Chapter 7

The Young integral

We can now come back to the problem that we discussed at the beginning of Chapter 6: given two continuous functions $X, Y\colon [0,T] \to \mathbb{R}$, how can we give a meaning to the integral $I_t = \int_0^t Y \, \mathrm{d}X$ for $t \in [0,T]$?

A natural answer, recall (6.3), is to look for a function $I\colon [0,T] \to \mathbb{R}$ satisfying

$$I_0 = 0, \qquad I_t - I_s = Y_s \, (X_t - X_s) + o(t-s), \qquad 0 \leqslant s \leqslant t \leqslant T\,. \tag{7.1}$$

As an application of the *Sewing Lemma* (Theorem 6.8), we can show that such a function I exists (and is necessarily unique) when X and Y are Hölder functions of exponents $\alpha, \beta \in \,]0,1]$ such that $\alpha + \beta > 1$. This leads to the notion of *Young integral*, to which this chapter is devoted.

Going beyond this setting, in order to treat the case $\alpha + \beta \leq 1$, will require the notion of *Rough Paths*, that we discuss in Chapter 8.

7.1. Construction of the Young integral

As we did in Chapter 6, it is convenient to rewrite (7.1) as follows: we look for a function $I\colon [0,T] \to \mathbb{R}$ satisfying

$$I_0 = 0, \qquad I_t - I_s = A_{st} + R_{st} \qquad \text{with} \qquad R_{st} = o(t-s), \tag{7.2}$$

where the *germ* $A\colon [0,T]^2_{\leqslant} \to \mathbb{R}$ is defined by

$$A_{st} = Y_s \, \delta X_{st} = Y_s \, (X_t - X_s)\,. \tag{7.3}$$

This is the framework of the *Sewing Lemma*, see Theorem 6.8, for which we need to fulfill the *coherence condition* (6.10), that is $\|\delta A\|_\eta < \infty$ for some $\eta > 1$ (we use the norms introduced in (1.9)). Recalling that

$$\delta A_{sut} := A_{st} - A_{su} - A_{ut} = -\delta Y_{su} \, \delta X_{ut}\,,$$

see (1.32), we can write for any $\alpha, \beta \in \,]0,1]$

$$|\delta A_{sut}| = |Y_u - Y_s| \, |X_t - X_u| \quad \Longrightarrow \quad \|\delta A\|_{\alpha+\beta} \leqslant \|\delta X\|_\alpha \, \|\delta Y\|_\beta\,. \tag{7.4}$$

As a consequence, it is natural to assume that $\|\delta X\|_\alpha < \infty$ and $\|\delta Y\|_\beta < \infty$ for α, $\beta \in \,]0,1]$ such that $\alpha + \beta > 1$.

We can now give a consistent definition of the integral $I_t = \int_0^t Y \, \mathrm{d}X$, known as *Young integral*, when X and Y are suitable Hölder functions.

THEOREM 7.1. (YOUNG INTEGRAL) *Fix* $\alpha,\beta\in\,]0,1]$ *with* $\alpha+\beta>1$. *For every* $(X, Y)\in\mathcal{C}^\alpha\times\mathcal{C}^\beta$ *there is a (necessarily unique) function* $I\colon[0,T]\to\mathbb{R}$ *which satisfies (7.1), i.e.*

$$I_0=0, \qquad I_t-I_s=Y_s\,(X_t-X_s)+o(t-s)\,. \tag{7.5}$$

The functon I*, called the* Young integral*, is also denoted by* $I_t=:\int_0^t Y\,\mathrm{d}X$.

The remainder $R_{st}:=I_t-I_s-Y_s\,(X_t-X_s)$ *satisfies the bound*

$$\|R\|_{\alpha+\beta}\leq K_{\alpha+\beta}\,\|\delta X\|_\alpha\,\|\delta Y\|_\beta\,, \tag{7.6}$$

where $K_\eta:=(1-2^{1-\eta})^{-1}$*, see (6.12). This yields* $I\in\mathcal{C}^\alpha$*, more precisely*

$$\|\delta I\|_\alpha\leq(\|Y\|_\infty+K_{\alpha+\beta}T^\beta\,\|\delta Y\|_\beta)\,\|\delta X\|_\alpha\,. \tag{7.7}$$

The Young integral $I=(I_t)_{t\in[0,T]}$*, as a function of* (X,Y)*, is a continuous bilinear map* $I\colon\mathcal{C}^\alpha\times\mathcal{C}^\beta\to\mathcal{C}^\alpha$.

Proof. Recalling (7.2)-(7.4), we have $\|\delta A\|_{\alpha+\beta}\leqslant\|\delta X\|_\alpha\|\delta Y\|_\beta<\infty$, that is $\delta A\in C_3^\eta$ with $\eta=\alpha+\beta>1$, where the spaces C_k^η were defined in (1.10). By the *Sewing Lemma*, see Theorem 6.8, there exists a (unique) function I which satisfies (6.11) and (6.12), hence (7.5) and (7.6) hold.

