Chapter 9
Geometric rough paths

9.1. Geometric rough paths

We recall that the set of smooth paths C^1 is not dense in $\mathcal{C}^\alpha$, but its closure is quite large, because it contains $\mathcal{C}^{\alpha'}$ for all $\alpha' > \alpha$. The situation is different for rough paths: the set $\mathcal{R}_{1,d}$ of canonical rough paths over smooth paths is again not dense in $\mathcal{R}_{\alpha,d}$, but its closure is a significantly smaller set, that we now describe.

Definition 9.1. *The closure of $\mathcal{R}_{1,d}$ in $\mathcal{R}_{\alpha,d}$ for $\alpha \in \left]\frac{1}{3}, 1\right]$ is denoted by $\mathcal{R}^{\mathrm{g}}_{\alpha,d}$ and its elements are called* geometric rough paths.

For smooth paths $f, g \in C^1$, the integration by parts formula holds:

$$\int_s^t f(u)\,\mathrm{d}g(u) = f(t)g(t) - f(s)g(s) - \int_s^t g(u)\,\mathrm{d}f(u)\,.$$

It follows that

$$\int_s^t (f(u) - f(s))\,\mathrm{d}g(u) + \int_s^t (g(u) - g(s))\,\mathrm{d}f(u) = (f(t) - f(s))(g(t) - g(s))\,.$$

We have seen in Proposition 7.7 that the same formula holds if $(f, g) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ with $\alpha + \beta > 1$ and the integral is in the Young sense.

Given a smooth path $X \in C^1$, define $\mathbb{X}^2$ by (8.26) as an ordinary integral (i.e. $(\mathbb{X}^1, \mathbb{X}^2)$ is the canonical rough path over X). The previous relation for $f = X_i$ and $g = X_j$ shows that

$$(\mathbb{X}^2_{st})^{ij} + (\mathbb{X}^2_{st})^{ji} = (\mathbb{X}^1_{st})^i(\mathbb{X}^1_{st})^j\,. \tag{9.1}$$

This relation is called the *shuffle relation*: for $i = j$ it identifies $\mathbb{X}^2_{ii}$ in terms of X_i:

$$(\mathbb{X}^2_{st})^{ii} = \frac{1}{2}((\mathbb{X}^1_{st})^i)^2\,, \tag{9.2}$$

while for $i \neq j$ it expresses $(\mathbb{X}^2)^{ij}$ in terms of $(\mathbb{X}^1)^i$, $(\mathbb{X}^1)^j$, $(\mathbb{X}^2)^{ji}$. Denoting by $\mathrm{Sym}(\mathbb{X}^2) := \frac{1}{2}(\mathbb{X}^2 + (\mathbb{X}^2)^T)$ the symmetric part of $\mathbb{X}^2$, we can rewrite the shuffle relation more compactly as follows:

$$\mathrm{Sym}(\mathbb{X}^2) = \frac{1}{2}\,\mathbb{X}^1 \otimes \mathbb{X}^1\,. \tag{9.3}$$

Definition 9.2. *Rough paths in $\mathcal{R}_{\alpha,d}$ that satisfy the shuffle relation (9.1)-(9.3) are called* weakly geometric *and denoted by $\mathcal{R}^{\mathrm{wg}}_{\alpha,d}$.*

Exercise 9.1. For $\alpha > \frac{1}{2}$ we have $\mathcal{R}_{\alpha,d} = \mathcal{R}^{\mathrm{wg}}_{\alpha,d}$ (every rough path is weakly geometric).

We can now show that the closure of $\mathcal{R}_{1,d}$ in $\mathcal{R}_{\alpha,d}$ is included in $\mathcal{R}^{\mathrm{wg}}_{\alpha,d}$.

