Chapter 6

The Sewing Lemma

We fix throughout the chapter a time horizon $T > 0$ and two continuous functions $X, Y: [0,T] \to \mathbb{R}$. In this setting the *integral*

$$\int_0^T Y_r \, \mathrm{d}X_r \tag{6.1}$$

can be defined as $\int_0^T Y_r \dot{X}_r \, \mathrm{d}r$ if X is differentiable or, more generally, as a Lebesgue integral if X is of bounded variation, so that $\mathrm{d}X$ is a signed measure. The key question we want to address is: *how to define the integral when X does not have such regularity?* This is an example of a more general problem: given a distribution (generalized function) $\dot{X}$ and a non-smooth function Y, how to define their product $Y\dot{X}$?

A motivation is given by $X = B$ with $(B_t)_{t \geq 0}$ a Brownian motion. In this special case, one can use probability theory to answer the question and define the integral in (6.1), but one sees that there are several possible definitions: for example Itô, Stratonovich, etc.

In this book, we are going to present the alternative answer provided by the theory of Rough Paths, originally introduced by Terry Lyons. This theory yields a robust construction of the integral in (6.1) and sheds a new "pathwise" light on stochastic integration.

The approach we follow is based on the *Sewing Lemma*, to which this chapter is devoted. In particular, we will show in Chapter 7 that the integral in (6.1) has a canonical definition (*Young integral*) when Y and X are Hölder continuous, under a constraint on their Hölder exponents. Going beyond this constraint requires Rough Paths, which will be studied in Chapter 8.

6.1. Local approximation

If X is of class C^1, we can define the integral function

$$I_t := \int_0^t Y_r \dot{X}_r \, \mathrm{d}r, \qquad t \in [0,T].$$

Then we have $I_0 = 0$ and for $0 \leqslant s \leqslant t \leqslant T$

$$I_t - I_s - Y_s (X_t - X_s) = \int_s^t (Y_r - Y_s) \, \dot{X}_r \, \mathrm{d}r = o(t - s) \tag{6.2}$$

as $t-s\to 0$, because $\dot{X}$ is bounded and $|Y_r - Y_s| = o(1)$ as $|r-s|\to 0$. Thus the integral function I_t satisfies

$$I_0 = 0, \qquad I_t - I_s = Y_s\,(X_t - X_s) + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T\,. \tag{6.3}$$

Remarkably, *the relation (6.3) characterizes* $(I_t)_{t\in[0,T]}$. Indeed, if I^1 and I^2 satisfy (6.3) with the same functions X, Y, their difference $\Delta := I^1 - I^2$ satisfies

$$|\Delta_t - \Delta_s| = o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T,$$

which implies $\frac{\mathrm{d}}{\mathrm{d}t}\Delta_t \equiv 0$ and then $\Delta_t = \Delta_0 = I_0^1 - I_0^2 = 0$ by (6.3). This simple result deserves to be stated in a separate

LEMMA 6.1. *Given any pair of functions* $X, Y\colon [0,T]\to\mathbb{R}$*, there can be at most one function* $I\colon [0,T]\to\mathbb{R}$ *satisfying (6.3).*

The formulation (6.3) is interesting also because the derivative $\dot{X}$ of X does not appear. Therefore, if we can find a function $I\colon [0,T]\to\mathbb{R}$ which satisfies (6.3), such a function is *unique* and we can take it as a *definition* of the integral (6.1).

We will see in Section 7.1 that this program can be accomplished when X and Y satisfy suitable Hölder regularity assumptions. In order to get there, in the next sections we will look at a more general problem.

6.2. A GENERAL PROBLEM

Let us generalise the problem (6.3). We define $A\colon [0,T]^2_{\leqslant}\to\mathbb{R}$ by setting for $0\leqslant s\leqslant t\leqslant T$

$$A_{st} := Y_s\,(X_t - X_s)\ . \tag{6.4}$$

We can then decouple (6.3) in two relations:

$$I_0 = 0, \qquad I_t - I_s = A_{st} + R_{st}\,, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{6.5}$$

$$R\colon [0,T]^2_{\leqslant}\to\mathbb{R}, \qquad R_{st} = o(t-s)\ . \tag{6.6}$$

The general problem is, given a continuous $A\colon [0,T]^2_{\leqslant}\to\mathbb{R}$, to find a pair of functions (I, R) satisfying (6.5)-(6.6). We call

- $A\colon [0,T]^2_{\leqslant}\to\mathbb{R}$ the *germ*,
- $I\colon [0,T]\to\mathbb{R}$ the *integral*,
- $R\colon [0,T]^2_{\leqslant}\to\mathbb{R}$ the *remainder*.