In order to prove (7.7), we note that

$$\begin{aligned}\|\delta I\|_\alpha &\leqslant \|A\|_\alpha+\|R\|_\alpha\leqslant\|Y\|_\infty\,\|\delta X\|_\alpha+T^\beta\,\|R\|_{\alpha+\beta}\\ &\leqslant \|Y\|_\infty\,\|\delta X\|_\alpha+T^\beta\,K_{\alpha+\beta}\,\|\delta X\|_\alpha\,\|\delta Y\|_\beta\,.\end{aligned}$$

Recalling Remark 1.4, in particular (1.15), this bound implies that I is a continuous function of (X,Y), as a map from $\mathcal{C}^\alpha\times\mathcal{C}^\beta$ to $\mathcal{C}^\alpha$.

We finally prove that the map $(X,Y)\mapsto I$ is bilinear: given $X,X'\in\mathcal{C}^\alpha$ and a fixed $Y\in\mathcal{C}^\beta$, if I satisfies (7.5) for (X,Y) and I' satisfies (7.5) for (X',Y), then for any $a,b\in\mathbb{R}$ the function $\hat{I}_t:=a\,I_t+b\,I_t'$ satisfies (7.5) for $(\hat{X}:=a\,X+b\,X',Y)$. Linearity with respect to Y is proved similarly. □

Remark 7.2. The setting of Theorem 7.1 provides a natural example of a germ $A_{st}:=Y_s\,\delta X_{st}$ which is *not* in C_2^η for any $\eta>1$ (excluding the trivial case when $Y\equiv0$ on the intervals where X is not constant, hence $A\equiv0$), but it satisfies $\delta A\in C_3^\eta$ with $\eta=\alpha+\beta>1$.

Remark 7.3. (BEYOND YOUNG) It is natural to wonder what happens in Theorem 7.1 for $(X,Y)\in\mathcal{C}^\alpha\times\mathcal{C}^\beta$ with $\alpha+\beta\leq1$. In this case, *there might be no solution to (6.5)-(6.6)*, because the necessary condition (6.9) in Lemma 6.5 can fail. For a simple example, consider $X_t=t^\alpha$ and $Y_t=t^\beta$ for $t\in[0,T]$ and note that for $s=0$ and $u=\frac{t}{2}$ we have by (1.32)

$$|\delta A_{sut}|=|\delta A_{0\frac{t}{2}t}|=\left|\delta Y_{0\frac{t}{2}}\right|\left|\delta X_{\frac{t}{2}t}\right|=\left(\frac{t}{2}\right)^\beta\left(t^\alpha-\left(\frac{t}{2}\right)^\alpha\right)\gtrsim t^{\alpha+\beta}, \tag{7.8}$$

which is not $o(t-s)=o(t)$ when $\alpha+\beta\leq1$.

In order to define a notion of integral $I_t = \int_0^t Y_s \, \mathrm{d}X_s$ when $(X, Y) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ with $\alpha + \beta \leq 1$, we need to relax condition (6.3), see Definition 8.1 below. This will lead to the notion of *Rough Paths*, described in Chapter 8.

7.2. Integral formulation of Young equations

In this section we explain why we call (2.4) a *Young* equation. In fact, we can interpret the finite difference equation (2.4) as an *integral equation*, using the Young integral of section 7.1.

PROPOSITION 7.4. *Let $Z \in \mathcal{C}^\alpha([0,T]; \mathbb{R}^k)$ with $\alpha > \frac{1}{2}$. Then Z satisfies (2.4) if and only if*

$$Z_t = Z_0 + \int_0^t \sigma(Z_s) \, \mathrm{d}X_s, \qquad t \in [0, T], \tag{7.9}$$

where the integral is in the Young sense.

Proof. We consider the germ $A_{st} := \sigma(Z_s) \, \delta X_{st}$, $0 \leqslant s \leqslant t \leqslant T$. By (7.4)

$$|\delta A_{sut}| = |\sigma(Z_u) - \sigma(Z_s)||X_t - X_u| \quad \Longrightarrow \quad \|\delta A\|_{2\alpha} \leqslant \|\nabla \sigma\|_\infty \|\delta X\|_\alpha \|\delta Z\|_\alpha .$$

Therefore we obtain that (2.4) is equivalent to (7.5) above. □

In the case $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, this argument does not work and the Young integral is not adapted, since the germ $A_{st} := \sigma(Z_s) \, \delta X_{st}$ has the property $\delta A \in C_3^{2\alpha}$ with $2\alpha \leqslant 1$, so that the Sewing Lemma can not be applied. However the equation (3.19) suggests *another* germ:

$$A_{st} := \sigma(Z_s) \, \mathbb{X}_{st}^1 + \sigma_2(Z_s) \, \mathbb{X}_{st}^2, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Note that $A = \delta Z - Z^{[3]}$, in the notation (3.19). Then by (3.27) we know that $\delta A \in C_3^{3\alpha}$. Therefore we can interpret the formula

$$\delta Z = A - \Lambda(\delta A)$$

as

$$Z_t = Z_0 + \int_0^t \sigma(Z_s) \, \mathrm{d}\mathbb{X}_s, \qquad 0 \leqslant t \leqslant T,$$

which for the moment is only a notation that will be made more precise in chapter 10.

7.3. Local existence via contraction

As an application of the estimates on the Young integral of Theorem 7.1, we want to give a local existence result for equation (2.4) which does not rely on compactness and which can be therefore used also in infinite dimension.