LEMMA 9.3. *Geometric rough paths are weakly geometric:* $\mathcal{R}^{\mathrm{g}}_{\alpha,d} \subset \mathcal{R}^{\mathrm{wg}}_{\alpha,d}$ *for any* $\alpha \in (\frac{1}{3}, 1)$*, with a strict inclusion.*

Proof. Canonical rough paths $(\mathbb{X}^1, \mathbb{X}^2) \in \mathcal{R}_{1,d}$ over smooth paths satisfy the shuffle relation (9.1)-(9.3). Geometric rough paths are by definition limits in $\mathcal{R}_{\alpha,d}$ of smooth paths in $\mathcal{R}_{1,d}$. Since convergence in $\mathcal{R}_{\alpha,d}$ implies pointwise convergence, geometric rough paths satisfy the shuffle relation too. This shows that $\mathcal{R}^{\mathrm{g}}_{\alpha,d} \subset \mathcal{R}^{\mathrm{wg}}_{\alpha,d}$.

To prove that the inclusion $\mathcal{R}^{\mathrm{g}}_{\alpha,d} \subset \mathcal{R}^{\mathrm{wg}}_{\alpha,d}$ is strict, it suffices to consider a weakly geometric rough path $(\mathbb{X}^1, \mathbb{X}^2) \in \mathcal{R}^{\mathrm{wg}}_{\alpha,d}$ which lies above a path $X \in \mathcal{C}^\alpha$ which is not in the closure of C^1. Such a path is not geometric (recall that $(\mathbb{X}^1_n, \mathbb{X}^2_n) \to (\mathbb{X}^1, \mathbb{X}^2)$ in $\mathcal{R}_{\alpha,d}$ implies $\mathbb{X}^1_n \to \mathbb{X}^1$ in C^α_2).

To prove the existence of such a rough path, in the one-dimensional case $d = 1$ it is enough to consider the one provided by (8.19), which is by construction weakly geometric, since the shuffle relation reduces to $\mathbb{X}^2_{st} := \frac{1}{2}(\mathbb{X}^1_{st})^2$. □

Although the inclusion $\mathcal{R}^{\mathrm{g}}_{\alpha,d} \subset \mathcal{R}^{\mathrm{wg}}_{\alpha,d}$ is strict, what is left out turns out to be not so large. More precisely, recalling that $\mathcal{R}^{\mathrm{g}}_{\alpha,d}$ is the closure of $\mathcal{R}_{1,d}$ in $\mathcal{R}_{\alpha,d}$, we have a result which is similar to what happens for Hölder spaces, with the important difference that the whole space $\mathcal{R}_{\alpha,d}$ is replaced by $\mathcal{R}^{\mathrm{wg}}_{\alpha,d}$. The proof is non-trivial and we omit it.

PROPOSITION 9.4. *For any* $\frac{1}{3} < \alpha' < \alpha < 1$ *one has* $\mathcal{R}^{\mathrm{wg}}_{\alpha,d} \subseteq \mathcal{R}^{\mathrm{g}}_{\alpha',d}$*. This means that for any* $\mathbb{X} \in \mathcal{R}^{\mathrm{wg}}_{\alpha,d}$ *there is a sequence* $\mathbb{X}_n \in \mathcal{R}_{1,d}$ *such that* $\mathbb{X}_n \to \mathbb{X}$ *in* $\mathcal{R}_{\alpha',d}$*.*

We stress that the notion of "weakly geometric" rough path depends only on the function $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$, but the notion of "geometric" rough path depends also on the chosen space $\mathcal{R}_{\alpha,d}$. Given a weakly geometric rough path $\mathbb{X} \in \mathcal{R}_{\alpha,d}$, even though $\mathbb{X}$ may fail to be geometric in $\mathcal{R}_{\alpha,d}$, it is certainly geometric in $\mathcal{R}_{\alpha',d}$ for all $\alpha' < \alpha$. In this sense, *every weakly geometric rough path is a geometric rough path, of a possibly slightly lower regularity.*

Finally we note the following

PROPOSITION 9.5. *Let* $\alpha \in \left(\frac{1}{2}, 1\right)$ *and* $X \in \mathcal{C}^\alpha([0,T]; \mathbb{R}^d)$*. The canonical* α*-rough path constructed in Lemma 8.14 is geometric.*

