Chapter 8

Rough paths

We have seen in Chapter 3 that it is possible to build a robust theory for a controlled equation of the form $\dot{Y}_t = \sigma(Y_t)\,\dot{X}_t$ with $X\colon [0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$ for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, provided we *choose* a function $\mathbb{X}^2\colon [0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$ satisfying for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$\delta\mathbb{X}^2_{sut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha},$$

see (3.13), where we denote $\mathbb{X}^1_{st} := \delta X_{st}$, $0 \leqslant s \leqslant t \leqslant T$. In coordinates, the former identity means

$$(\delta\mathbb{X}^2)^{ij}_{sut} = \delta X^i_{su}\,\delta X^j_{ut}, \qquad |(\mathbb{X}^2_{st})^{ij}| \lesssim |t-s|^{2\alpha}, \qquad i,j \in \{1,\ldots,d\}. \tag{8.1}$$

In Section 3.2 we left the problem of the existence of such a function $\mathbb{X}^2$ open.

We recall that, for X of class C^1, we have a natural choice for $\mathbb{X}^2$ given by

$$(\mathbb{X}^2_{st})^{ij} := \int_s^t (X^i_r - X^i_s)\,\dot{X}^j_r\,\mathrm{d}r, \qquad 0 \leqslant s \leqslant t \leqslant T,$$

see (3.9). In Lemma 7.6 we saw that, for $\alpha > \frac{1}{2}$ and $X \in \mathcal{C}^\alpha([0,T];\mathbb{R}^d)$, the (uniquely defined) Young integral $I^{ij}_t := \int_0^t X^i\,\mathrm{d}X^j$ satisfies

$$R^{ij}_{st} := I^{ij}_t - I^{ij}_s - X^i_s\,(X^j_t - X^j_s) = \int_s^t (X^i_r - X^i_s)\,\mathrm{d}X^j_r, \qquad |R^{ij}_{st}| \lesssim |t-s|^{2\alpha},$$

where the integral in the right-hand side is again of the Young type and $2\alpha > 1$.

There is a clear resemblance between the two last expressions, and indeed for $\alpha > \frac{1}{2}$ we show in Lemma 8.14 below that setting $(\mathbb{X}^2_{st})^{ij} := R^{ij}_{st}$ we obtain (8.1) and this is the only possible choice.

If now $\alpha \leqslant \frac{1}{2}$, neither of these formulae is well-defined, because for $2\alpha \leqslant 1$ we are not in the setting of the Young integral. However, we have seen in Chapter 3 that the bound $|\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}$ is enough for the whole theory of existence, uniqueness and stability of the rough equation (3.19) to work, even if $2\alpha \leqslant 1$.

This suggests that, for every $i,j \in \{1,\ldots,d\}$, the function $(\mathbb{X}^2_{st})^{ij}$ can be interpreted as the remainder R^{ij} associated with an integral I^{ij} of (X^i, X^j), where we *weaken* our requirements with respect to the Young integral, namely we only require that

$$I^{ij}_t - I^{ij}_s - X^i_s\,(X^j_t - X^j_s) = (\mathbb{X}^2_{st})^{ij}, \qquad |(\mathbb{X}^2_{st})^{ij}| \lesssim |t-s|^{2\alpha},$$

and now $2\alpha \leqslant 1$. Therefore the choice of the rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ over X is equivalent to the choice of a *generalised integral* $I = \int_0^\cdot X \otimes \mathrm{d}X \in \mathcal{C}^\alpha([0,T];\mathbb{R}^d \otimes \mathbb{R}^d)$, and in this case $\mathbb{X}^2$ plays the role of a generalised remainder with respect to the germ $(s,t) \mapsto X_s \otimes (X_t - X_s)$.

In this chapter we explore these notions and explain them in greater detail.

8.1. INTEGRAL BEYOND YOUNG

Let us fix $(X,Y)\in\mathcal{C}^\alpha\times\mathcal{C}^\beta$. We saw in Theorem 7.3 that when $\alpha+\beta>1$ we can define the integral $I_t=\int_0^t Y\,\mathrm{d}X$ as the unique function which solves

$$I_0=0\,,\qquad \delta I_{st}=Y_s\,\delta X_{st}+R_{st},\qquad R_{st}=o(t-s)\,. \tag{8.2}$$

This was based on the observation that for the germ $A_{st}:=Y_s\,\delta X_{st}$ we have

$$\delta A_{sut}=-\delta Y_{su}\,\delta X_{ut}\qquad\Longrightarrow\qquad \|\delta A\|_{\alpha+\beta}\leqslant\|\delta X\|_\alpha\|\delta Y\|_\beta\,.$$

Therefore if $\eta:=\alpha+\beta>1$ we have $\|\delta A\|_\eta<\infty$, i.e. the germ A is coherent, see Definition 6.7, and the Sewing Lemma can be applied, see Theorem 6.8.

We now focus on the regime $\alpha+\beta\leqslant 1$. As we have already seen in (7.8) above, there exist germs A which allow *no function I solving (8.2)*. Indeed, we recall that choosing $X_t=t^\alpha$ and $Y_t=t^\beta$, $t\in[0,T]$, then the germ $A_{st}:=Y_s\,\delta X_{st}$ satisfies $|\delta A_{0\frac{t}{2}t}|\gtrsim t^{\alpha+\beta}$, see (7.8), and therefore the necessary condition (6.9) in Lemma 6.5 is not satisfied.

A solution is to relax the requirement $R_{st}=o(t-s)$ in (8.2), say to

$$\exists\eta\leqslant 1:\qquad |R_{st}|\lesssim|t-s|^\eta. \tag{8.3}$$

Arguing as in Lemma 6.5, this would imply

$$|\delta R_{sut}|\lesssim|t-s|^\eta+|u-s|^\eta+|t-u|^\eta\lesssim|u-s|^\eta+|t-u|^\eta$$

since $\eta\leqslant 1$. On the other hand, by Proposition 6.4 we have $|\delta R_{sut}|=|\delta A_{sut}|\lesssim|u-s|^\beta|t-u|^\alpha$. Choosing $|u-s|=|t-u|$ shows that the best we can hope for in (8.3) is $\eta=\alpha+\beta$.

Summarizing, given $(X,Y)\in\mathcal{C}^\alpha\times\mathcal{C}^\beta$ with $\alpha+\beta\leqslant 1$, it is natural to wonder whether there exists a function I which satisfies the following weakening of (8.2)

$$I_0=0\,,\qquad \delta I_{st}=Y_s\,\delta X_{st}+R_{st}\,,\qquad |R_{st}|\lesssim|t-s|^{\alpha+\beta}\,. \tag{8.4}$$

This would provide a "generalised notion of integral" $\int_0^\cdot Y\,\mathrm{d}X$. This justifies the following

DEFINITION 8.1. *Fix $\alpha,\beta\in(0,1)$ with $\alpha+\beta\leqslant 1$. Given $(X,Y)\in\mathcal{C}^\alpha\times\mathcal{C}^\beta$, if there exists a function $I\colon[0,T]\to\mathbb{R}$ which satisfies*

$$I_t-I_s=Y_s\,(X_t-X_s)+O(|t-s|^{\alpha+\beta})\qquad \textit{uniformly as } |t-s|\to 0, \tag{8.5}$$

we say that I is a generalised integral of Y in $\mathrm{d}X$.

We stress that this new definition of integral extends the previous one (8.2) for $(X,Y)\in\mathcal{C}^\alpha\times\mathcal{C}^\beta$ with $\alpha+\beta>1$, because the term $o(t-s)$ is actually $O(|t-s|^{\alpha+\beta})$ in this case, by the key estimate for the Young integral (or, equivalently, for the sewing map).