Let $Z_0 \in \mathbb{R}^k$ and $X \in \mathcal{C}^\alpha([0,T]; \mathbb{R}^d)$ be given, $\sigma: \mathbb{R}^k \to \mathbb{R}^d \otimes (\mathbb{R}^d)^*$ smooth and the unknown $Z: [0,T] \to \mathbb{R}^k$ is such that $\sigma(Z) \in \mathcal{C}^\alpha$ and $2\alpha > 1$, so that the right-hand side of (7.9) can be interpreted as a Young integral. We want now to show the following

THEOREM 7.5. (CONTRACTION FOR YOUNG DIFFERENTIAL EQUATIONS) *Let* σ: $\mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ *be of class* C^2 *with* $\nabla \sigma$ *and* $\nabla^2 \sigma$ *bounded. Let* $\alpha \in]\frac{1}{2}, 1]$ *and* $X \in \mathcal{C}^\alpha([0,T];\mathbb{R}^d)$ *fixed. It* $T > 0$ *is small enough, then for any* $Z_0 \in \mathbb{R}$ *there exists a unique* $Z \in \mathcal{C}^\alpha([0,T];\mathbb{R}^k)$ *which satisfies (7.9).*

Proof. For all $f \in \mathcal{C}^\alpha([0,T];\mathbb{R}^k)$ we have

$$|\sigma(f_t) - \sigma(f_s)| \leqslant \|\nabla \sigma\|_\infty |f_t - f_s|$$

so that

$$\|\delta \sigma(f)\|_\alpha \leqslant \|\nabla \sigma\|_\infty \|\delta f\|_\alpha.$$

By (7.7) with $\alpha = \beta$ we obtain for all $f \in \mathcal{C}^\alpha$ satisfying (7.9)

$$\|\delta f\|_\alpha \leqslant (|\sigma(f_0)| + (1 + K_{2\alpha}) T^\alpha \|\nabla \sigma\|_\infty \|\delta f\|_\alpha) \|\delta X\|_\alpha$$

since

$$\|\sigma(f)\|_\infty \leqslant |\sigma(f_0)| + T^\alpha \|\delta \sigma(f)\|_\alpha.$$

Therefore, if T satisfies

$$T^\alpha \leqslant \frac{1}{2} \frac{1}{(1 + K_{2\alpha}) \|\nabla \sigma\|_\infty \|\delta X\|_\alpha}$$

then we have the following a priori estimate on solutions to (7.9)

$$\|\delta Z\|_\alpha \leqslant 2|\sigma(Z_0)| \, \|\delta X\|_\alpha .$$

We fix such T and we set $\mathcal{C}^\alpha(Z_0) := \{ f \in \mathcal{C}^\alpha \colon f_0 = Z_0, \|\delta f\|_\alpha \leqslant 2|\sigma(Z_0)| \, \|\delta X\|_\alpha \}$. Then we define $\Lambda \colon \mathcal{C}^\alpha \to \mathcal{C}^\alpha$ given by

$$\Lambda(f) := h, \qquad h_t := Z_0 + \int_0^t \sigma(f_s) \, \mathrm{d}X_s, \qquad t \in [0,T].$$

It is easy to see, arguing as above, that Λ acts on $\mathcal{C}^\alpha(Z_0)$, namely $\Lambda \colon \mathcal{C}^\alpha(Z_0) \to \mathcal{C}^\alpha(Z_0)$. Note that the map $\mathcal{C}^\alpha(Z_0) \times \mathcal{C}^\alpha(Z_0) \ni (a,b) \mapsto \|\delta a - \delta b\|_\alpha$ defines a distance on $\mathcal{C}^\alpha(Z_0)$ which induces the same topology as $\|\cdot\|_{\mathcal{C}^\alpha}$. We want to show that Λ is a contraction for this distance if T is small enough. By (7.7) we have for $\alpha = \beta$

$$\begin{aligned} \|\delta \Lambda(a) - \delta \Lambda(b)\|_\alpha &\leqslant (\|\sigma(a) - \sigma(b)\|_\infty + K_{2\alpha} T^\alpha \|\delta \sigma(a) - \delta \sigma(b)\|_\alpha) \|\delta X\|_\alpha . \\ &\leqslant T^\alpha (1 + K_{2\alpha}) \|\delta X\|_\alpha \|\delta \sigma(a) - \delta \sigma(b)\|_\alpha . \end{aligned}$$

We now need to estimate $\|\delta \sigma(a) - \delta \sigma(b)\|_\alpha$. By Lemma 2.8

$$\|\delta \sigma(a) - \delta \sigma(b)\|_\alpha \leqslant \|\nabla \sigma\|_\infty \|\delta a - \delta b\|_\alpha + \|\nabla^2 \sigma\|_\infty (\|\delta a\|_\alpha + \|\delta b\|_\alpha) \|a - b\|_\infty.$$

Since, as usual, $\|a - b\|_\infty \leqslant T^\alpha \|\delta a - \delta b\|_\alpha$, we obtain

$$\|\delta \sigma(a) - \delta \sigma(b)\|_\alpha \leqslant (\|\nabla \sigma\|_\infty + T^\alpha \|\nabla^2 \sigma\|_\infty (\|\delta a\|_\alpha + \|\delta b\|_\alpha)) \|\delta a - \delta b\|_\alpha. \tag{7.10}$$

Therefore, for all $a, b \in \mathcal{C}^\alpha(Z_0)$

$$\|\delta \Lambda(a) - \delta \Lambda(b)\|_\alpha \leqslant C_T \|\delta a - \delta b\|_\alpha,$$

where $C_T := T^\alpha (1 + K_{2\alpha}) \|\delta X\|_\alpha (\|\nabla \sigma\|_\infty + T^\alpha \|\nabla^2 \sigma\|_\infty 4 |\sigma(Z_0)| \, \|\delta X\|_\alpha)$. It is now enough to consider T small enough so that $C_T < 1$. □

7.4. PROPERTIES OF THE YOUNG INTEGRAL

The Young integral $\int_0^t Y\,\mathrm{d}X$, defined in Theorem 7.1, shares many properties with the classical Riemann-Lebesgue integral, that we now discuss.