Proof. We recall that by the Chen relation

$$\delta(\mathbb{X}^2)^{ij}_{sut} = \delta X^i_{su}\, \delta X^j_{ut}, \qquad \delta(\mathbb{X}^2)^{ji}_{sut} = \delta X^j_{su}\, \delta X^i_{ut},$$

so that

$$\delta[(\mathbb{X}^2)^{ij} + (\mathbb{X}^2)^{ji}]_{sut} = \delta X^i_{su}\, \delta X^j_{ut} + \delta X^j_{su}\, \delta X^i_{ut}.$$

On the other hand by a simple computation

$$\delta[\delta X^i\, \delta X^j]_{sut} = \delta X^i_{su}\, \delta X^j_{ut} + \delta X^j_{su}\, \delta X^i_{ut}.$$

Therefore $(\mathbb{X}^2)^{ij} + (\mathbb{X}^2)^{ji} - \delta X^i \, \delta X^j = \delta f$ for some $f \in C_1$ such that $\delta f \in C_2^{2\alpha}$. Since $2\alpha > 1$, we obtain that $\delta f \equiv 0$. □

Note that Proposition 9.5 can be seen as a consequence of the integration by parts formula satisfied by the Young integral, see Proposition 7.7.

9.2. THE STRATONOVICH ROUGH PATH

Let $(B_t)_{t \geqslant 0}$ be a d-dimensional Brownian motion. We have seen in Theorem 4.2 that the $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$, defined by

$$\mathbb{B}^1_{st} := \delta B_{st}, \qquad \mathbb{B}^2_{st} := \int_s^t \mathbb{B}^1_{sr} \otimes \mathrm{d}B_r, \qquad 0 \leqslant s \leqslant t \leqslant T,$$

with an Itô integral, defines a.s. a α-rough path for all $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, that we can call the *Itô rough path*. As in section 5.1 we modify now this definition and we set

$$\bar{\mathbb{B}}^1_{st} := \delta B_{st}, \qquad \bar{\mathbb{B}}^2_{st} := \int_s^t \bar{\mathbb{B}}^1_{sr} \otimes \circ \mathrm{d}B_r, \qquad 0 \leqslant s \leqslant t \leqslant T,$$

where $\circ$ denotes Stratonovich integration, namely

$$\bar{\mathbb{B}}^1_{st} := \delta B_{st}, \qquad \bar{\mathbb{B}}^2_{st} := \int_s^t (B_r - B_s) \otimes \mathrm{d}B_r + \frac{t-s}{2} I, \qquad 0 \leqslant s \leqslant t \leqslant T,$$

with I the identity matrix in $\mathbb{R}^d \otimes \mathbb{R}^d$. By Lemma 8.15, $\bar{\mathbb{B}} = (\bar{\mathbb{B}}^1, \bar{\mathbb{B}}^2)$ defines a α-rough path for all $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, that we call the *Stratonovich rough path*. Now we show that $\bar{\mathbb{B}}$ is geometric. We recall that the integration by parts formula reads in this case

$$B^i_t \, B^j_t - B^i_s \, B^j_s = \int_s^t B^i_r \circ \mathrm{d}B^j_r + \int_s^t B^j_r \circ \mathrm{d}B^i_r, \qquad 0 \leqslant s \leqslant t.$$

Moreover

$$\int_s^t B^i_s \circ \mathrm{d}B^j_r = B^i_s \, (B^j_t - B^j_s).$$

Therefore

$$\begin{aligned} (\bar{\mathbb{B}}^2_{st})^{ij} + (\bar{\mathbb{B}}^2_{st})^{ji} &= B^i_t \, B^j_t - B^i_s \, B^j_s - B^i_s \, (B^j_t - B^j_s) - B^j_s \, (B^i_t - B^i_s) \\ &= (B^i_t - B^i_s)(B^j_t - B^j_s) = [\bar{\mathbb{B}}^1_{st} \otimes \bar{\mathbb{B}}^1_{st}]^{ij}. \end{aligned}$$

As in the remark following Proposition 9.5, also in the case of the Stratonovich rough path an integration by parts formula is at the heart of the geometric property.