On the positive side, *there is always existence for (8.4)* if $\alpha+\beta<1$. This is a non-trivial result, due (in a more general setting) to Lyons and Victoir. We state this as a separate result, which is a consequence of Proposition 8.5 below.

LEMMA 8.2. *Let* $(X,Y)\in\mathcal{C}^\alpha\times\mathcal{C}^\beta$ *with* $\alpha+\beta<1$. *There exists* $(I,R)\in\mathcal{C}^\alpha\times C_2^{\alpha+\beta}$ *satisfying (8.4).*

Remark 8.3. It is an easy observation that *uniqueness can not hold for (8.4)*. Indeed, given I which solves (8.4), any function of the form $I'_t:=I_t+h_t-h_0$ with $h\in\mathcal{C}^{\alpha+\beta}$ still solves (8.4). As a matter of fact, *all solutions are of this form*, because given two solutions I,I' of (8.4), with corresponding R,R', their difference $h:=I'-I$ must satisfy $|\delta h_{st}|=|R'_{st}-R_{st}|\lesssim|t-s|^{\alpha+\beta}$.

Remark 8.4. An integral I as in Definition 8.1 is necessarily of class $\mathcal{C}^\alpha$ by (8.5).

We state now a result which implies Lemma 8.2 above.

PROPOSITION 8.5. (PARAINTEGRAL) *Fix* $\alpha,\beta\in(0,1)$ *with* $\alpha+\beta<1$. *There exists a (non unique) bilinear and continuous map* $J_\prec\colon\mathcal{C}^\alpha\times\mathcal{C}^\beta\to C_2^{\alpha+\beta}$ *such that*

$$\|J_\prec(X,Y)\|_{\alpha+\beta}\leqslant\mathsf{C}\,\|\delta X\|_\alpha\,\|\delta Y\|_\beta, \tag{8.6}$$

for a suitable $\mathsf{C}=\mathsf{C}(\alpha,\beta,T)$, *with the property that, for all* $s<u<t$,

$$\delta J_\prec(X,Y)_{sut}=\delta Y_{su}\,\delta X_{ut}. \tag{8.7}$$

The proof of Proposition 8.5 is postponed to Section 8.9 below.

Remark 8.6. Let $\alpha,\beta\in(0,1)$ with $\alpha+\beta\leqslant1$. Finding a generalised integral of Y in $\mathrm{d}X$ for $(X,Y)\in\mathcal{C}^\alpha\times\mathcal{C}^\beta$ as in Definition 8.1 is equivalent to finding $R_{st}\in C_2^{\alpha+\beta}$ such that

$$\delta R_{sut}=\delta Y_{su}\,\delta X_{ut}, \tag{8.8}$$

$$R\in C_2^{\alpha+\beta}. \tag{8.9}$$

Indeed, if we define $A_{st}:=Y_s\,\delta X_{st}$, relation (8.8) implies that $\delta(A+R)=0$, hence there exists $I\colon[0,T]\to\mathbb{R}$ which satisfies $\delta I=A+R$, which is exactly relation (8.5).

By Proposition 8.5 and Remark 8.6, if $\alpha,\beta\in(0,1)$ and $\alpha+\beta<1$, any $(X,Y)\in\mathcal{C}^\alpha\times\mathcal{C}^\beta$ admits an integral I as in Definition 8.1.

8.2. A NEGATIVE RESULT

We show that the usual integral $I(f,g)=\int_0^t f_s g'_s\,\mathrm{d}s$, when $g\in C^1$, cannot be extended to a continuous operator on $\mathcal{C}^{\alpha'}\times\mathcal{C}^{\beta'}$, when $\alpha'+\beta'<1$.

LEMMA 8.7. *Set* $[0,T]=[0,1]$ *and define, for* $\alpha,\beta\in(0,1)$,

$$f_n(t):=\frac{1}{n^\alpha}\cos(nt),\qquad g_n(t):=\frac{1}{n^\beta}\sin(nt).$$

Then $f_n\rightsquigarrow_\alpha 0$ *and* $g_n\rightsquigarrow_\beta 0$ *(recall Definition 7.15), more precisely:*

$$\|f_n\|_\infty\to0,\qquad\|\delta f_n\|_\alpha\leqslant2;\qquad\qquad\|g_n\|_\infty\to0,\qquad\|\delta g_n\|_\beta\leqslant2. \tag{8.10}$$

(In particular, $f_n \to 0$ in $\mathcal{C}^{\alpha'}$ and $g_n \to 0$ in $\mathcal{C}^{\beta'}$ for any $\alpha' < \alpha$ and $\beta' < \beta$.)

However, if we fix $\alpha + \beta \leqslant 1$, we have $I(f_n, g_n) \not\to 0$, because

$$\forall t \in [0,1]: \qquad \lim_{n\to\infty} I(f_n, g_n)_t = \begin{cases} +\infty & \text{if } \alpha+\beta<1 \\ \frac{1}{2}t & \text{if } \alpha+\beta=1 \\ 0 & \text{if } \alpha+\beta>1 \end{cases}.$$

Proof. Note that $\|f_n\|_\infty = n^{-\alpha}$ and $\|f_n'\|_\infty = n^{1-\alpha}$, hence

$$|f_{n_t} - f_{n_s}| \leqslant \min\{\|f_n'\|_\infty |t-s|, 2\|f_n\|_\infty\} \leqslant \min\{n^{1-\alpha}|t-s|, 2\, n^{-\alpha}\}.$$

Since $\min\{x,y\} \leqslant x^\gamma y^{1-\gamma}$, for any $\gamma \in [0,1]$, choosing $\gamma = \alpha$ we obtain

$$|f_n(t) - f_n(s)| \leqslant 2^{1-\alpha}\, |t-s|^\alpha,$$

hence $\|\delta f_n\|_\alpha \leqslant 2^{1-\alpha} \leqslant 2$. Similar arguments apply to g_n, proving (8.10).

Next we observe that $\frac{1}{2\pi}\int_0^{2\pi}\cos^2(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\sin^2(x)\,\mathrm{d}x = \frac{1}{2}$. Then, for fixed $t > 0$, as $n \to \infty$

$$\int_0^{nt} \cos^2(x)\,\mathrm{d}x = \int_0^{2\pi\lfloor\frac{nt}{2\pi}\rfloor} \cos^2(x)\,\mathrm{d}x + O(1) = \frac{1}{2}2\pi\left\lfloor\frac{nt}{2\pi}\right\rfloor + O(1) = \frac{t}{2}n + O(1).$$

It follows that

$$I(f_n, g_n)_t = \frac{n}{n^{\alpha+\beta}}\int_0^t \cos^2(ns)\,\mathrm{d}s = \frac{1}{n^{\alpha+\beta}}\int_0^{nt}\cos^2(x)\,\mathrm{d}x \sim \frac{t}{2}\, n^{1-(\alpha+\beta)}. \qquad \square$$

8.3. A CHOICE

We have seen in (7.11) above that, given $(X,Y) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ with $\alpha+\beta>1$, we have an *explicit formula* for the remainder $R_{st} = I_t - I_s - Y_s(X_t - X_s)$, given by

$$R_{st} = \int_s^t (Y_u - Y_s)\,\mathrm{d}X_u, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{8.11}$$

where $I_t = \int_0^t Y_u\,\mathrm{d}X_u$ is the *unique* function given by the Young integral of Theorem 7.1. Moreover $R_{st} = \int_s^t (Y_u - Y_s)\,\mathrm{d}X_u$ is the unique function in C_2 which satisfies

$$R \in C_2^{\alpha+\beta}, \qquad \delta R_{sut} = \delta Y_{su}\,\delta X_{ut}, \qquad 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{8.12}$$