A elementary but useful observation is that $\int_0^t Y\,\mathrm{d}X$ is a linear function of Y (for fixed X) and a linear function of X (for fixed Y), by bilinearity.

For an interval $[s,t] \subset [0,T]$ we will use the notation

$$I_t - I_s =: \int_s^t Y\,\mathrm{d}X\,.$$

If the integrand $Y_u = c$ is constant for all $u \in [s,t]$, then $\int_s^t Y\,\mathrm{d}X = c\,(X_t - X_s)$, which follows directly from (7.5). As a corollary, we obtain the following useful formula for the remainder.

LEMMA 7.6. *Let $(X,Y) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ for $\alpha, \beta \in]0,1]$ with $\alpha + \beta > 1$ and let $I_t := \int_0^t Y_u\,\mathrm{d}X_u$ be the Young integral, see Theorem 7.1. Then the remainder*

$$R_{st} := I_t - I_s - Y_s\,(X_t - X_s), \qquad 0 \leqslant s \leqslant t \leqslant T,$$

admits the explicit formula

$$R_{st} = \int_s^t (Y_u - Y_s)\,\mathrm{d}X_u, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{7.11}$$

where the right hand side is a Young integral.

Proof. By linearity and the basic property mentioned above, we obtain

$$\int_s^t (Y_u - Y_s)\,\mathrm{d}X_u = \int_s^t Y_u\,\mathrm{d}X_u - \int_s^t Y_s\,\mathrm{d}X_u = I_t - I_s - Y_s\,(X_t - X_s) = R_{st}\,. \qquad \square$$

An important property is *integration by parts*, which follows by the uniqueness of the solution for the problem (6.5)-(6.6), recall Lemma 6.2.

PROPOSITION 7.7. (INTEGRATION BY PARTS) *Fix $\alpha, \beta \in]0,1]$ with $\alpha + \beta > 1$. For all $(X,Y) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ the Young integral satisfies*

$$\int_0^t X\,\mathrm{d}Y + \int_0^t Y\,\mathrm{d}X = X_t Y_t - X_0 Y_0\,. \tag{7.12}$$

Proof. Let us set $I_t' := \int_0^t X\,\mathrm{d}Y + \int_0^t Y\,\mathrm{d}X$. By the property (7.5) we have

$$I_t' - I_s' = \underbrace{Y_s(X_t - X_s) + X_s(Y_t - Y_s)}_{A_{st}} + o(t-s)\,.$$

Next we set $I_t'' := X_t\,Y_t - X_0\,Y_0$ and note that, by direct computation,

$$I_t'' - I_s'' = \underbrace{Y_s(X_t - X_s) + X_s(Y_t - Y_s)}_{A_{st}} + \underbrace{(X_t - X_s)(Y_t - Y_s)}_{R_{st}}\,,$$

where $|R_{st}| \leqslant \|\delta X\|_\alpha \|\delta Y\|_\beta |t-s|^{\alpha+\beta} = o(t-s)$. By Lemma 6.2, for any germ A, there can be at most one function I which satisfies $\delta I_{st} = A_{st} + o(t-s)$ (6.5)-(6.6), hence $I' = I''$. □

We next discuss the *chain rule*.

PROPOSITION 7.8. (CHAIN RULE) *Let $X \in \mathcal{C}^\alpha$ with $\alpha \in]\frac{1}{2}, 1]$. Let $\varphi: \mathbb{R} \to \mathbb{R}$ be differentiable with $\varphi' \in \mathcal{C}^\gamma(\mathbb{R})$, for $\gamma \in]0,1]$ such that $\gamma > \frac{1}{\alpha} - 1$ (a sufficient condition is that $\varphi \in C^2$). Then $\varphi'(X) = \varphi' \circ X \in \mathcal{C}^{\alpha\gamma}$ and*

$$\varphi(X_t) - \varphi(X_0) = \int_0^t \varphi'(X)\,\mathrm{d}X\,, \tag{7.13}$$

where the right hand side is a Young integral.