On the other hand, the Itô rough path is *not* geometric, since the integration by parts formula with Itô integrals reads for $i = j$

$$(B^i_t)^2 - (B^i_s)^2 = 2 \int_s^t B^i_r \, \mathrm{d}B^i_r + (t - s), \qquad 0 \leqslant s \leqslant t,$$

and moreover we have

$$\int_s^t B^i_s \, \mathrm{d}B^i_r = B^i_s \, (B^i_t - B^i_s).$$

Therefore by the definition of $\mathbb{B}^2_{st}$

$$\begin{aligned} 2(\mathbb{B}^2_{st})^{ii} &= (B^i_t)^2 - (B^i_s)^2 - 2B^i_s(B^i_t - B^i_s) - (t-s) \\ &= (B^i_t - B^i_s)^2 - (t-s) \\ &= [\mathbb{B}^1_{st} \otimes \mathbb{B}^1_{st}]^{ii} - (t-s) \\ &\neq [\mathbb{B}^1_{st} \otimes \mathbb{B}^1_{st}]^{ii}. \end{aligned}$$

Note that for $i \neq j$ we do obtain $(\mathbb{B}^2_{st})^{ij} + (\mathbb{B}^2_{st})^{ji} = [\mathbb{B}^1_{st} \otimes \mathbb{B}^1_{st}]^{ij}$.

9.3. Non-geometric rough paths

We next consider generic rough paths. These cannot be approximated by canonical rough paths over smooth paths. However we have

Lemma 9.6. *Given an arbitrary rough path $(\mathbb{X}^1, \mathbb{X}^2) \in \mathcal{R}_{\alpha,d}$ lying above X, there is always a weakly geometric rough path $(\mathbb{X}^1, \tilde{\mathbb{X}}^2) \in \mathcal{R}^{\mathrm{wg}}_{\alpha,d}$ lying above the same path X.*

Proof. It suffice to define $\tilde{\mathbb{X}}^2_{ij} := \mathbb{X}^2_{ij}$ for all $i > j$ and use the shuffle relation to define the remaining entries of $\tilde{\mathbb{X}}^2$, i.e. $\tilde{\mathbb{X}}^2_{ii} := \frac{1}{2}(\mathbb{X}^1_i)^2$ and $\tilde{\mathbb{X}}^2_{ij} := \mathbb{X}^1_i \, \mathbb{X}^1_j - \mathbb{X}^2_{ji}$ for all $i < j$. In this way $(\mathbb{X}^1, \tilde{\mathbb{X}}^2)$ satisfies the shuffle relation by construction and it is easy to check that $\tilde{\mathbb{X}}^2 \in C_2^{2\alpha}$.

It remains to prove that the Chen relation (8.21) holds for $(\mathbb{X}^1, \tilde{\mathbb{X}}^2)$, that is

$$\delta\tilde{\mathbb{X}}^2_{ij}(s,u,t) = \mathbb{X}^1_i(s,u)\, \mathbb{X}^1_j(u,t).$$

If $i > j$ this holds because $\tilde{\mathbb{X}}^2_{ij} = \mathbb{X}^2_{ij}$, so we only need to consider $i = j$ and $i < j$. Note that if we define $A_{st} := \delta f_{st}\, \delta g_{st}$, for arbitrary $f, g\colon [a,b] \to \mathbb{R}$, we have

$$\begin{aligned} \delta A_{sut} &= \delta f_{st}\, \delta g_{st} - \delta f_{su}\, \delta g_{su} - \delta f_{ut}\, \delta g_{ut} \\ &= (\delta f_{su} + \delta f_{ut})\, \delta g_{st} - \delta f_{su}\, \delta g_{su} - \delta f_{ut}\, \delta g_{ut} \\ &= \delta f_{su}\, \delta g_{ut} + \delta g_{su} \delta f_{ut}. \end{aligned}$$

Applying this to $f = X^i$ and $g = X^j$ yields, for $i < j$,

$$\begin{aligned} \delta\tilde{\mathbb{X}}^2_{ij}(s,u,t) &= \delta(\mathbb{X}^1_i\, \mathbb{X}^1_j - \mathbb{X}^2_{ji})(s,u,t) \\ &= \mathbb{X}^1_i(s,u)\, \mathbb{X}^1_j(u,t) + \mathbb{X}^1_j(s,u)\, \mathbb{X}^1_i(u,t) - \mathbb{X}^1_j(s,u)\, \mathbb{X}^1_i(u,t) \\ &= \mathbb{X}^1_i(s,u)\, \mathbb{X}^1_j(u,t). \end{aligned}$$

Similarly, choosing $f = g = X_i$ gives $\delta\tilde{\mathbb{X}}^2_{ii}(s,u,t) = \mathbb{X}^1_i(s,u)\mathbb{X}^1_i(u,t)$. □

As a corollary, we obtain a useful approximation result.