In the regime $\alpha+\beta<1$, the Young integral is not available anymore. However by Proposition 8.5 we know that we can find an integral $I \in \mathcal{C}^\alpha$ in the sense of Definition 8.1 by setting

$$\delta I_{st} := Y_s(X_t - X_s) - J_\prec(X,Y)_{st},$$

where $J_\prec$ is the paraintegral of Proposition 8.5, see also Remark 8.6. This shows that, in this setting, the remainder $R_{st} = I_t - I_s - Y_s(X_t - X_s)$ is not given by an explicit formula like (8.11) (which is now ill-defined), rather we have

$$R = -J_\prec(X,Y).$$

However formula (8.11) suggests that we can *define*

$$\int_s^t (Y_u - Y_s)\,\mathrm{d}X_u := R_{st} = -J_{\prec}(X,Y)_{st}, \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{8.13}$$

In other words, the left hand side of (8.13) is *chosen* to be equal to the remainder R associated with the integral I as in (8.4). We recall that $R = -J_{\prec}(X,Y)$ satisfies

$$R \in C_2^{\alpha+\beta}, \qquad \delta R_{sut} = \delta Y_{su}\,\delta X_{ut}, \qquad 0 \leqslant s \leqslant u \leqslant t \leqslant T. \tag{8.14}$$

The difference between formula (8.14) and formula (8.12), is that in the former $\alpha + \beta < 1$ while in the latter $\alpha + \beta > 1$. Accordingly, in (8.14) the function R is *not* uniquely determined, while in (8.12) it is.

The comparison between formula (8.14) and formula (8.12), and the explicit expression (8.11) in the case $\alpha + \beta > 1$ show that (8.13) is a reasonable *definition* of the function $(s,t) \mapsto \int_s^t (Y_u - Y_s)\,\mathrm{d}X_u$ in the setting $\alpha + \beta \leqslant 1$.

We also stress that R in (8.14) *can not be uniquely determined*. Indeed, by Remark 8.3, we have infinitely many possible choices given by

$$R' = R + \delta h, \qquad h \in \mathcal{C}^{\alpha+\beta}, h_0 = 0. \tag{8.15}$$

Remark 8.8. In the special case $X = Y$ and $\alpha = \beta \leqslant \frac{1}{2}$, (8.4) becomes

$$I_0 = 0\,, \qquad \delta I_{st} = X_s\,\delta X_{st} + R_{st}\,, \qquad |R_{st}| \lesssim |t-s|^{2\alpha}\,. \tag{8.16}$$

Now the germ is $A_{st} = X_s(X_t - X_s)$ and we have a simple canonical solution which does not rely on the paraintegral and is given by

$$I_t := \frac{1}{2}(X_t^2 - X_0^2), \qquad R_{st} := \frac{1}{2}(X_t - X_s)^2,$$

since

$$\underbrace{\frac{1}{2}\,(X_t^2 - X_s^2)}_{I_t - I_s} = \underbrace{X_s(X_t - X_s)}_{A_{st}} + \underbrace{\frac{1}{2}(X_t - X_s)^2}_{R_{st}}.$$

As we have seen in (7.15)-(7.16), if $\alpha > \frac{1}{2}$ then (I,R) is the *only* solution of (8.16) and moreover

$$R_{st} = \int_s^t (X_r - X_s)\,\mathrm{dX}_r$$

where the integral is in the Young sense. If $\alpha \leqslant \frac{1}{2}$, then we have infinitely many possible solutions (I', R').

8.4. One-dimensional rough paths

We have seen at the beginning of this chapter that for every $i,j \in \{1,\ldots,d\}$, the function $(\mathbb{X}_{st}^2)^{ij}$ plays the role of the remainder R^{ij} associated with a generalised integral I^{ij} of (X^i, X^j) in the sense of Definition 8.1 with $\alpha = \beta < \frac{1}{2}$: in other words the choice of $\mathbb{X}^2$ is *equivalent* to the choice of integrals (in the sense of Definition 8.1) $I^{ij} \in \mathcal{C}^\alpha$ for all $i,j \in \{1,\ldots,d\}$, such that

$$I_0^{ij} = 0\,, \qquad \delta I_{st}^{ij} = X_s^i\,\delta X_{st}^j + (\mathbb{X}_{st}^2)^{ij}\,, \qquad |(\mathbb{X}_{st}^2)^{ij}| \lesssim |t-s|^{2\alpha}\,,$$

or, in more compact notations,

$$I_0 = 0, \qquad \delta I_{st} = X_s \otimes \mathbb{X}^1_{st} + \mathbb{X}^2_{st}, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}. \tag{8.17}$$

Existence of $\mathbb{X}^2$ satisfying (8.17) with $\alpha < \frac{1}{2}$ is therefore granted by Lemma 8.2, e.g. via the paraintegral of Theorem 8.5. We also know that in the regime $\alpha < \frac{1}{2}$ we have infinitely many possible choices for $(I, \mathbb{X}^2)$, all of the form (8.15) above.

Suppose first that we are in the setting $d = 1$. Then Definition 3.2 becomes

DEFINITION 8.9. *Let $\alpha \in]\frac{1}{3}, \frac{1}{2}]$ and $X: [0,T] \to \mathbb{R}$ of class $\mathcal{C}^\alpha$. A α-*Rough Path *over X is a pair $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2) \in C_2^\alpha \times C_2^{2\alpha}$ such that*

$$\mathbb{X}^1_{st} = X_t - X_s, \qquad \delta \mathbb{X}^2_{sut} = \mathbb{X}^1_{su}\, \mathbb{X}^1_{ut}. \tag{8.18}$$

We recall that the conditions $X \in \mathcal{C}^\alpha$ and $\mathbb{X}^1 = \delta X \in C_2^\alpha$ are equivalent, and that $(\mathbb{X}^1, \mathbb{X}^2) \in C_2^\alpha \times C_2^{2\alpha}$ is equivalent to

$$|\mathbb{X}^1_{st}| \lesssim |t-s|^\alpha, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}.$$

We have seen in Chapter 3 that it is possible to build an integration theory for every choice of the α-rough path $\mathbb{X}$ over X. In this theory we can recover existence *and* uniqueness of the integral function $\int_0^\cdot Y \, \mathrm{d}X$ for a large class of choices of Y. For this we have to give very different roles to the integrator X and to the integrand Y, whereas in the case of the Young integral the two functions play a symmetric role: X will be a component of a rough path and Y a component of a *controlled path*, see Chapter 10.

We note that the algebraic condition $\delta \mathbb{X}^2_{sut} = \mathbb{X}^1_{su}\, \mathbb{X}^1_{ut}$ is *non-linear*, which implies that α-rough paths do not form a vector subspace of $C_2^\alpha \times C_2^{2\alpha}$.

For all $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, given any *real-valued* path $X \in \mathcal{C}^\alpha([0,T]; \mathbb{R})$, there is always a rough path lying above X. Indeed, $I_t := \frac{1}{2} X_t^2$ is a generalised integral of X in $\mathrm{d}X$ integral in the sense of Definition 8.1, because

$$\delta I_{st} = \frac{1}{2}(X_t^2 - X_s^2) = X_s\, \delta X_{st} + \frac{1}{2}(\delta X_{st})^2 = X_s\, \delta X_{st} + O(|t-s|^{2\alpha}).$$

Then, by Remark 8.8, we can define a rough path $\mathbb{X}$ by setting

$$\mathbb{X}^2_{st} = \frac{1}{2}(\delta X_{st})^2. \tag{8.19}$$

More directly, note that (8.19) satisfies the Chen relation (8.21), and clearly $\mathbb{X}^2 \in C_2^{2\alpha}$.