Proof. It is easy to see that $\varphi'(X) \in \mathcal{C}^{\alpha\gamma}$, which implies that $\int_0^t \varphi'(X)\,\mathrm{d}X$ is well-defined as a Young integral, since $\alpha + \alpha\gamma > 1$. By the definition (7.5) of the Young integral, proving (7.13) amounts to showing that

$$|\varphi(X_t) - \varphi(X_s) - \varphi'(X_s)\ (X_t - X_s)| \lesssim o(t-s).$$

By the classical Lagrange theorem, if, say, $X_t > X_s$, then

$$\varphi(X_t) - \varphi(X_s) - \varphi'(X_s)(X_t - X_s) = (\varphi'(\xi) - \varphi'(X_s))(X_t - X_s)$$

with $\xi \in]X_s, X_t[$. Since $\varphi' \in \mathcal{C}^\gamma$ and $X \in \mathcal{C}^\alpha$, it follows that

$$|\varphi(X_t) - \varphi(X_s) - \varphi'(X_s)\,(X_t - X_s)| \lesssim |X_t - X_s|^{\gamma+1} = o(t-s)$$

since $\gamma + 1 > \frac{1}{\alpha} \geqslant 1$. This completes the proof. □

More generally, we have

COROLLARY 7.9. *In the same setting of Proposition 7.8, for all $s \leqslant t$*

$$\varphi(X_t) - \varphi(X_s) = \varphi'(X_s)(X_t - X_s) + \int_s^t (\varphi'(X_r) - \varphi'(X_s))\,\mathrm{d}X_r\,. \tag{7.14}$$

Proof. It is enough to note that, by (7.13),

$$\begin{aligned}\varphi(X_t) - \varphi(X_s) &= \int_s^t \varphi'(X_r)\,\mathrm{d}X_r \\ &= \varphi'(X_s)(X_t - X_s) + \int_s^t (\varphi'(X_r) - \varphi'(X_s))\,\mathrm{d}X_r\,,\end{aligned}$$

where all integrals are in the Young sense. □

In particular, for $X \in \mathcal{C}^\alpha$ with $\alpha > \frac{1}{2}$, we have

$$\frac{X_t^2}{2} - \frac{X_s^2}{2} = X_s(X_t - X_s) + \int_s^t (X_r - X_s)\,\mathrm{d}X_r, \tag{7.15}$$

which can be rewritten as follows:

$$\int_s^t (X_r - X_s)\,\mathrm{d}X_r = \frac{X_t^2}{2} - \frac{X_s^2}{2} - X_s\,(X_t - X_s) = \frac{(X_t - X_s)^2}{2}\,. \tag{7.16}$$

7.5. More on Hölder spaces

We discuss further properties of the Hölder spaces $\mathcal{C}^\alpha$ for $\alpha \in (0,1)$ (excluding the case $\alpha = 1$ of Lipschitz functions). These will be useful in the next Section 7.6, when we discuss the uniqueness of the Young integral.

Let us denote by C^∞ the space of infinitely differentiable functions. We note that $C^\infty \subset \mathcal{C}^\alpha$ for every $\alpha \in (0,1)$, but C^∞ *is not dense in* $\mathcal{C}^\alpha$.

Theorem 7.10. *For any $\alpha \in (0,1)$, the closure of C^∞ in $\mathcal{C}^\alpha$ is the subset $\mathcal{C}_0^\alpha$ defined by*

$$\mathcal{C}_0^\alpha \;:= \{ f\colon [0,T] \to \mathbb{R} : \quad |f(t) - f(s)| = o(|t-s|^\alpha) \text{ uniformly as } |t-s| \to 0 \}\,.$$

Remark 7.11. Note that $f \in \mathcal{C}_0^\alpha$ if and only if

$$\forall \epsilon > 0 \quad \exists \delta_\epsilon > 0: \qquad |f(t) - f(s)| \le \epsilon |t-s|^\alpha \qquad \text{for } |t-s| \le \delta_\epsilon\,, \tag{7.17}$$

which implies (exercise) that $C^1 \subset \mathcal{C}_0^\alpha \subset \mathcal{C}^\alpha$ for $\alpha \in (0,1)$. It follows that the closure of C^1 in $\mathcal{C}^\alpha$ is again $\mathcal{C}_0^\alpha$, simply because $C^\infty \subset C^1 \subset \mathcal{C}_0^\alpha$.

Exercise 7.1. Prove that $C^1 \subset \mathcal{C}_0^\alpha$ and $\mathcal{C}_0^\alpha \subset \mathcal{C}^\alpha$ for $\alpha \in (0,1)$ (inclusions are strict).

We stress that the subset $\mathcal{C}_0^\alpha$ is strictly included in $\mathcal{C}^\alpha$, but what is left out is not so large, in the following sense.

Exercise 7.2. Prove that $\mathcal{C}^{\alpha'} \subset \mathcal{C}_0^\alpha$ for $0 < \alpha < \alpha' < 1$ (the inclusion is strict).

The proof of Theorem 7.10, which we defer to Section 7.7, is based on the following classical approximation result (also proved in Section 7.7).

Lemma 7.12. *For any continuous $f\colon [0,T] \to \mathbb{R}$ there is a sequence $f_n \in C^\infty$ such that $\|f_n - f\|_\infty \to 0$. One can take f_n with the same modulus of continuity as f, in the following sense: given an arbitrary function $h(\cdot)$,*

$$\begin{aligned} \textit{if} \quad & |f(t) - f(s)| \le h(t-s) \qquad \forall s,t \in [0,T]\,, \\ \textit{then} \quad & |f_n(t) - f_n(s)| \le h(t-s) \qquad \forall s,t \in [0,T], \quad \forall n \in \mathbb{N}\,. \end{aligned} \tag{7.18}$$

It follows that $\|\delta f_n\|_\alpha \le \|\delta f\|_\alpha$ for all $n \in \mathbb{N}$ and $\alpha \in (0,1)$.

