6 NOV 2024

Pavia – Milano Bicocca – INdAM Ph.D. program in Mathematics

Course Title. *Stochastic Analysis and Applications*

Teacher(s). Francesco Caravenna (Milano-Bicocca)

Overview. This course presents modern topics in differential equations driven by irregular functions (continuous yet non-differentiable) that lead to the foundational concept of *rough paths*. This generalization offers new insights into the classical theory of stochastic integration with respect to Brownian motion. We may also explore applications to singular (stochastic) partial differential equations.
The course's analytic core requires few prerequisites. Applications in stochastic integration require measure-theoretic probability, and familiarity with Brownian motion is helpful (interested students lacking some background are encouraged to contact me).

When. Wed 14:30-17:00 (Nov 2024 – Jan 2025)
First lecture: Nov 6, 2024

Where. Department of Mathematics and Applications, University of Milano-Bicocca
U5 building (via Cozzi 55, Milano), room ~~3014~~ **2109 (2nd floor)**

Webpage of the course. (Guest access with key `SAA-2425`)
https://elearning.unimib.it/course/view.php?id=58890

Streaming. (Webex link)
https://unimib.webex.com/unimib-it/j.php?MTID=m70a34a9a5bbf8cbd22d531252932b59e

Contacts. francesco.caravenna@unimib.it

Abstract.

- Introduction to singular differential equations, the sewing bound
- Difference equations: the Young case
- Difference equations: the rough case
- Stochastic differential equations
- The sewing lemma and the Young integral
- Rough paths and rough integration
- Examples and applications

References.

- Peter Friz and Martin Hairer. *A Course on Rough Paths*. Second Edition, Springer (2020)
- Francesco Caravenna, Massimiliano Gubinelli, Lorenzo Zambotti. *Ten Lectures on Rough Paths*. Lecture Notes.

- T. Lyons. Differential equations driven by rough signals. Rev. Mat. Iberoam. 14 (1998), 215-310
- M. Gubinelli. Controlling Rough Paths. J. Funct. Anal. 216 (2004), 86-140.
- A.M. Davie. Differential Equations Driven by Rough Paths: An Approach via Discrete Approximations. Appl. Math. Research Express (2008), abm 009.
- P. Friz, M. Hairer. A course on rough paths. Second Edition. Springer (2020).
- F.C., M. Gubinelli, L. Zambotti. Ten lectures on rough paths. Work in progress.

0_ MOTIVATION

Controlled ODEs (ordinary differential equations)

$$\boxed{\dot{Z}_t = \sigma(Z_t) \cdot \dot{X}_t} \qquad (*)$$

GIVEN: $X = (X_t)_{t \in [0,T]}$ $\boxed{X : [0,T] \longrightarrow \mathbb{R}^d}$

(Fix $T > 0$).

UNKNOWN: $Z = (Z_t)_{t \in [0,T]}$ $\boxed{Z : [0,T] \to \mathbb{R}^k}$

GIVEN: $\sigma(\cdot)$, $\sigma : \mathbb{R}^k \to \underbrace{\mathcal{L}(\mathbb{R}^d, \mathbb{R}^k)}_{\mathbb{R}^k \otimes (\mathbb{R}^d)^* \simeq \mathbb{R}^{k \times d}}$

(Fixed: $d, k \in \mathbb{N}$, $T > 0$).

As written, $(*)$ makes sense for $X \in C^1$.

GOAL OF THIS COURSE: make sense of $(*)$ for (continuous, but) NON-DIFFERENTIABLE X.

Key motivation: $X = B = (B_t)_{t \geq 0}$ Brownian motion. (BM)

Setting: $X = (X_t)_{t \in [0,T]}$ is α-HÖLDER ($X \in \mathcal{C}^\alpha$):

$$\boxed{\exists\, C < \infty : \quad |X_t - X_s| \leq C \cdot (t-s)^\alpha \quad \forall\, 0 \leq s < t \leq T}$$

Here $\alpha \in (0,1]$ ($\alpha = 1$: Lipschitz functions).