We are going to present conditions which allow to solve this problem.

Note that *we always have uniqueness*. Indeed, given (I^1, R^1) and (I^2, R^2) which solve (6.5)-(6.6) for the same A, by the same arguments which lead to Lemma 6.1 we have $\frac{\mathrm{d}}{\mathrm{d}t}(I_t^1 - I_t^2) \equiv 0$, hence $I^1 = I^2$ and then $R^1 = R^2$ by (6.5). We record this as

LEMMA 6.2. *Given any germ* A*, there can be at most one pair of functions* (I, R) *satisfying (6.5)-(6.6).*

6.3. AN ALGEBRAIC LOOK

We first focus on relation (6.5) alone. For a fixed germ A, this equation has infinitely many solutions (I,R), because given *any* I we can simply *define* R so as to fulfill (6.5). Interestingly, all solutions admit an algebraic characterization in terms of R alone.

LEMMA 6.3. *Fix a function $A\in C_2$.*

1. *If a pair $(I,R)\in C_1\times C_2$ satisfies (6.5), then R satisfies*

$$(\delta R)_{sut}=-(\delta A)_{sut},\qquad \forall 0\leqslant s\leqslant u\leqslant t\leqslant T\,. \tag{6.7}$$

2. *Viceversa, given any function $R\in C_2$ which satisfies (6.7), if we set $I_t:=A_{0t}+R_{0t}$, the pair $(I,R)\in C_1\times C_2$ satisfies (6.5).*

Proof. Relation (6.5) clearly implies (6.7), simply because $\delta(\delta I)=0$. Viceversa, given R satisfying (6.7), we can define $L_{st}:=A_{st}+R_{st}$ so that

$$L_{st}-L_{su}-L_{ut}=0.$$

Applying this formula to $(s',u',t')=(0,s,t)$, we obtain that $I_t:=L_{0t}$ satisfies

$$I_t-I_s=L_{0t}-L_{0s}=L_{st}=A_{st}+R_{st}$$

and the proof is complete because $I_0:=L_{00}=A_{00}+R_{00}=0$, which follows by (6.7) for $s=u=0$. $\square$

We can now rephrase Lemma 6.3 as follows.

PROPOSITION 6.4. *Fix $A\in C_2$. Finding a pair $(I,R)\in C_1\times C_2$ satisfying (6.5) is equivalent to finding $R\in C_2$ such that*

$$\delta R_{sut}=-\delta A_{sut}\,,\qquad \forall\, 0\leqslant s\leqslant u\leqslant t\leqslant T. \tag{6.8}$$

6.4. ENTERS ANALYSIS: THE SEWING LEMMA

So far we have analyzed (6.5). We now let (6.6) enter the game, i.e. we look for a pair of functions $(I,R)\in C_1\times C_2$ which fulfills (6.5)-(6.6), given a (general) germ $A\in C_2$.

We stress that condition (6.6) is essential to ensure *uniqueness*: without it, equation (6.5) admits infinitely many solutions, as discussed before Lemma 6.3. When we couple (6.5) with (6.6), uniqueness is guaranteed by Lemma 6.2, but *existence* is no longer obvious. This is what we now focus on.

We start with a simple necessary condition.