Proposition 9.7. *For any rough path $(\mathbb{X}^1, \mathbb{X}^2) \in \mathcal{R}_{\alpha,d}$, there is a function $f \in \mathcal{C}^{2\alpha}([0,T]; \mathbb{R}^d \otimes \mathbb{R}^d)$ and a sequence of canonical rough paths over smooth paths $(\mathbb{X}^1_n, \mathbb{X}^2_n) \in \mathcal{R}_{1,d}$ such that*

$$(\mathbb{X}^1_n, \mathbb{X}^2_n + \delta f) \to (\mathbb{X}^1, \mathbb{X}^2) \qquad \text{in } \mathcal{R}_{\alpha',d}, \qquad \forall \alpha' \in \left(\frac{1}{3}, \alpha\right).$$

Proof. By Lemma 9.6 there is a weakly geometric rough path $(\mathbb{X}^1, \tilde{\mathbb{X}}^2)$ lying above the same path X. Then $\mathbb{X}^2 - \tilde{\mathbb{X}}^2 = \delta f$ for some $f \in \mathcal{C}^{2\alpha}([0,T]; \mathbb{R}^d \otimes \mathbb{R}^d)$, by Lemma 8.15. By Proposition 9.4, there is a sequence $(\mathbb{X}^1_n, \mathbb{X}^2_n) \in \mathcal{R}_{1,d}$ such that $(\mathbb{X}^1_n, \mathbb{X}^2_n) \to (\mathbb{X}^1, \tilde{\mathbb{X}}^2)$ in $\mathcal{R}_{\alpha',d}$, for any $\alpha' < \alpha$. It follows that $(\mathbb{X}^1_n, \mathbb{X}^2_n + \delta f) \to (\mathbb{X}^1, \tilde{\mathbb{X}}^2 + \delta f) = (\mathbb{X}^1, \mathbb{X}^2)$. □

9.4. PURE AREA ROUGH PATHS

Given $X \in \mathcal{C}^\alpha$, we have defined in Definition 3.2 the subset $\mathcal{R}_{\alpha,d}(X)$ of rough paths $(\mathbb{X}^1, \mathbb{X}^2) \in \mathcal{R}_{\alpha,d}$ lying above X, i.e. such that $\mathbb{X}^1 = \delta X$. The case of $\mathbb{X}^1 \equiv 0$ is particularly interesting:

DEFINITION 9.8. *The elements of $\mathcal{R}_{\alpha,d}(0)$, i.e. those of the form $\mathbb{X} = (0, \mathbb{X}^2)$, are called* pure area *rough paths.*

Pure area rough paths are very explicit. Let us denote by $(\mathbb{R}^{d\times d})^{\mathrm{a}}$ the subspace of $\mathbb{R}^{d\times d}$ given by antisymmetric matrices.

LEMMA 9.9. $\mathbb{X} = (0, \mathbb{X}^2)$ *is a pure area α-rough path if and only if $\mathbb{X}^2 = \delta f$, for some $f \in \mathcal{C}^{2\alpha}([0,T]; \mathbb{R}^{d\times d})$. Such rough path is weakly geometric if and only if $\mathbb{X}^2_{st} \in (\mathbb{R}^{d\times d})^{\mathrm{a}}$, i.e. $\mathbb{X}^2_{st}$ is an antisymmetric matrix, for all $s,t \in [0,T]^2_{\leqslant}$; equivalently, we can take $f \in \mathcal{C}^{2\alpha}([0,T]; (\mathbb{R}^{d\times d})^{\mathrm{a}})$.*