8.5. THE VECTOR CASE

Let us consider now a *vector valued* path $X: [0,T] \to \mathbb{R}^d$, with $X_t = (X_t^1, \ldots, X_t^d)$. We suppose that X is of class $\mathcal{C}^\alpha$, namely that $X^i \in \mathcal{C}^\alpha$ for all $i = 1, \ldots, d$, with $\alpha > \frac{1}{3}$.

We can now generalise Definition 8.9 to the vector case. The multi-dimensional case $d\geq 2$ is sensibly richer, because off-diagonal terms $\int X^i\,\mathrm{d}X^j$ with $i\neq j$ do not have explicit candidates as in (8.19).

Definition 8.10. *Let $\alpha\in\,]\frac{1}{3},\frac{1}{2}]$, $d\geq 1$ and $X\colon[0,T]\to\mathbb{R}^d$ of class $\mathcal{C}^\alpha$. A α-*Rough Path *on $\mathbb{R}^d$ over X is a pair $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$, with*

- $\mathbb{X}^1=(\delta X^i)_{i=1,\ldots,d}\in C_2^\alpha([0,T];\mathbb{R}^d)$
- $\mathbb{X}^2=(R^{ij})_{i,j=1,\ldots,d}\in C_2^{2\alpha}([0,T]^2_\leqslant;\mathbb{R}^d\otimes\mathbb{R}^d)$

such that

$$(\delta\mathbb{X}^2_{sut})^{ij}=(\mathbb{X}^1_{su})^i\,(\mathbb{X}^1_{ut})^j, \tag{8.20}$$

or equivalently

$$\mathbb{X}^2_{st}-\mathbb{X}^2_{su}-\mathbb{X}^2_{ut}=\mathbb{X}^1_{su}\otimes\mathbb{X}^1_{ut}. \tag{8.21}$$

We denote by $\mathcal{R}_{\alpha,d}$ the space of α-rough paths on $\mathbb{R}^d$ and by $\mathcal{R}_{\alpha,d}(X)$ the set of α-rough paths over X.

The condition (8.20)-(8.21) is the celebrated *Chen relation*. As in the one-dimensional case, existence of $\mathbb{X}^2$ satisfying (8.20)-(8.21) with $\alpha<\frac{1}{2}$ is therefore granted by Lemma 8.2, e.g. via the paraintegral of Theorem 8.5. We also know that in the regime $\alpha<\frac{1}{2}$ we have infinitely many possible choices for $(I,\mathbb{X}^2)$, all of the form (8.15) above.

We are going to see in Chapter 10 that it is possible to build an integration theory for every choice of an α-rough path $\mathbb{X}$. Again, we note that the condition (8.20)-(8.21) is *non-linear*, which implies that α-rough paths do not form a vector space.

The following exercise is a simple summary of the discussion at the beginning of this chapter.

Exercise 8.1. Given a α-rough path $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ over X in $\mathbb{R}^d$, a process $I\in\mathcal{C}^\alpha([0,T];\mathbb{R}^d\otimes\mathbb{R}^d)$ satisfying (8.17) is a generalised integral of X in $\mathrm{d}X$ in the sense of Definition 8.1.

Viceversa, given $X\in\mathcal{C}^\alpha([0,T];\mathbb{R}^d)$ and an integral $I\in\mathcal{C}^\alpha([0,T];\mathbb{R}^d\otimes\mathbb{R}^d)$ of X in $\mathrm{d}X$, in the sense of Definition 8.1, defining $\mathbb{X}^2$ by (8.17) we obtain a α-rough path $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ over X in $\mathbb{R}^d$.

In the multi-dimensional case $X\in\mathcal{C}^\alpha([0,T];\mathbb{R}^d)$ with $d\geq 2$, building a rough path over X is non-trivial, because one has to define off-diagonal integrals $\int X^i\,\mathrm{d}X^j$ for $i\neq j$. However, by the results we have proved on the existence of the paraintegral in Proposition 8.5, we can easily deduce the following.

Proposition 8.11. *For any $d\in\mathbb{N}$, $\alpha\in(\frac{1}{3},\frac{1}{2})$ and $X\in\mathcal{C}^\alpha([0,T];\mathbb{R}^d)$, there is a α-rough path $\mathbb{X}$ which lies above X (hence, by Lemma 8.15, there are infinitely many of them).*

Proof. For any fixed $i,j\in\{1,\ldots,d\}$, let I^{ij} be a generalised integral of X^i in $\mathrm{d}X^j$ in the sense of Definition 8.1, whose existence is guaranteed by the paraintegral of Proposition 8.5. Then, by Exercise 8.1, defining $\mathbb{X}^2$ by (8.17) we obtain a rough path $\mathbb{X}$ which lies above X. □

We conclude with an elementary observation, that will be useful later. By Exercise 8.1, any α-rough path $\mathbb{X}$ over $X \in \mathcal{C}^\alpha([0,T];\mathbb{R}^d)$ determines an integral I of (X, X), given by (8.17). Applying the latter relation in a telescopic fashion, we can write

$$I_t = \sum_{[t_i, t_{i+1}] \in \mathcal{P}} (X_{t_i} \delta X_{t_i t_{i+1}} + \mathbb{X}^2_{t_i t_{i+1}}), \tag{8.22}$$

where $\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_k = t\}$ is an arbitrary partition of $[0, t]$. We will see in Chapter 10 below that a generalization of (8.22), when we also take the limit of vanishing mesh $|\mathcal{P}| \to 0$, is the correct recipe for building "Riemann-sums", in order to define a generalised integral of h in $\mathrm{d}X$ in the sense of Definition 8.1 for a wide class of functions h.

8.6. Distance on rough paths

We denote by $\mathcal{R}_{\alpha,d}$ the set of all α-rough paths in $\mathbb{R}^d$. For $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2) \in \mathcal{R}_{\alpha,d}$ we set

$$\|\mathbb{X}\|_{\mathcal{R}_{\alpha,d}} := \|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha} = \sup_{0 \le s < t \le T} \frac{|\mathbb{X}^1_{st}|}{|t-s|^\alpha} + \sup_{0 \le s < t \le T} \frac{|\mathbb{X}^2_{st}|}{|t-s|^{2\alpha}}. \tag{8.23}$$

We stress that $\mathcal{R}_{\alpha,d}$ *is not a vector space*, because the Chen relation (8.21) is not linear. However, it is meaningful to define for $\mathbb{X}, \bar{\mathbb{X}} \in \mathcal{R}_{\alpha,d}$

$$d_{\mathcal{R}_{\alpha,d}}(\mathbb{X}, \bar{\mathbb{X}}) := \|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha}. \tag{8.24}$$

Exercise 8.2. $d_{\mathcal{R}_{\alpha,d}}$ is a distance on $\mathcal{R}_{\alpha,d}$.