We will see that there is a big difference between:

$\boxed{\alpha > \frac{1}{2}}$ "YOUNG CASE" $\to$ "easy" Young (1936)

$\boxed{\alpha \le \frac{1}{2}}$ "ROUGH CASE" $\to$ "challenging"

- $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ $\to$ FOCUS ON THIS (includes BM)
- $\alpha \in (\frac{1}{4}, \frac{1}{3}]$
- ....

Idea: rephrase (*) in alternative ways which make sense when X is non-differentiable.

We will reformulate (*) as a FINITE-DIFFERENCE EQUATION:

$$(*) \quad \dot{Z}_t = \sigma(Z_t) \cdot \dot{X}_t$$

$$(*') \quad Z_t - Z_s = \sigma(Z_s) \cdot (X_t - X_s) + \underbrace{\{\dots\}}_{\substack{\text{ADDITIONAL EXPLICIT TERMS} \\ \text{(NOT INVOLVING DIRECTLY } \dot{X}_t\text{)}}} + o(t-s) \quad \text{for } 0 \le s < t \le T$$

We will show that such a formulation is powerful enough to prove WELL-POSEDNESS RESULTS (existence, uniqueness, a priori estimates, continuous dependence on initial data, ...) for an interesting class of non-differentiable paths X.

- If $\alpha > \frac{1}{2}$ (Young case), then we can take $\{\cdots\} = 0$ in $(*')$ and we can prove well-posedness for generic paths X of class $\mathcal{C}^\alpha$.

- If $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ (Rough case), in general $(*')$ with $\{\cdots\} = 0$ has NO SOLUTION. However, if we specify a suitable notion of "$\int_0^t X_u^i \, dX_u^j$", then we can prove well-posedness for $(*')$ for a natural, explicit choice of $\{\cdots\}$.

In case $X = B$ is BM, we can choose any stochastic integral we like (Ito, Stratonovich, ...) for $\int_0^t X_u \, dX_u$ and this (non canonical) choice will lead to a corresponding solution of $(*')$ which will be shown to coincide with the solution given by the "classical" theory of stochastic differential equations.

This is the plan for Lectures 1 to 4.

(Assigning a path $X = (X_t)_{t \geq 0}$ along with its iterated integrals $\int_0^t X_u^i \, dX_u^j$ defines a ROUGH PATH)

The following lectures are devoted to an alternative (and more customary) reformulation of $(*)$.

$$(*) \quad \dot{Z}_t = \sigma(Z_t) \cdot \dot{X}_t$$

$$(*'') \quad Z_t = Z_0 + \mathcal{I}_t(\sigma(Z), X)$$

$$\text{where } \mathcal{I}_t(Y, X) := \text{"} \int_0^t Y_u \, dX_u \text{"}$$

Strategy:

- Define a suitable notion of "integral" $\mathcal{I}_t(Y, X)$ (for non-differentiable X), for a wide enough class $\mathcal{H}$ of integrands $Y = (Y_u)_{u \geq 0}$.

- Prove that the maps $Z \mapsto \sigma(Z)$ and $Y \mapsto (\mathcal{I}_t(Y, X))_{t \geq 0}$ are CONTINUOUS with respect to a suitable metric on $\mathcal{H}$.

- Finally show that the continuous map

$$Z = (Z_t)_{t \in [0,T]} \mapsto (z_0 + \mathcal{I}_t(\sigma(Z), X))_{t \in [0,T]}$$

is actually a CONTRACTION on $\mathcal{H}$, if we fix $T > 0$ small enough $\Rightarrow \forall z_0 \in \mathbb{R}^k \; \exists!$ fixed point of the map above, i.e. a solution of $(*'')$.

We will show that this stategy can be pursued in the context of ROUGH PATHS and the resulting solution Z of $(*'')$ coincides with the solution of $(*')$.

We will see that the first approach is more "elementary" and, actually, more powerful, because it requires less assumptions (close to optimal) on $\sigma(\cdot)$.