Remark 7.13. Lemma 7.12 holds with no change for functions $f\colon [0,T] \to R$, where R is an arbitrary Banach space. One only needs a notion of integral $\int_0^T f_s\,\mathrm{d}s$ when f is continuous, and for this one can take the Riemann integral, i.e. the limit of Riemann sums $\sum_i f(t_i)(t_{i+1} - t_i)$ along partitions (t_i) of $[0,T]$ with vanishing mesh $\max_i |t_{i+1} - t_i| \to 0$ (one can check that such Riemann sums form a Cauchy family). This integral satisfies the key usual properties: $f \mapsto \int_0^T f_s\,\mathrm{d}s$ is linear, $|\int_0^T f_s\,\mathrm{d}s| \le \int_0^T |f_s|\,\mathrm{d}s$ and $\int_0^T f'_s\,\mathrm{d}s = f_T - f_0$.

7.6. Uniqueness of the Young integral

Throughout this section we denote by I_t^{Young} the *Young integral* $I_t = \int_0^t Y \, \mathrm{d}X$ built in Theorem 7.1. We want to compare it with the *classical integral*

$$I_t^{\text{classical}} := \int_0^t Y_u \, \dot{X}_u \, \mathrm{d}u$$

which is defined for continuous Y and continuously differentiable $X \in C^1$.

We remarked in (6.2)-(6.3) that $I_t^{\text{classical}}$ satisfies property (7.5), therefore $I_t^{\text{classical}}$ coincides with I_t^{Young} when $(X, Y) \in C^1 \times \mathcal{C}^\beta$, for any $\beta \in]0, 1]$. In other terms, the *Young integral is an extension of the classical integral*.

We can be more precise: by Theorem 7.1, for $\alpha, \beta \in]0, 1]$ with $\alpha + \beta > 1$, the Young integral $I^{\text{Young}} = (I_t^{\text{Young}})_{t \in [0,T]}$ is a continuous bilinear map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^\alpha$. This means that I^{Young} is a *continuous* extension of the classical integral $I^{\text{classical}}$ defined on $C^1 \times \mathcal{C}^\beta$. It would be tempting to state that it is the *unique* continuous extension, but *this is not true*, because $C^1 \subset \mathcal{C}^\alpha$ is not dense in $\mathcal{C}^a$ (see Theorem 7.10 and Remark 7.11).

Interestingly, it is possible to characterize the Young integral as the unique continuous extension of $I^{\text{classical}}$, if we let the exponent α vary. Given $\bar{\alpha} \in]0, 1[$, we define the space

$$\mathcal{C}^{>\bar{\alpha}} := \bigcup_{\alpha \in]\bar{\alpha}, 1]} \mathcal{C}^\alpha$$

and we agree that $f_n \to f$ in $\mathcal{C}^{>\bar{\alpha}}$ if and only if $f_n \to f$ in $\mathcal{C}^\alpha$ for some $\alpha > \bar{\alpha}$. The basic observation is that C^1 *is dense in* $\mathcal{C}^{>\bar{\alpha}}$: for any $f \in \mathcal{C}^{>\bar{\alpha}}$ we can find a sequence $f_n \in C^1$ such that $f_n \to f$ in $\mathcal{C}^{>\bar{\alpha}}$.[7.1]

If we fix $\bar{\alpha} = 1 - \beta$, for $\beta \in]0, 1]$, the Young integral $I^{\text{Young}} = (I_t^{\text{Young}})_{t \in [0,T]}$ is a continuous map from $\mathcal{C}^{>(1-\beta)} \times \mathcal{C}^\beta$ to $\mathcal{C}^{>(1-\beta)}$, by Theorem 7.1.

These observations yield immediately the following result.

Proposition 7.14. (Characterization of the Young integral, I) *Fix any $\beta \in]0, 1]$. The Young integral $I^{\text{Young}} = (I_t^{\text{Young}})_{t \in [0,T]}$, viewed as a map from $\mathcal{C}^{>(1-\beta)} \times \mathcal{C}^\beta$ to $\mathcal{C}^{>(1-\beta)}$, is the unique continuous extension of the classical integral $I^{\text{classical}} = (I_t^{\text{classical}})_{t \in [0,T]}$ defined on $C^1 \times \mathcal{C}^\beta$.*

Explicitly, I^{Young} is the unique map $I: \mathcal{C}^{>(1-\beta)} \times \mathcal{C}^\beta \to \mathcal{C}^{>(1-\beta)}$ such that:

- $I_t = I_t^{\text{classical}} = \int_0^t Y_u \, \dot{X}_u \, \mathrm{d}u$ *for $X \in C^1$;*
- *if $(X_n, Y_n) \to (X, Y)$ in $\mathcal{C}^\alpha \times \mathcal{C}^\beta$, for some $\alpha > 1 - \beta$, then we have the convergence $I(X_n, Y_n) \to I(X, Y)$ in $\mathcal{C}^{\alpha'}$ for some $\alpha' > 1 - \beta$.*

Alternatively, we can characterize the Young integral as the unique continuous extension of the classical integral on $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ for fixed α, provided we consider a weaker notion of convergence on $\mathcal{C}^\alpha$.