LEMMA 6.5. *For (6.5)-(6.6) to admit a solution, it is necessary that the germ A satisfies*

$$|\delta A_{sut}|=o(t-s),\qquad \textit{for } 0\leqslant s\leqslant u\leqslant t\leqslant T. \tag{6.9}$$

Proof. If (6.5) admits a solution, by Proposition 6.4 we have $|\delta A_{sut}| = |\delta R_{sut}|$. If furthermore R satisfies (6.6), we must have for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$|\delta R_{sut}| \leq |R_{st}| + |R_{su}| + |R_{ut}| = o(t-s) + o(u-s) + o(t-u) = o(t-s)\,. \qquad \square$$

Remark 6.6. Choosing $u = s$ in (6.9) we obtain that $-A_{ss} = o(t-s)$, which means that $A_{ss} = 0$. Therefore a necessary condition for (6.5)-(6.6) to admit a solution is that A *vanishes on the diagonal of* $[0,T]^2_{\leqslant}$.

Remarkably, the necessary condition in Lemma 6.5 is close to being sufficient: it is enough to upgrade $o(t-s)$ in $O((t-s)^\eta)$ for some $\eta > 1$. This is the content of the celebrated *Sewing Lemma*, which we next present.

We have seen in the Sewing bound (Theorem 1.9) that any $R \in C_2$ such that $R_{st} = o(t-s)$ for $0 \leqslant s \leqslant t \leqslant T$ satisfies an a priori estimate $\|R\|_\eta \leqslant K_\eta \|\delta R\|_\eta$ for any $\eta > 1$. Of course, this estimate is only interesting if $\|\delta R\|_\eta < \infty$ for some $\eta > 1$. This property, that we call *coherence*, is at the heart of the celebrated Sewing Lemma (Gubinelli [2], Feyel-de La Pradelle [1]), as it provides a sufficient condition on the germ A for the solution of (6.5)-(6.6).

DEFINITION 6.7. (COHERENCE) *A germ $A \in C_2$ is called* coherent *if, for some $\eta > 1$, it satisfies $\delta A \in C_3^\eta$, i.e. $\|\delta A\|_\eta < \infty$. More explicitly:*

$$\exists \eta \in (1,\infty): \qquad |\delta A_{sut}| \lesssim |t-s|^\eta\,, \qquad 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{6.10}$$

THEOREM 6.8. (SEWING LEMMA) *For any coherent germ $A \in C_2$ there exists a (unique) function $I\colon [0,T] \to \mathbb{R}$ such that $|A_{st} - \delta I_{st}| = o(t-s)$; equivalently, there exists a unique pair $(I,R) \in C_1 \times C_2$ such that*

$$I_0 = 0, \qquad I_t - I_s = A_{st} + R_{st} \qquad \text{with} \qquad R_{st} = o(t-s)\,. \tag{6.11}$$

- *The "remainder" $R_{st} := \delta I_{st} - A_{st}$ satisfies the Sewing Bound:*

$$\|R\|_\eta \leq K_\eta\, \|\delta A\|_\eta \qquad \text{where} \qquad K_\eta := (1 - 2^{1-\eta})^{-1}\,. \tag{6.12}$$

- *The* integral $I \in C_1$ *is the limit of* Riemann sums *of the germ:*

$$I_t := \lim_{|\mathcal{P}| \to 0} \sum_{i=0}^{\#\mathcal{P}-1} A_{t_i t_{i+1}} \tag{6.13}$$

  along arbitrary partitions $\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_k = t\}$ *of* $[0,t]$ *with vanishing mesh* $|\mathcal{P}| := \max_{i=0,\ldots,k-1} |t_{i+1} - t_i| \to 0$ *(we set* $\#\mathcal{P} := k$*).*

The Sewing Lemma is a cornerstone of the theory of *Rough Paths*, to be introduced in Chapter 8. We will already see in Chapter 7 an interesting application to *Young integrals*. The (instructive) proof of Theorem 6.8 is postponed to Section 6.6.

Remark 6.9. For a fixed partition $\mathcal{P}$ of $[0,t]$ we have, by $\delta I_{st} = A_{st} + R_{st}$,

$$I_t = \sum_{i=0}^{\#\mathcal{P}-1} A_{t_i t_{i+1}} + \sum_{i=0}^{\#\mathcal{P}-1} R_{t_i t_{i+1}}.$$

Therefore, (6.13) is equivalent to

$$\lim_{|\mathcal{P}|\to 0} \sum_{i=0}^{\#\mathcal{P}-1} R_{t_i t_{i+1}} = 0$$

which is the reason why one wants the remainder R to be small close to the diagonal. The information $R_{st} = o(t-s)$ is not enough in general to obtain the existence of (I, R), while the stronger estimate $|R_{st}| \lesssim |t-s|^\eta$ is sufficient.