When we talk of convergence in $\mathcal{R}_{\alpha,d}$, we mean with respect to the distance $d_{\mathcal{R}_{\alpha,d}}$. Note that $d_{\mathcal{R}_{\alpha,d}}$ is equal on $\mathcal{R}_{\alpha,d}$ to the distance induced by the natural norm $\|F\|_\alpha + \|G\|_{2\alpha}$ for $(F, G) \in C_2^\alpha \times C_2^{2\alpha}$. In particular $\mathbb{X}_n = (\mathbb{X}^1_n, \mathbb{X}^2_n) \to \mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ in $\mathcal{R}_{\alpha,d}$ if and only if $\mathbb{X}^1_n \to \mathbb{X}^1$ in C_2^α and $\mathbb{X}^2_n \to \mathbb{X}^2$ in $C_2^{2\alpha}$.

LEMMA 8.12. *The metric space $(\mathcal{R}_{\alpha,d}, d_{\mathcal{R}_{\alpha,d}})$ is complete.*

Proof. Let $(\mathbb{X}_n)_{n\in\mathbb{N}} \subset \mathcal{R}_{\alpha,d}$ be a Cauchy sequence. Then, by definition of $d_{\mathcal{R}_{\alpha,d}}$, for every $\epsilon > 0$ there is $\bar{n}_\epsilon < \infty$ such that for all $n, m \ge \bar{n}_\epsilon$ and $0 \le s < t \le T$

$$|\mathbb{X}^1_n(s,t) - \mathbb{X}^1_m(s,t)| \le \epsilon |t-s|^\alpha, \quad |\mathbb{X}^2_n(s,t) - \mathbb{X}^2_m(s,t)| \le \epsilon |t-s|^{2\alpha}. \tag{8.25}$$

Note that

$$d_{\mathcal{R}_{\alpha,d}}(\mathbb{X}, \bar{\mathbb{X}}) \ge \frac{\|\mathbb{X}^1 - \bar{\mathbb{X}}^1\|_\infty}{T^\alpha} + \frac{\|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_\infty}{T^{2\alpha}}.$$

It follows that the sequences of continuous functions $(\mathbb{X}^1_n)_{n\in\mathbb{N}}$ and $(\mathbb{X}^2_n)_{n\in\mathbb{N}}$ are Cauchy in the sup-norm, hence there are continuous functions $\mathbb{X}^1$ and $\mathbb{X}^2$ such that $\|\mathbb{X}^1_n - \mathbb{X}^1\|_\infty \to 0$ and $\|\mathbb{X}^2_n - \mathbb{X}^2\|_\infty \to 0$. In particular, we have pointwise convergence $\mathbb{X}^1_m(s,t) \to \mathbb{X}^1(s,t)$ and $\mathbb{X}^2_m(s,t) \to \mathbb{X}^2(s,t)$ as $m \to \infty$. Taking this limit in (8.25) shows that $d_{\mathcal{R}_{\alpha,d}}(\mathbb{X}_n, \mathbb{X}) \le 2\epsilon$ for all $n \ge \bar{n}_\epsilon$. □

This allows to rephrase the continuity result of section 3.7. We fix

$$D \geq \|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty + \|\nabla^3\sigma\|_\infty + \|\nabla\sigma_2\|_\infty + \|\nabla^2\sigma_2\|_\infty.$$

We obtain from Proposition 3.11

Proposition 8.13. *We suppose that $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$ and $\sigma: \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$ is of class C^3, with $\|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty + \|\nabla^3\sigma\|_\infty + \|\nabla\sigma_2\|_\infty + \|\nabla^2\sigma_2\|_\infty < +\infty$ (without boundedness assumptions on σ and σ_2). For $\mathbb{X} \in \mathcal{R}_{\alpha,d}$ and $Z_0 \in \mathbb{R}^k$ we denote by Z: $[0,T] \to \mathbb{R}^k$ the unique solution to equation (3.19)*

$$Z^{[3]}_{st} = o(t-s), \qquad Z^{[3]}_{st} = \delta Z_{st} - \sigma(Z_s)\, \mathbb{X}^1_{st} - \sigma_2(Z_s)\, \mathbb{X}^2_{st},$$

Then the map $\mathbb{R}^k \times \mathcal{R}_{\alpha,d} \ni (Z_0, \mathbb{X}) \mapsto Z \in \mathcal{C}^\alpha$ is locally Lipschitz continuous.

8.7. Canonical rough paths for $\alpha > \frac{1}{2}$

Let $\frac{1}{3} < \alpha' \leqslant \frac{1}{2} < \alpha < 1$. Then it is well known that $\mathcal{C}^\alpha \subset \mathcal{C}^{\alpha'}$. Therefore, if $X \in \mathcal{C}^\alpha([0,T]; \mathbb{R}^d)$ we have in particular $X \in \mathcal{C}^{\alpha'}([0,T]; \mathbb{R}^d)$ and therefore there is a α'-rough path $\mathbb{X}$ over X. However, is there a α-rough path over X? Note that we have restricted Definition 8.10 to the range $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, while here we are discussing the existence of $\mathbb{X}^2$: $[0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$ satisfying the Chen relation (8.21) and

$$|\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}$$

where now $\alpha > \frac{1}{2}$.

Lemma 8.14. *Let $\alpha \in \left(\frac{1}{2}, 1\right]$. For every $X \in \mathcal{C}^\alpha([0,T]; \mathbb{R}^d)$, there is a unique $\mathbb{X}^2$: $[0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$ satisfying the Chen relation (8.21) and such that $\mathbb{X}^2 \in C_2^{2\alpha}$. We have the explicit formula*

$$\mathbb{X}^2_{st} = \int_s^t \mathbb{X}^1_{su} \otimes \mathrm{d}X_u, \qquad \mathbb{X}^1_{st} = \delta X_{st}, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{8.26}$$

where the integral is in the Young sense. Moreover the map $\mathcal{C}^\alpha \ni X \mapsto \mathbb{X}^2 \in C_2^{2\alpha}$ is continuous (in particular, locally Lipschitz-continuous).

Proof. It is easy to check that $\mathbb{X}^2$ in (8.26) satisfies the Chen relation (8.18), thanks to the bi-linearity of the Young integral. Indeed, we can rewrite (8.26) as

$$\mathbb{X}^2_{st} = \int_s^t X_u \otimes \mathrm{d}X_u - X_s \otimes (X_t - X_s)\,, \tag{8.27}$$

hence for $s \leqslant u \leqslant t$ we have that

$$\begin{aligned}
(\delta\mathbb{X}^2)_{sut} &= -X_s \otimes (X_t - X_s) + X_s \otimes (X_u - X_s) + X_u \otimes (X_t - X_u) \\
&= -X_s \otimes (X_t - X_u) + X_u \otimes (X_t - X_u) \\
&= \delta X_{su} \otimes \delta X_{ut}\,.
\end{aligned}$$

We show now that $\mathbb{X}^2 \in C_2^{2\alpha}$. We recall that the Young integral satisfies the following key estimate, for $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$ with $\alpha + \beta > 1$:

$$\left| \int_s^t f \, \mathrm{d}g - f_s (g_t - g_s) \right| \leqslant c_{\alpha+\beta} |t-s|^{\alpha+\beta} .$$

Choosing $f = X^i$ and $g = X^j$ shows that $\mathbb{X}^2$, given by (8.27), is $O(|t-s|^{2\alpha})$. Finally, we prove the continuity of $\mathcal{C}^\alpha \ni X \mapsto \mathbb{X}^2 \in C_2^{2\alpha}$. Given $X, \bar{X} \in \mathcal{C}^\alpha$ and the respective $\mathbb{X}^2, \bar{\mathbb{X}}^2 \in C_2^{2\alpha}$, we have

$$\mathbb{X}^2_{st} - \bar{\mathbb{X}}^2_{st} = \int_s^t (\mathbb{X}^1_{su} - \bar{\mathbb{X}}^1_{su}) \otimes \mathrm{d}X_u + \int_s^t \bar{\mathbb{X}}^1_{su} \otimes \mathrm{d}(X - \bar{X})_u,$$

with all integrals in the Young sense. Then by the Sewing Lemma

$$\|\mathbb{X}^2 - \bar{\mathbb{X}}^2\|_{2\alpha} \leqslant K_{2\alpha}(\|\delta X\|_\alpha + \|\delta \bar{X}\|_\alpha) \|\delta X - \delta \bar{X}\|_\alpha.$$

The proof is complete. □

Therefore, we could extend Definition 8.10 to α-rough paths for $\alpha \in \left(\frac{1}{3}, 1\right]$. For $\alpha \in \left(\frac{1}{2}, 1\right]$ and $X \in \mathcal{C}^\alpha([0,T]; R^d)$ there is a unique α-rough path over X, which we call the *canonical rough path* over X.