7.1. If $f \in \mathcal{C}^\alpha$ with $\alpha > \bar{\alpha}$, by Exercise 7.2 we have $f \in \mathcal{C}_0^{\alpha'}$ for any $\alpha' \in]\bar{\alpha}, \alpha[$, then by Theorem 7.10 we can find $f_n \in C^\infty$ such that $f_n \to f$ in $\mathcal{C}^{\alpha'}$, hence $f_n \to f$ in $\mathcal{C}^{>\bar{\alpha}}$.

Definition 7.15. *Fix $\alpha \in]0,1]$. Given $f_n, f: [0,T] \to \mathbb{R}$, with $n \in \mathbb{N}$, we write*

$$f_n \rightsquigarrow_\alpha f \qquad \Longleftrightarrow \qquad \| f_n - f \|_\infty \to 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}} \| \delta f_n \|_\alpha < \infty\,. \tag{7.19}$$

In other terms, $f_n \rightsquigarrow_\alpha f$ if and only if $f_n \to f$ in the sup-norm and, moreover, the sequence f_n is bounded in $\mathcal{C}^\alpha$.

We leave it as an exercise to check some basic properties.

Exercise 7.3. Fix $\alpha \in]0,1]$ and let $f_n, f: [0,T] \to \mathbb{R}$, with $n \in \mathbb{N}$. Prove the following.

1. If $f_n \rightsquigarrow_\alpha f$, then $f \in \mathcal{C}^\alpha$; more precisely $\| \delta f \|_\alpha \le \sup_{n \in \mathbb{N}} \| \delta f_n \|_\alpha < \infty$.
2. If $f_n \rightsquigarrow_\alpha f$, then $f_n \to f$ in $\mathcal{C}^{\alpha'}$ for any $\alpha' < \alpha$, but not necessarily $f_n \to f$ in $\mathcal{C}^\alpha$.
3. If $f_n \rightsquigarrow_\alpha f$ and $\varphi: \mathbb{R} \to \mathbb{R}$ is Lipschitz, then $\varphi(f_n) \rightsquigarrow_\alpha \varphi(f)$.
4. In the definition (7.19) of $f_n \rightsquigarrow_\alpha f$, the uniform convergence $\| f_n - f \|_\infty \to 0$ can be replaced by pointwise convergence: $f_n(t) \to f(t)$ for every $t \in [0,T]$.

We can now provide the following characterization of the Young integral.

Theorem 7.16. (Characterization of the Young integral, II) *Fix α, $\beta \in]0,1]$ with $\alpha + \beta > 1$. The Young integral $I^{\text{Young}} = (I_t^{\text{Young}})_{t \in [0,T]}$ is the unique map $I: \mathcal{C}^\alpha \times \mathcal{C}^\beta \to \mathcal{C}^\alpha$ such that:*

1. *$I_t = I_t^{\text{classical}} = \int_0^t Y_u \dot{X}_u \, \mathrm{d}u$ for $X \in C^1$;*
2. *if $X_n \rightsquigarrow_\alpha X$ and $Y_n \rightsquigarrow_\beta Y$, we have $I(X_n, Y_n) \rightsquigarrow_\alpha I(X, Y)$.*

Proof. We already know that the Young integral I^{Young} satisfies property 1. Let us show that it also satisfies property 2: given $X_n \rightsquigarrow_\alpha X$ and $Y_n \rightsquigarrow_\beta Y$, we need to prove that

$$I^{\text{Young}}(X_n, Y_n) \rightsquigarrow_\alpha I^{\text{Young}}(X, Y)\,. \tag{7.20}$$

Let us fix $\alpha' < \alpha$, $\beta' < \beta$ such that we still have $\alpha' + \beta' > 1$. We know by Exercise 7.3 that $X_n \to X$ in $\mathcal{C}^{\alpha'}$ and $Y_n \to Y$ in $\mathcal{C}^{\beta'}$. Since the Young integral is a continuous bilinear operator $I^{\text{Young}}: \mathcal{C}^{\alpha'} \times \mathcal{C}^{\beta'} \to \mathcal{C}^{\beta'}$, we have the convergence $I^{\text{Young}}(X_n, Y_n) \to I^{\text{Young}}(X, Y)$ in $\mathcal{C}^{\alpha'}$, which implies

$$\| I^{\text{Young}}(X_n, Y_n) - I^{\text{Young}}(X, Y) \|_\infty \to 0\,.$$

To prove (7.20), it remains to observe that, by (7.7),

$$\sup_n \| I^{\text{Young}}(X_n, Y_n) \|_\alpha \leqslant \sup_n \left(\| Y_n \|_\infty + K_{\alpha+\beta} T^\alpha \, \| \delta Y_n \|_\beta \right) \| X_n \|_\alpha < \infty\,.$$

We next consider an operator $I: \mathcal{C}^\alpha \times \mathcal{C}^\beta \to \mathcal{C}^\alpha$ which satisfies properties 1 and 2 and we show that it must coincide with the Young integral I^{Young}. Given $X \in C^\alpha$ and $Y \in C^\beta$, by Lemma 7.12 we can construct a sequence $(X_n) \subset C^1$ with $\| X_n - X \|_\infty \to 0$ and $\| X_n \|_\alpha \leqslant \| X \|_\alpha$. By property 2 we have $I(X_n, Y) \rightsquigarrow_\alpha I(X, Y)$ and $I^{\text{Young}}(X_n, Y) \rightsquigarrow_\alpha I^{\text{Young}}(X, Y)$, which implies pointwise convergence: for any $t \in [0,T]$

$$I_t(X, Y) = \lim_n I_t(X_n, Y) \qquad \text{and} \qquad I_t^{\text{Young}}(X, Y) = \lim_n I_t^{\text{Young}}(X_n, Y)\,.$$

By property 1 we have $I_t(X_n, Y) = I_t^{\text{Young}}(X_n, Y)$ for any n, hence

$$I_t(X, Y) = I_t^{\text{Young}}(X, Y) \qquad \forall t \in [0, T],$$

which completes the proof. □

7.7. Two technical proofs

We give here the proof of Theorem 7.10 and Lemma 7.12.