6.5. The Sewing Map

Given a coherent germ A, by Theorem 6.8 we can find an integral I and a remainder R which solve (6.5)-(6.6). We now look closer at the remainder R.

Lemma 6.10. *In the setting of Theorem 6.8,* the remainder R is a function of δA*: given two coherent germs A, A' with $\delta A = \delta A'$, the corresponding remainders R, R' coincide. Moreover, the map $\delta A \mapsto R$ is linear.*

Proof. By Proposition 6.4 we have $\delta(R - R') = \delta(A' - A) = 0$, hence $R - R' = \delta f$ for some $f \in C_1$ (see Remark 1.10). Both $|R_{st}|$ and $|R'_{st}|$ are $o(|t-s|)$ by (6.6), hence $|f_t - f_s| = o(|t-s|)$. Then f must be constant by Lemma 6.1 and therefore $R = R'$. Linearity of the map $\delta A \mapsto R$ is easy. ☐

Since R is a function of δA, we introduce a specific notation for this map:

$$R = -\Lambda(\delta A)$$

where the minus sign is for later convenience.

Let us describe more precisely this map Λ. Throughout the following discussion, we fix arbitrarily $\eta \in (1, \infty)$.

- *Domain.* The map Λ is defined on δA for coherent germs A, see Definition 6.7. The domain of Λ is then $C_3^\eta \cap \delta C_2$, where we denote by $\delta C_2 \subseteq C_3$ the image of the space C_2 under the operator δ in (1.23).
- *Codomain.* The map Λ sends δA to $-R$, and we have $|R_{st}| \lesssim |t-s|^\eta$, see (6.12). A natural choice of codomain for Λ is then C_2^η.
- *Characterization.* In view of Proposition 6.4 and Lemma 6.2, the function $-R = \Lambda(\delta A)$ is characterized by the properties

$$\delta(-R) = \delta A\,, \qquad |R_{st}| = o(t-s)\,.$$

  The second condition is already enforced by our choice C_2^η of codomain for Λ, which yields $|R_{st}| \lesssim |t-s|^\eta$ (with $\eta > 1$). The first relation can be rewritten as $\delta(\Lambda(B)) = B$ for all B in the domain of Λ, that is $\delta \circ \Lambda$ *is the identity map*.

In conclusion, we have proved the following result.

Theorem 6.11. (Sewing Map) *Let $\eta \in (1, \infty)$. There exists a unique map*

$$\Lambda \colon C_3^\eta \cap \delta C_2 \longrightarrow C_2^\eta,$$

called the Sewing Map*, such that* $\delta \circ \Lambda = \mathrm{id}$ *is the identity on* $C_3^\eta \cap \delta C_2$.

- *The map* Λ *is linear and satisfies*

$$\|\Lambda(B)\|_\eta \leqslant K_\eta \|B\|_\eta \qquad \forall B \in C_3^\eta \cap \delta C_2 \,, \tag{6.14}$$

where K_η *is the same constant as in (6.12).*

- *Given a coherent germ* $A \in C_2$*, i.e. such that* $\delta A \in C_3^\eta$*, the unique solution* (I, R) *of (6.5)-(6.6) is* $R := -\Lambda(\delta A)$ *and* $I_t := A_{0t} + R_{0t}$.

6.6. Proof of the Sewing Lemma

We prove the Sewing Lemma, i.e. Theorem 6.8.

Proof. We fix a germ $A \in C_2$ with $\|\delta A\|_\eta < \infty$ for some $\eta > 1$ (we do *not* require $A_{ab} = o(b-a)$). Our goal is to build a function $I: [0, T] \to \mathbb{R}$ such that $|\delta I_{st} - A_{st}| = o(t-s)$. Uniqueness of I follows by Lemma 6.2, while the bound (6.12) follows by the Sewing Bound (1.26) applied to $R_{st} := \delta I_{st} - A_{st}$ (note that $\delta R = -\delta A$, because $\delta \circ \delta = 0$).