While for $\alpha > \frac{1}{2}$ there is a unique rough path lying above a given path $X \in \mathcal{C}^\alpha$, for $\alpha \leqslant \frac{1}{2}$ there are infinitely many of them, that can be characterized explicitly.

LEMMA 8.15. *Let $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ be a α-rough path in $\mathbb{R}^d$, with $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$. Then $\bar{\mathbb{X}} = (\mathbb{X}^1, \bar{\mathbb{X}}^2)$ is a α-rough path if and only if for some $f \in \mathcal{C}^{2\alpha}([0,T]; \mathbb{R}^d \otimes \mathbb{R}^d)$ one has $\bar{\mathbb{X}}^2 = \mathbb{X}^2 + \delta f$, that is*

$$\bar{\mathbb{X}}^2_{st} = \mathbb{X}^2_{st} + f_t - f_s, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Proof. By assumption $\mathbb{X}^2$ and $\bar{\mathbb{X}}^2$ satisfy the Chen relation (8.21). If $\bar{\mathbb{X}}^2 = \mathbb{X}^2 + \delta f$ then $\mathbb{X}^2 \in C_2^{2\alpha}$ if and only if $\delta \mathbb{X}^2 = \delta \bar{\mathbb{X}}^2$ and $\bar{\mathbb{X}}^2 \in C_2^{2\alpha}$. Therefore, if $\mathbb{X}$ is a α-rough path then so is $\bar{\mathbb{X}}$.

Viceversa, if $\bar{\mathbb{X}}$ is a α-rough path, then $\delta \mathbb{X}^2 = \delta \bar{\mathbb{X}}^2$ because both $\mathbb{X}$ and $\bar{\mathbb{X}}$ satisfy the Chen relation (8.21) with the same $\mathbb{X}^1$, hence $\bar{\mathbb{X}}^2 = \mathbb{X}^2 + \delta f$ for some f. Since both $\mathbb{X}^2, \bar{\mathbb{X}}^2$ belong to $C_2^{2\alpha}$, then also $\delta f \in C_2^{2\alpha}$, which is the same as $f \in \mathcal{C}^{2\alpha}$. □

Remark 8.16. We mainly work with α-Hölder rough pats for $\alpha \in (\frac{1}{3}, \frac{1}{2})$, excluding the boundary case $\alpha = \frac{1}{2}$ for technical reasons. Let us stress that, by doing so, *we are not throwing away any rough paths, but only giving up a tiny amount of regularity*, because any $\frac{1}{2}$-rough path is a α-rough path, for any $\alpha < \frac{1}{2}$.

To summarize, the situation is the following:

1. For $\alpha \in \left(\frac{1}{2}, 1\right]$ and $X \in \mathcal{C}^\alpha([0,T]; \mathbb{R}^d)$ there is a unique α-rough path over X
2. For $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and $X \in \mathcal{C}^\alpha([0,T]; \mathbb{R}^d)$, there are infinitely many α-rough paths over X

3. For $\alpha = \frac{1}{2}$, either there is no α-rough path over X, or there are infinitely many of them.

In the range $\alpha \in \left(\frac{1}{2}, 1\right]$, the unique α-rough path $\mathbb{X}$ above X can be called the *canonical rough path* over X. We let $\mathcal{R}_{1,d}$ be the set of all canonical rough paths over paths $X \in C^1$ (see Lemma 8.14).

8.8. Lack of continuity

We have seen in Lemma 8.14 that, for $\alpha > \frac{1}{2}$, the map $\mathcal{C}^\alpha \ni X \mapsto \mathbb{X}^2 \in C_2^{2\alpha}$ is continuous. It is a crucial fact that this continuity property can *not* be extended to $\alpha \leqslant \frac{1}{2}$, as shown by the next example.

For $n \in \mathbb{N}$ consider the smooth paths $X_n^1, X_n^2 \colon [0,1] \to \mathbb{R}$

$$X_n^1(t) := \frac{1}{\sqrt{n}} \cos(nt), \qquad X_n^2(t) := \frac{1}{\sqrt{n}} \sin(nt).$$

We have already shown in Lemma 8.7 that $X_n^1 \to 0$ and $X_n^2 \to 0$ in $\mathcal{C}^\alpha$, for all $\alpha \in (0, \frac{1}{2})$. More precisely, we have shown that $X_n^1 \rightsquigarrow_{\frac{1}{2}} 0$ and $X_n^2 \rightsquigarrow_{\frac{1}{2}} 0$, by showing that $\|\delta X_n^1\|_{\frac{1}{2}} \leq 2$, $\|\delta X_n^2\|_{\frac{1}{2}} \leq 2$ for all $n \in \mathbb{N}$ and, obviously, $\|X_n^1\|_\infty \to 0$, $\|X_n^2\|_\infty \to 0$. Next we set

$$I_n^{ij}(t) := \int_0^t X_n^i(u)\, \mathrm{d}X_n^j(u), \qquad \text{for } i, j \in \{1, 2\},$$

and correspondingly

$$\begin{aligned} (\mathbb{X}_n^2)_{st}^{ij} &= \\ &= \int_s^t (X_n^i(u) - X_n^i(s))\, \mathrm{d}X_n^j(u) = I_n^{ij}(t) - I_n^{ij}(s) - X_n^i(s)(X_n^j(t) - X_n^j(s)). \end{aligned} \tag{8.28}$$

It is not difficult to show that $(\mathbb{X}_n^2)^{ij} \to (\mathbb{X}^2)^{ij}$ in C_2^θ, for any $\theta \in (0,1)$, where we define

$$(\mathbb{X}^2)_{st}^{ij} = \begin{pmatrix} 0 & \dfrac{t-s}{2} \\ -\dfrac{t-s}{2} & 0 \end{pmatrix} = \begin{cases} \frac{t-s}{2} & \text{if } i=1, j=2 \\ -\frac{t-s}{2} & \text{if } i=2, j=1 \\ 0 & \text{if } i=j \end{cases}. \tag{8.29}$$

As a consequence, for any $\alpha \in (\frac{1}{3}, \frac{1}{2})$, we have $\mathbb{X}_n^1 \to 0$ in $\mathcal{C}^\alpha$ and $\mathbb{X}_n^2 \to \mathbb{X}^2$ in $C_2^{2\alpha}$, that is *the canonical rough path* $(\mathbb{X}_n^1, \mathbb{X}_n^2)$ *converge in* $\mathcal{R}_{\alpha,d}$ *to the rough path* $(0, \mathbb{X}^2)$.