Proof. Since $(0,0)$ is a rough path, it follows by Lemma 8.15 that for all (pure area) rough paths $(0, \mathbb{X}^2)$ we have $\mathbb{X}^2 = \delta f$ for some $f \in \mathcal{C}^{2\alpha}$. We may assume that $f(0) = 0$ (just redefine $f(t)$ as $f(t) - f(0)$). Since $\mathbb{X}^1 = 0$, the shuffle relation (9.3) becomes $\mathrm{Sym}(\mathbb{X}^2) = 0$, i.e. $\mathbb{X}^2_{st}$ is an antisymmetric matrix. Then $f(t) = f(t) - f(0) = \mathbb{X}^2_{0t}$ is antisymmetric too. □

Note that the set $\mathcal{R}_{\alpha,d}(0)$ of pure area rough paths is a *vector space*, because the Chen relation (8.21) reduces to the linear relation $\delta\mathbb{X}^2 = 0$. Here is the link with general rough paths.

PROPOSITION 9.10. *The set $\mathcal{R}_{\alpha,d}(X)$ of rough paths laying above a given path X is an affine space, with associated vector space $\mathcal{R}_{\alpha,d}(0)$, the space of pure area rough paths.*

Proof. Given rough paths $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ and $\bar{\mathbb{X}} = (\mathbb{X}^1, \bar{\mathbb{X}}^2)$ lying above the same path X, their difference $\mathbb{X} - \bar{\mathbb{X}} = (0, \mathbb{X}^2 - \bar{\mathbb{X}}^2)$ is a pure area rough path, because it satisfies the Chen relation $\delta(\mathbb{X}^2 - \bar{\mathbb{X}}^2) = 0$ (since $\delta\mathbb{X}^2 = \mathbb{X}^1 \otimes \mathbb{X}^1 = \delta\bar{\mathbb{X}}^2$).

Alternatively, Lemma 8.15 yields $\mathbb{X}^2 - \bar{\mathbb{X}}^2 = \delta f$ for some $f \in \mathcal{C}^{2\alpha}$, hence $(0, \mathbb{X}^2 - \bar{\mathbb{X}}^2)$ is a pure area rough path by Lemma 9.9. □

We have seen in Section 8.8 how pure area rough paths can arise concretely as limits of canonical rough paths associated with smooth paths.

9.5. DOSS-SUSSMANN

In this section we suppose that $\sigma:\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$ is such that for all $i\in\{1,...,k\}$ the $d\times d$ matrix $((\sigma_2)^i_{j\ell})_{j\ell}$ is symmetric, namely by (3.5) we have the *Frobenius condition*

$$\sum_{a=1}^{k}\sigma^a_\ell(y)\,\partial_a\sigma^i_j(y)=\sum_{a=1}^{k}\sigma^a_j(y)\,\partial\sigma^i_\ell(y),\qquad\forall y\in\mathbb{R}^k, i\in\{1,...,k\}, j,\ell\in\{1,...,d\}.\quad(9.4)$$

If we introduce the vector fields $(X_j)_{j=1,\ldots,d}$ on $\mathbb{R}^k$:

$$X_j\,f:=\sum_{a=1}^{k}\sigma^a_j\,\partial_a f,\qquad f\in C^\infty(\mathbb{R}^k),$$

then the Frobenius condition (9.4) is equivalent to the commutation relation

$$X_j\circ X_\ell=X_\ell\circ X_j,\qquad j,\ell\in\{1,\ldots,d\}.$$

For example, if $k=d=2$ and we consider

$$\sigma^i_j(y)=\mathbb{1}_{\{i=j\}}\,y_i,$$

namely

$$\sigma(y)=\begin{pmatrix} y_1 & 0\\ 0 & y_2\end{pmatrix},\qquad y=(y_1,y_2)\in\mathbb{R}^2,$$

then

$$\partial_a\sigma^i_j(y)=\mathbb{1}_{\{i=j=a\}},$$

and

$$(\sigma_2)^i_{j\ell}(y)=\sum_{a=1}^{2}\partial_a\sigma^i_j(y)\,\sigma^a_\ell(y)=\mathbb{1}_{\{i=j=\ell\}}\,y_i,$$

which is clearly symmetric in (j,ℓ).