We fix $0 \leqslant s < t \leqslant T$. Given a partition $\mathcal{P} = \{s = t_0 < t_1 < \ldots < t_m = t\}$ of $[s, t]$, let us define $I_{\mathcal{P}}(A) := \sum_{i=0}^{m-1} A_{t_i t_{i+1}}$ as in (1.20). The following bound holds:

$$|I_{\mathcal{P}}(A) - A_{st}| \leq C_\eta \, \|\delta A\|_\eta \, (t-s)^\eta \qquad \text{with} \qquad C_\eta := \sum_{n \geq 1} \frac{2^\eta}{n^\eta} < \infty, \tag{6.15}$$

as we showed in the proof of Theorem 1.18, see (1.46), which applies to any function $A = (A_{s,t})$. Similarly, if $\mathcal{Q} \supseteq \mathcal{P}$ is another partition of $[s, t]$,

$$\begin{aligned}
|I_{\mathcal{Q}}(A) - I_{\mathcal{P}}(A)| &\leqslant \sum_{i=0}^{\#\mathcal{P}-1} |I_{\mathcal{Q} \cap [t_i, t_{i+1}]}(A) - A_{t_i t_{i+1}}| \\
&\leqslant C_\eta \, \|\delta A\|_\eta \sum_{i=0}^{\#\mathcal{P}-1} (t_{i+1} - t_i)^\eta \\
&\leqslant C_\eta \, \|\delta A\|_\eta \, |\mathcal{P}|^{\eta-1} \sum_{i=0}^{\#\mathcal{P}-1} (t_{i+1} - t_i) \\
&\leqslant C_\eta \, \|\delta A\|_\eta T \, |\mathcal{P}|^{\eta-1}
\end{aligned}$$

where we recall that $|\mathcal{P}| := \max_i (t_{i+1} - t_i)$. Finally, if $\mathcal{P}$ and $\mathcal{P}'$ are *arbitrary partitions*, setting $\mathcal{Q} := \mathcal{P} \cup \mathcal{P}'$ and applying the triangle inequality yields

$$|I_{\mathcal{P}'}(A) - I_{\mathcal{P}}(A)| \leq C_\eta \, \|\delta A\|_\eta T \, (|\mathcal{P}|^{\eta-1} + |\mathcal{P}'|^{\eta-1}) \,.$$

This shows that *the family* $I_{\mathcal{P}}(A)$ *is Cauchy as* $|\mathcal{P}| \to 0$ (for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $|\mathcal{P}|, |\mathcal{P}'| \leq \delta_\epsilon$ implies $|I_{\mathcal{P}'}(A) - I_{\mathcal{P}}(A)| \leq \epsilon$), hence it admits a limit as $|\mathcal{P}| \to 0$, that we call J_{st}.

We now *define* $I_t := J_{0t}$. We claim that

$$I_t - I_s = J_{st} \qquad \text{for all} \quad 0 \leq s < t \leq T \,.$$

Indeed, if we consider partitions $\mathcal{P}'$ on $[0,s]$ and $\mathcal{P}$ of $[s,t]$, then $\mathcal{P}'' := \mathcal{P} \cup \mathcal{P}'$ is a partition of $[0,t]$ such that $I_{\mathcal{P}''}(A) - I_{\mathcal{P}'}(A) = I_{\mathcal{P}}(A)$, and taking the limit of vanishing mesh we get $J_{0t} - J_{0s} = J_{st}$, that is the claim.

Finally, taking the limit of relation (6.15), since $I_{\mathcal{P}}(A) \to J_{st} = I_t - I_s$, we obtain our goal $|\delta I_{st} - A_{st}| \lesssim (t-s)^{\eta} = o(t-s)$. This completes the proof, since (6.13) holds by construction. □

Remark 6.12. Taking the limit of (6.15) gives

$$|R_{st}| \leq C_\eta \, \|\delta A\|_\eta \, |t-s|^\eta \,, \qquad R_{st} := \delta I_{st} - A_{st} \,, \qquad 0 \leqslant s < t \leqslant T \,,$$

which is the bound (6.12) with K_η replaced by the worse constant C_η. This is because the estimate (6.15) holds for arbitrary partitions.