Let us prove that $(\mathbb{X}_n^2)^{ij} \to (\mathbb{X}^2)^{ij}$ in C_2^θ, for any $\theta \in (0,1)$. We have already shown the pointwise (actually uniform) convergence $I_n^{12}(t) \to \frac{1}{2} t$. With similar arguments, one shows the uniform convergence $I_n^{ij} \to I^{ij}$ defined by

$$I^{ij}(t) = \begin{pmatrix} 0 & \dfrac{t}{2} \\ -\dfrac{t}{2} & 0 \end{pmatrix} = \begin{cases} \frac{t}{2} & \text{if } i=1, j=2 \\ -\frac{t}{2} & \text{if } i=2, j=1 \\ 0 & \text{if } i=j \end{cases}.$$

It follows by (8.28) that we have the uniform convergence $(\mathbb{X}_n^2)_{st}^{ij} \to I^{ij}(t) - I^{ij}(s) = (\mathbb{X}^2)_{st}^{ij}$. To prove convergence in C_2^θ, it suffices to show a uniform "Lipschitz-like" bound $|(\mathbb{X}_n^2)_{st}^{ij}| \leq 2\,|t-s|$, which is easy:

$$\begin{aligned} |(\mathbb{X}_n^2)_{st}^{ij}| &\leq \int_s^t |X_n^i(u) - X_n^i(s)|\,|(X_n^j)'(u)|\,\mathrm{d}u \\ &\leq 2\,\|X_n^i\|_\infty\,\|(X_n^j)'\|_\infty |t-s| \\ &= 2\frac{1}{\sqrt{n}}\frac{n}{\sqrt{n}}|t-s| \\ &= 2\,|t-s|\,. \end{aligned}$$

8.9. Proof of Proposition 8.5

Given continuous functions $X, Y\colon [0,T] \to \mathbb{R}$, let us define $R^1, R^2 \in C_2$

$$R^1(X,Y)_{st} := -Y_s\,\delta X_{st}\,, \qquad R^2(X,Y)_{st} := X_t\,\delta Y_{st}\,, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{8.30}$$

and note that

$$R_{st}^2 = R_{st}^1 + \delta(XY)_{st}\,.$$

Recalling Remark 8.6, it is easy to check that R^1 and R^2 satisfy

$$\delta R^1(X,Y)_{sut} = \delta R^2(X,Y)_{sut} = \delta Y_{su}\,\delta X_{ut}\,. \tag{8.31}$$

However, neither R^1 nor R^2 are in $C_2^{\alpha+\beta}$ in general, because we can only estimate

$$\|R^1\|_\alpha \leqslant \|Y\|_\infty\,\|\delta X\|_\alpha\,, \qquad \|R^2\|_\beta \leqslant \|X\|_\infty\,\|\delta Y\|_\beta\,. \tag{8.32}$$

We are going to show that, by combining R^1 and R^2 in a suitable way, one can build R which satisfies both (8.8) and (8.9). This yields the existence of an integral.

We start with a technical approximation lemma.

Lemma 8.17. *Given $f \in \mathcal{C}^\alpha$, there is a sequence $(\tilde{f}_n)_n \subset C^\infty$ such that*

$$f(x) = f(0) + \sum_{n\geq 0} \tilde{f}_n(x)\,, \qquad \forall x \in [0,T]\,. \tag{8.33}$$

One can choose $\tilde{f}_n$ so that for every $n \geq 0$

$$\|\tilde{f}_n\|_\infty \leqslant C\,\|\delta f\|_\alpha\,2^{-n\alpha}\,, \qquad \|\tilde{f}_n'\|_\infty \leqslant C\,\|\delta f\|_\alpha\,2^{n(1-\alpha)}\,, \tag{8.34}$$

where $C \in (0,\infty)$ depends only on T (e.g. one can take $C = 2(T^\alpha + 1)$).

Proof. We may assume without loss of generality that $f(x) = 0$ (it suffices to redefine $f(x)$ as $f(x) - f(0)$, which does not change $\|\delta f\|_\alpha$.)

We extend $f\colon \mathbb{R} \to \mathbb{R}$ (e.g. with $f(x) := f(0)$ for $x \leqslant 0$ and $f(x) := f(T)$ for $x \geq T$) so that $\|f\|_\alpha$ is not changed. Then we fix a probability density $\phi\colon [-1,1] \to [0,\infty)$ with $\phi \in C^1$ and for $n \geq 0$ we define the rescaled density

$$\phi_n(x) := 2^n \phi(2^n x)\,.$$

Next, for $n \geq 0$, we set $f_n(x) := (f * \phi_n)(x)$, that is

$$\begin{aligned} f_n(x) &:= \int_{\mathbb{R}} f(z)\, \phi_n(x-z)\, \mathrm{d}z = \int_{\mathbb{R}} f(x-z)\, \phi_n(z)\, \mathrm{d}z \\ &= \int_{\mathbb{R}} f(x - \tfrac{z}{2^n}) \phi(z)\, \mathrm{d}z\,. \end{aligned} \tag{8.35}$$

It is easy to check that $\|f_n - f\|_\infty \to 0$. Next we define

$$\tilde{f}_0(x) := f_0(x)\,, \qquad \text{for } k \geq 1\text{:} \quad \tilde{f}_k(x) := f_k(x) - f_{k-1}(x)\,.$$

Note that $\sum_{k=0}^n \tilde{f}_k = f_n$, hence relation (8.33) is proved (we recall that $f(0) = 0$).

We now prove the first relation in (8.34). Since $f(0) = 0$, for all $x \in [0, T]$ we can write

$$\begin{aligned} |\tilde{f}_0(x)| = |f_0(x)| &\leqslant \int_{\mathbb{R}} |f(x-z)|\, \phi(z)\, \mathrm{d}z = \int_{\mathbb{R}} |f(x-z) - f(0)| \phi(z)\, \mathrm{d}z \\ &\leqslant \|\delta f\|_\alpha \int_{\mathbb{R}} |x - z|^\alpha\, \phi(z)\, \mathrm{d}z \leqslant (T^\alpha + 1)\, \|\delta f\|_\alpha\,, \end{aligned}$$

where for the last inequality we have used $(x+y)^\alpha \leqslant x^\alpha + y^\alpha$ (for $\alpha < 1$ and $x, y \geq 0$), $x \leqslant T$ and $\int_{\mathbb{R}} |z|^\alpha\, \phi(z)\, \mathrm{d}z \leqslant \int_{[-1,1]} \phi(z)\, \mathrm{d}z = 1$, because ϕ is a density supported on $[-1, 1]$. For $k \geq 1$ we estimate

$$\begin{aligned} |\tilde{f}_k(x)| &= |f_k(x) - f_{k-1}(x)| \\ &\leqslant \int_{\mathbb{R}} |f(x - \tfrac{z}{2^k}) - f(x - \tfrac{z}{2^{k-1}})|\, \phi(z)\, \mathrm{d}z \\ &\leqslant 2^{-k\alpha}\, \|\delta f\|_\alpha \end{aligned}$$

again because $\int_{\mathbb{R}} |z|^\alpha \phi(z)\, \mathrm{d}z \leqslant 1$. We have proved the first relation in (8.34).