If the Frobenius condition (9.4) holds and $\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2)$ is a weakly geometric α-rough path, we obtain

$$\begin{aligned}(\sigma_2(y)\,\mathbb{X}^2)^i &= \sum_{a,b=1}^{2}(\sigma_2)^i_{ab}(y)(\mathbb{X}^2)^{ba}\\ &= \sum_{a,b=1}^{2}\frac{1}{2}\{(\sigma_2)^i_{ab}+(\sigma_2)^i_{ba}\}(y)(\mathbb{X}^2)^{ba}\\ &= \frac{1}{2}\sum_{a,b=1}^{2}(\sigma_2)^i_{ab}(y)\{(\mathbb{X}^2)^{ab}+(\mathbb{X}^2)^{ba}\}\\ &= \frac{1}{2}\sum_{a,b=1}^{2}(\sigma_2)^i_{ab}(y)\,(\mathbb{X}^1)^a(\mathbb{X}^1)^b \qquad (9.5)\\ &= \frac{1}{2}(\sigma_2(y)\,(\mathbb{X}^1\otimes\mathbb{X}^1))^i.\end{aligned}$$

In this case it turns out that the solution Z to the associated finite difference equation is a function of $\mathbb{X}^1$ alone since (3.19) is equivalent to

$$|Z^{[3]}_{st}| \lesssim |t-s|^{3\alpha}, \qquad Z^{[3]}_{st} = \delta Z_{st} - \sigma(Z_s)\,\mathbb{X}^1_{st} - \frac{1}{2}\,\sigma_2(Z_s)\,(\mathbb{X}^1_{st}\otimes\mathbb{X}^1_{st}). \tag{9.6}$$

Arguing as in the proof of the proof of Theorem 3.11, it can be seen that the map $(Z_0,\mathbb{X}^1)\mapsto Z$ is continuous.

Proposition 9.11. *Let $M>0$ and let us suppose that $\mathbb{X}$ is a weakly geometric rough path and $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$ satisfies the Frobenius condition (9.4). If*

$$\max\{|\sigma(Z_0)| + |\sigma(\bar Z_0)| + |\sigma_2(\bar Z_0)|, \|\mathbb{X}^1\|_\alpha, \|\bar{\mathbb{X}}^1\|_\alpha\} \leqslant M,$$

then for every $T>0$ there are $\hat\tau_{M,D,T}, C_{M,D,T}>0$ such that for $\tau\in\,]0,\hat\tau_{M,D,T}]$

$$\begin{aligned}&\|Z-\bar Z\|_{\infty,\tau} + \|\delta Z - \delta\bar Z\|_{\alpha,\tau} + \|Z^{[2]} - \bar Z^{[2]}\|_{2\alpha,\tau} \leqslant\\ &\leqslant C_{M,D,T}\,(|Z_0-\bar Z_0| + \|\mathbb{X}^1-\bar{\mathbb{X}}^1\|_\alpha).\end{aligned}$$

In particular, the solution to (3.19) depends only on $\mathbb{X}^1$ if $\mathbb{X}$ is a geometric rough path.

Proof. The proof follows from the same arguments as in the proof of Theorem 3.11, if one uses the algebraic relations for $Y^{[3]} := Z^{[3]} - \bar Z^{[3]}$ and $\delta Y^{[3]} := \delta Z^{[3]} - \delta\bar Z^{[3]}$ obtained by replacing $\mathbb{X}^2$ with $\frac{1}{2}\mathbb{X}^1\otimes\mathbb{X}^1$, as in

$$\begin{aligned} Z^{[3]}_{st} &= \delta Z_{st} - \sigma(Z_s)\,\mathbb{X}^1_{st} - \frac{1}{2}\,\sigma_2(Z_s)\,(\mathbb{X}^1_{st}\otimes\mathbb{X}^1_{st}),\\ \delta Z^{[3]}_{sut} &= (\sigma(Z_u) - \sigma(Z_s) - \sigma_2(Z_s)\,\mathbb{X}^1_{su})\,\mathbb{X}^1_{ut} + \frac{1}{2}\,\delta\sigma_2(Z)_{su}\,(\mathbb{X}^1_{ut}\otimes\mathbb{X}^1_{ut}),\end{aligned}$$

