0,a+b=1} \left(\frac{a\,b}{a\wedge b}\right)^{\alpha\wedge\eta} a^{\eta-\alpha\wedge\eta}\, b^{2\alpha-\alpha\wedge\eta} \leqslant 1.$$

Now by (4.36)

$$\begin{aligned} \mathbb{E}[|\hat{R}_{st}|^p] &\leqslant \mathbb{E}\left[\left(\int_s^t (\delta h_{su} - h^1_s\, \mathbb{B}^1_{su})^2\, \mathrm{d}u\right)^{\frac{p}{2}}\right] \\ &\leqslant c_p\, \mathbb{E}\left[\|\delta h - h^1\mathbb{B}^1\|^p_{\eta+\alpha} \left(\int_s^t (u-s)^{2(\eta+\alpha)}\, \mathrm{d}u\right)^{\frac{p}{2}}\right] \\ &\leqslant c_p\, \mathbb{E}[\|\delta h - h^1\mathbb{B}^1\|^p_{\eta+\alpha}]\, (t-s)^{p\left(\eta+\alpha+\frac{1}{2}\right)}. \end{aligned}$$

By Proposition 4.8, we have $Q_\gamma < \infty$ a.s. for any $\gamma < \eta + \alpha + \frac{1}{2} - \frac{1}{p}$. Plugging $\gamma = \eta + 2\alpha$ we get $\alpha < \frac{1}{2} - \frac{1}{p}$, which is satisfied for p large enough, since $\alpha < \frac{1}{2}$. □

Finally, we give a proof of (half of) Burkholder-Davies-Gundy inequality for $p \geq 2$.

PROPOSITION 4.12. *For all $p \geq 2$ there is a constant $c_p < \infty$ such that for all $0 \leqslant s \leqslant t \leqslant T$*

$$\mathbb{E}\bigg[\bigg|\int_s^t y_u \, \mathrm{d}B_u\bigg|^p\bigg] \leqslant c_p \, \mathbb{E}\bigg[\bigg(\int_s^t y_u^2 \, \mathrm{d}u\bigg)^{\frac{p}{2}}\bigg]$$

for any progressively measurable process such that $\mathbb{P}$-a.s. $\int_0^1 y_u^2 \, \mathrm{d}u < \infty$.

Proof. To simplify the notation we set $s = 0$ and $m_t := \int_0^t y_u \, \mathrm{d}B_u$.

First we make the additional assumptions that $\mathbb{E}\big[\int_0^1 y_u^2 \mathrm{d}u\big] < \infty$ and m is bounded by some deterministic constant. By the Itô formula applied to m_t, we get

$$\mathrm{d}|m_t|^p = p|m_t|^{p-1}\mathrm{sgn}(m_t) y_t \, \mathrm{d}B_t + \frac{p(p-1)}{2}|m_t|^{p-2} y_t^2 \, \mathrm{d}t.$$

In general $(\int_0^t |m_u|^{p-1}\mathrm{sgn}(m_u) y_u \, \mathrm{d}B_u)_t$ is a local martingale, but under our additional assumptions it is a true martingale with zero expectation, because $\mathbb{E}[\int_0^1 |m_u|^{2(p-1)} \, y_u^2 \mathrm{d}u] < \infty$ (recall that m is bounded). Consequently

$$\mathbb{E}[|m_t|^p] = \frac{p(p-1)}{2} \, \mathbb{E}\bigg[\int_0^t |m_u|^{p-2} \, y_u^2 \, \mathrm{d}u\bigg].$$

If we set $|\bar{m}_t| := \sup_{u \leqslant t} |m_u|$, we obtain by Hölder

$$\begin{aligned}
\mathbb{E}[|m_t|^p] &\leqslant \frac{p(p-1)}{2}\mathbb{E}\bigg[|\bar{m}_t|^{p-2}\int_0^t y_u^2 \, \mathrm{d}u\bigg] \\
&\leqslant \frac{p(p-1)}{2} \, \mathbb{E}[|\bar{m}_t|^p]^{1-\frac{2}{p}} \mathbb{E}\bigg[\bigg(\int_0^t y_u^2 \, \mathrm{d}u\bigg)^{\frac{p}{2}}\bigg]^{\frac{2}{p}}. \qquad (4.38)
\end{aligned}$$

Since $(|m_t|)_{t \geq 0}$ is submartingale bounded in L^p with continuous trajectories, by Doob L^p inequality we have: $\mathbb{E}[|\bar{m}_t|^p] \leqslant (\frac{p}{p-1})^p \, \mathbb{E}[|m_t|^p]$. Plugging the above in (4.38) we conclude:

$$\mathbb{E}\bigg[\bigg|\int_0^t y_u \, \mathrm{d}B_u\bigg|^p\bigg] \leqslant c_p \, \mathbb{E}\bigg[\bigg(\int_0^t y_u^2 \, \mathrm{d}u\bigg)^{\frac{p}{2}}\bigg].$$

As far as the general case is concerned, let us define

$$\tau^n = \inf\{t \geq 0 \colon |m_t| > n\} \wedge \inf\bigg\{t \geq 0 \colon \int_0^t y_u^2 \, \mathrm{d}u > n\bigg\}$$

Note that $\mathbb{P}$-a.s. τ^n is a non decreasing sequence of stopping times, with $\tau^n = \infty$ for n large enough. We denote $y_t^n := y \, \mathbb{1}_{[0,\tau^n]}(t)$ and $m_t^n := \int_0^t y_u^n \mathrm{d}B_u$. By construction, y^n and m^n satisfy our additional assumptions. Since $m_t^n = m_{t \wedge \tau^n}$ a.s., we have

$$\begin{aligned}
\mathbb{E}\bigg[\bigg|\int_0^{t \wedge \tau^n} y_u \, \mathrm{d}B_u\bigg|^p\bigg] &\leqslant c_p \, \mathbb{E}\bigg[\bigg(\int_0^t y_u^2 \, \mathbf{1}_{[0,\tau^n]}(u) \, \mathrm{d}u\bigg)^{\frac{p}{2}}\bigg] \\
&\leqslant c_p \, \mathbb{E}\bigg[\bigg(\int_0^t y_u^2 \, \mathrm{d}u\bigg)^{\frac{p}{2}}\bigg].
\end{aligned}$$

Finally we notice that by Fatou's Lemma

$$\begin{aligned}\mathbb{E}\Bigg[\bigg|\int_0^t y_u \,\mathrm{d}B_u\bigg|^p\Bigg] &= \mathbb{E}\Bigg[\liminf_{n\to\infty}\bigg|\int_0^{t\wedge\tau^n} y_u \,\mathrm{d}B_u\bigg|^p\Bigg] \\ &\leqslant \liminf_{n\to\infty}\mathbb{E}\Bigg[\bigg|\int_0^{t\wedge\tau^n} y_u \,\mathrm{d}B_u\bigg|^p\Bigg] \\ &\leqslant c_p\,\mathbb{E}\Bigg[\bigg(\int_0^t y_u^2 \,\mathrm{d}u\bigg)^{\frac{p}{2}}\Bigg].\end{aligned}$$

The proof is complete. □