Proof of Lemma 7.12. We extend $f: [0, T] \to \mathbb{R}$ to a function defined on the whole real line, by setting $f(t) = f(0)$ for $t < 0$ and $f(t) = f(T)$ for $t > T$.

Let us fix a C^∞ function $\varphi: \mathbb{R} \to \mathbb{R}$ supported in $[-1, 1]$ with unit integral: $\int_{\mathbb{R}} \varphi(u)\, \mathrm{d}u = 1$. Note that $\varphi_n(t) := n\varphi(nt)$ is supported in $\left[-\frac{1}{n}, \frac{1}{n}\right]$ and also has unit integral: $\int_{\mathbb{R}} \varphi_n(u)\, \mathrm{d}u = 1$. We then define $f_n = \varphi_n * f$, that is

$$f_n(t) := \int_{\mathbb{R}} \varphi_n(t - u)\, f(u)\, \mathrm{d}u.$$

It is a classical result that $f_n \in C^\infty$ (we can differentiate inside the integral by dominated convergence, since f is bounded).

We next write

$$f_n(t) = \int_{\mathbb{R}} \varphi_n(u)\, f(t - u)\, \mathrm{d}u = \int_{\mathbb{R}} \varphi(v)\, f\Big(t - \frac{v}{n}\Big)\, \mathrm{d}v,$$

which implies $\|f_n - f\|_\infty \leq \sup_{t \in \mathbb{R}, |u| \leq 1} \left|f\left(t - \frac{v}{n}\right) - f(t)\right|$ (since φ has unit integral), hence $\|f_n - f\|_\infty \to \infty$. Property (7.18) is also directly checked. □

Proof of Theorem 7.10. First we show that $\mathcal{C}_0^\alpha$ is closed in $\mathcal{C}^\alpha$: given f_n in $\mathcal{C}_0^\alpha$ and $f \in \mathcal{C}^\alpha$ such that $\|f_n - f\|_\alpha \to 0$, we need to show that $f \in \mathcal{C}_0^\alpha$, that is (7.17) holds. For $s < t$ and $n \in \mathbb{N}$ we can write, by the triangle inequality,

$$\frac{|f(t) - f(s)|}{(t - s)^\alpha} \leq \|\delta f - \delta f_n\|_\alpha + \frac{|f_n(t) - f_n(s)|}{(t - s)^\alpha}. \tag{7.21}$$

Fix $n = \bar{n}_\epsilon$ such that $\|\delta f_{\bar{n}_\epsilon} - \delta f\|_\alpha < \frac{\epsilon}{2}$. Since $f_{\bar{n}_\epsilon} \in \mathcal{C}_0^\alpha$, by (7.17) we can fix $\delta_\epsilon > 0$ such that for $|t - s| \leq \delta$ the last term in (7.21) is $\leq \frac{\epsilon}{2}$ and we are done.

It remains to show that, for any $f \in \mathcal{C}_0^\alpha$, there is a sequence $f_n \in C^\infty$ such that $\|f_n - f\|_\infty + \|\delta f_n - \delta f\|_\alpha \to 0$ (recall Remark 1.4). We define $f_n \in C^\infty$ as in Lemma 7.12, so we only need to show that $\|\delta f_n - \delta f\|_\alpha \to 0$.

Since $f \in \mathcal{C}_0^\alpha$, property (7.17) holds. The same property holds replacing for f_n, uniformly for $n \in \mathbb{N}$, thanks to relation (7.18). This means that for any $\epsilon > 0$, for all $0 \leqslant s < t \leqslant T$ with $|t - s| \leqslant \delta_\epsilon$, and for any $n \in \mathbb{N}$, we can write

$$\frac{|(f_n - f)(t) - (f_n - f)(s)|}{(t - s)^\alpha} \leqslant \frac{|f_n(t) - f_n(s)|}{(t - s)^\alpha} + \frac{|f(t) - f(s)|}{(t - s)^\alpha} \leqslant 2\,\epsilon.$$

If we fix $\bar{n}_\epsilon > 0$ such that $\|f_n - f\|_\infty \leqslant \epsilon\, (\delta_\epsilon)^\alpha$ for all $n \geq \bar{n}_\epsilon$, for $|t - s| > \delta_\epsilon$ we get

$$\frac{|(f_n - f)(t) - (f_n - f)(s)|}{(t - s)^\alpha} \leqslant \frac{2\|f_n - f\|_\infty}{(\delta_\epsilon)^\alpha} \leqslant \epsilon.$$

Altogether, the previous relations show that $\|\delta f_n - \delta f\|_\alpha \leqslant 2\epsilon$ for $n \geq \bar{n}_\epsilon$. This implies that $\|\delta f_n - \delta f\|_\alpha \to 0$. □