and analogously for $\bar Z^{[3]}, \delta\bar Z^{[3]}$. One can also note the simple estimate

$$\|\mathbb{X}^1\otimes\mathbb{X}^1 - \bar{\mathbb{X}}^1\otimes\bar{\mathbb{X}}^1\|_{2\alpha} \leqslant \|\mathbb{X}^1-\bar{\mathbb{X}}^1\|_\alpha(\|\mathbb{X}^1\|_\alpha + \|\bar{\mathbb{X}}^1\|_\alpha).$$

The rest of the proof is identical to that of Theorem 3.11. □

Remark 9.12. Doss and Sussmann prove a continuity result in the sup-norm.

9.6. Lack of continuity (again)

In section 9.5 we have seen that, under appropriate conditions on σ, the map $(Z_0, \mathbb{X}^1)\mapsto Z$ is continuous if $\mathbb{X} = (\mathbb{X}^1,\mathbb{X}^2)$ varies in the class of weakly geometric rough paths. In this section we show that this is not a general fact, and the continuity result of Proposition 3.11 can not be improved in general.

More precisely, we consider the sequence $\mathbb{X}_n = (\mathbb{X}^1_n, \mathbb{X}^2_n)$ such that $\mathbb{X}^1_n\to 0$, $\mathbb{X}^2_n\to\mathbb{X}^2\neq 0$ constructed in Section 8.8 and we provide an explicit $\sigma\colon\mathbb{R}^k\to\mathbb{R}^k\otimes(\mathbb{R}^d)^*$ one such that the solution Z^n to the finite difference equation

$$\delta Z^n_{st} - \sigma(Z^n_s)\,(\mathbb{X}^1_n)_{st} - \sigma_2(Z^n_s)(\mathbb{X}^2_n)_{st} = o(t-s)$$

is *not* a continuous function of $(Z_0,\mathbb{X}^1)$ (but only of $(Z_0,\mathbb{X}^1,\mathbb{X}^2)$).

For $y_1, y_2 \in \mathbb{R}$, $\sigma: \mathbb{R}^2 \to \mathbb{R}^2 \otimes (\mathbb{R}^2)^*$, we set

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \sigma(y) := \begin{pmatrix} y_2 & 0 \\ 0 & y_1 \end{pmatrix}.$$

In coordinates,

$$\sigma^i_j(y) = \mathbb{1}_{\{i=j=1\}}\, y_2 + \mathbb{1}_{\{i=j=2\}}\, y_1.$$

If we compute the partial derivative

$$\frac{\partial \sigma^i_j(y)}{\partial y_a} = \mathbb{1}_{\{i=j\neq a\}}, \qquad a \in \{1, 2\},$$

we obtain the expression for σ_2

$$(\sigma_2)^i_{j\ell}(y) = \sum_{a=1}^{2} \partial_a \sigma^i_j(y)\, \sigma^a_\ell(y) = \mathbb{1}_{\{i=j\neq \ell\}}\, y_\ell.$$

Note that σ_2 is not symmetric with respect to (j, ℓ) i.e. $(\sigma_2)^i_{j\ell} \neq (\sigma_2)^i_{\ell j}$, namely it does not satisfy the Frobenius condition (9.4). By taking $\mathbb{X}^2$ from Section 8.8, we compute

$$(\sigma_2(y)\, \mathbb{X}^2)^i = \sum_{a,b=1}^{2} (\sigma_2)^i_{ab}(y)(\mathbb{X}^2)^{ba} = \frac{t-s}{2}\, (\mathbb{1}_{\{i=2\}}\, y_1 - \mathbb{1}_{\{i=1\}}\, y_2).$$

Since we have already shown that $\mathbb{X}^1 = 0$, we get

$$\dot{y} = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y,$$

we can conclude that the solution y is in the form of exponential different from a constant.