We finally prove the second relation in (8.34). Note that

$$f_n'(x) = \int_{\mathbb{R}} f(z)\, \phi_n'(x-z)\, \mathrm{d}z = 2^n \int_{\mathbb{R}} f(x - \tfrac{z}{2^n})\, \phi'(z)\, \mathrm{d}z\,,$$

which has the same form as $f_n(x)$, see the last integral in (8.35), just with an extra multiplicative factor 2^n and with ϕ replaced by ϕ'. Arguing as before, we obtain

$$|\tilde{f}_0'(x)| = |f_0'(x)| \leqslant (T^\alpha + 1) \left(\int_{[-1,1]} |\phi'(z)|\, \mathrm{d}z \right) \|\delta f\|_\alpha\,,$$

$$|\tilde{f}_k'(x)| = |f_k'(x) - f_{k-1}'(x)| \leqslant 2^{k(1-\alpha)} \left(\int_{[-1,1]} |\phi'(z)|\, \mathrm{d}z \right) \|\delta f\|_\alpha,$$

for $k \geq 1$. We can choose ϕ to be symmetric, decreasing on $[0, 1]$, with $\phi(0) = 1$ and $\phi(1) = 0$, so that

$$\int_{[-1,1]} |\phi'(z)|\, \mathrm{d}z = 2 \int_0^1 (-\phi'(z))\, \mathrm{d}z = 2\, (\phi(0) - \phi(1)) = 2\,,$$

and this completes the proof. □

Proof of Proposition 8.5. The existence of an integral is an immediate consequence of Remark 8.6, because if we define $R_{st} := J_\prec(X, Y)_{st}$, then both relations (8.8) and (8.9) are satisfied.

It remains to build $J_{\prec}$. Let us write, applying Lemma 8.17,

$$X(x) = X(0) + \sum_{m\geq 0} \tilde{X}_n(x)\,, \qquad Y(x) = Y(0) + \sum_{n\geq 0} \tilde{Y}_m(x)\,.$$

Recalling (8.30), we define

$$J_{\prec}(X,Y) := \sum_{0\leqslant m\leqslant n} R^1(\tilde{X}_n, \tilde{Y}_m) + \sum_{0\leqslant n<m} R^2(\tilde{X}_n, \tilde{Y}_m)\,. \tag{8.36}$$

We show below that the series converge uniformly. Note that $\sum_{n\geq 0}\tilde{X}_n(x) = X(x) - X(0)$, hence $\sum_{n\geq 0}\delta\tilde{X}_n = \delta(X - X(0)) = \delta X$, and similarly for Y. Applying (8.31), we get

$$\begin{aligned}\delta J_{\prec}(X,Y)_{sut} &= \sum_{0\leqslant m\leqslant n} (\delta\tilde{Y}_n)_{su}\,(\delta\tilde{X}_m)_{ut} + \sum_{0\leqslant n<m} (\delta\tilde{Y}_n)_{su}\,(\delta\tilde{X}_m)_{ut}\\ &= \left(\sum_{n\geq 0}(\delta\tilde{Y}_n)_{su}\right)\left(\sum_{m\geq 0}(\delta\tilde{X}_m)_{ut}\right) = \delta Y_{su}\,\delta X_{ut}\,,\end{aligned}$$

which proves (8.7). We now prove (8.6). Note that, by (8.34),

$$|(\delta\tilde{X}_n)_{st}| \leqslant \|\tilde{X}_n'\|_\infty\,|t-s| \leqslant C\,\|\delta X\|_\alpha\,2^{-\alpha n}(2^n\,|t-s|)\,,$$

but at the same time, always by (8.34),

$$|(\delta\tilde{X}_n)_{st}| \leqslant |\tilde{X}_n(s)| + |\tilde{X}_n(t)| \leqslant 2\,\|\tilde{X}_n\|_\infty \leqslant 2C\,\|\delta X\|_\alpha\,2^{-\alpha n}\,.$$

Altogether, using the notation $x\wedge y := \min\{x,y\}$,

$$|(\delta\tilde{X}_n)_{st}| \leqslant 2C\,\|\delta X\|_\alpha\,2^{-\alpha n}\,(2^n|t-s|\wedge 1)\,.$$

Similarly

$$|(\delta\tilde{Y}_m)_{st}| \leqslant 2C\,\|\delta Y\|_\beta\,2^{-\beta m}\,(2^m|t-s|\wedge 1)\,.$$

Recalling (8.30) and applying again (8.34), we get

$$\begin{aligned}|R^1(\tilde{X}_n,\tilde{Y}_m)_{st}| &\leqslant \|\tilde{Y}_m\|_\infty\,|(\delta\tilde{X}_n)_{st}|\\ &\leqslant 2C^2\,\|\delta X\|_\alpha\,\|\delta Y\|_\beta\,2^{-\alpha n}\,2^{-\beta m}(2^n|t-s|\wedge 1)\,.\end{aligned}$$

and similarly

$$\begin{aligned}|R^2(\tilde{X}_n,\tilde{Y}_m)_{st}| &\leqslant \|\tilde{X}_n\|_\infty\,|(\delta\tilde{Y}_m)_{st}|\\ &\leqslant 2C^2\,\|\delta X\|_\alpha\,\|\delta Y\|_\beta\,2^{-\alpha n}\,2^{-\beta m}\,(2^m|t-s|\wedge 1)\end{aligned}$$

These relations show that the series in (8.36) converge indeed uniformly. We now plug these estimates into (8.36), getting

$$\begin{aligned}|J_{\prec}(X,Y)_{st}| \leqslant\; 2C^2\,\|\delta X\|_\alpha\,\|\delta Y\|_\beta\Bigg(&\sum_{0\leqslant m\leqslant n} 2^{-\alpha n}\,2^{-\beta m}\,(2^m|t-s|\wedge 1)\\ &+ \sum_{0\leqslant n<m} 2^{-\alpha n}\,2^{-\beta m}\,(2^n|t-s|\wedge 1)\Bigg).\end{aligned} \tag{8.37}$$

Let us set for convenience

$$\bar{k} = \bar{k}_{st} := \log_2 \frac{1}{|t-s|},$$

so that $2^m|t-s| \leqslant 2$ if and only if $m \leqslant \bar{k}$. Since $\sum_{n=m}^{\infty} 2^{-\alpha n} \leqslant \frac{1}{1-2^{-\alpha}} 2^{-\alpha m}$, the first sum in (8.37) can be bounded as follows (neglecting the prefactor $(1-2^{-\alpha})^{-1}$):

$$\begin{aligned}\sum_{m\geq 0} 2^{-(\alpha+\beta)m}(2^m|t-s| \wedge 1) &\leqslant |t-s| \sum_{0\leqslant m<\bar{k}} 2^{(1-\alpha-\beta)m} + \sum_{m\geq \bar{k}} 2^{-(\alpha+\beta)m} \\ &\leqslant |t-s| \frac{2^{(1-\alpha-\beta)\bar{k}}}{2^{1-\alpha-\beta}-1} + \frac{2^{-(\alpha+\beta)\bar{k}}}{1-2^{-(\alpha+\beta)}} \\ &\leqslant \left\{ \frac{1}{2^{1-\alpha-\beta}-1} + \frac{1}{1-2^{-(\alpha+\beta)}} \right\} |t-s|^{\alpha+\beta}.\end{aligned}$$

The same estimates apply to the second sum in (8.37), hence (8.6) is proved. □

Remark 8.18. In the previous proof, if $\alpha+\beta=1$, then we have

$$\sum_{0\leqslant m<\bar{k}} \underbrace{2^{(1-\alpha-\beta)m}}_{=1} = \bar{k} = \log_2 \frac{1}{|t-s|}$$

and therefore we obtain, instead of (8.6), that

$$|J_{\prec}(f,g)|_{st} \lesssim |t-s| \log \frac{1}{|t-s|}, \qquad 0 \leqslant s < t \leqslant T.$$