1 - THE SEWING BOUND

Goal: introduce a key tool, the SEWING BOUND.
Motivation: prove uniqueness for our equation

$$(\ast) \quad \dot{Z}_t = \sigma(Z_t) \cdot \dot{X}_t$$

Fix $T > 0$, $d, k \in \mathbb{N}$.
Fix a path $X = (X_t)_{t \in [0,T]}$, $X : [0,T] \to \mathbb{R}^d$, of class $\mathcal{C}^\alpha$ (we will later require $\alpha > \frac{1}{2}$).

Fix $\sigma : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k)$ continuous (we will later take it linear).

Idea: assume for the moment that $X \in C^1$.
Assume that $Z = (Z_t)_{t \in [0,T]}$, $Z : [0,T] \to \mathbb{R}^k$ is of class C^1 and solves $(\ast)$ in the classical sense:

$$(\ast) \quad \dot{Z}_t = \sigma(Z_t) \cdot \dot{X}_t \qquad \forall t \in [0,T].$$

Then $\forall\, 0 \leq s < t \leq T$

$$Z_t - Z_s = \int_s^t \underbrace{\sigma(Z_u)}_{\sigma(Z_s) + \{\sigma(Z_u) - \sigma(Z_s)\}} \cdot \dot{X}_u \, du$$

$$= \sigma(Z_s) \cdot (X_t - X_s) + \int_s^t \{\sigma(Z_u) - \sigma(Z_s)\} \cdot \dot{X}_u \, du$$

Thus

$$(*') \quad Z_t - Z_s = \sigma(Z_s)\cdot(X_t - X_s) + o(t-s), \quad 0 \le s < t \le T.$$

We showed that, if $X \in C^1$, any solution Z of $(*)$ also satisfies $(*')$.

Vice versa, if $X \in C^1$, any solution of $(*')$ also solves $(*)$ (exercise!).

However, $(*')$ does not contain $\dot{X}$ (unlike $(*)$) hence it is meaningful for non-differentiable X.

We will prove in the next lecture WELL-POSEDNESS for $(*')$ when $X \in \mathcal{C}^\alpha$ with $\alpha \in (\frac{1}{2}, 1]$ (Young case) and $\sigma(\cdot)$ is regular enough.

<u>Remark</u> Whenever we write $o(t-s)$ we always mean uniformly for $0 \le s < t \le T$, i.e.

$$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 : \quad \forall\, 0 \le s < t \le T, \ t-s < \delta_\varepsilon \Rightarrow |o(t-s)| < \varepsilon(t-s).$$

<u>Lemma</u>. Let X and $\sigma(\cdot)$ be continuous. Then any solution Z of $(*')$ is continuous, more precisely:

$$|Z_t - Z_s| \le C\,|X_t - X_s| + |o(t-s)| \quad \text{for some } C < \infty.$$

Proof: By $(*')$, the stated bound holds with

$$C := \|\sigma(Z)\|_\infty := \sup_{0\le t\le T} |\sigma(Z_t)|$$

It remains to show that $C<\infty$. Since $\sigma(\cdot)$ is continuous, it suffices to show that $\|Z\|_\infty < \infty$.

Fix $\varepsilon = 1$ and let $\delta_1 > 0$: $t-s \le \delta_1 \Rightarrow |o(t-s)| < 1$.

If $0 < b-a \le \delta_1$ then $(0 \le a < b \le T)$

$$\sup_{t\in[a,b]} |Z_t| \le |Z_a| + \sup_{t\in[a,b]} \underbrace{|Z_t - Z_a|}_{\sigma(Z_a)\cdot(X_t - X_a) + o(t-a)}$$

$$\le |Z_a| + |\sigma(Z_a)|\cdot(2\|X\|_\infty + 1).$$

FUNCTION SPACES

$$[0,T]^n_\le := \{(t_1,\dots,t_n) : 0 \le t_1 \le t_2 \le \dots \le t_n \le T\}$$

$$C_n := \{\text{continuous } F : [0,T]^n_\le \to \mathbb{R}^K\}$$

Fix $\eta \in (0,\infty)$. We equip C_2 and C_3 with a norm

$$F \in C_2: \qquad \|F\|_\eta := \sup_{0\le s<t\le T} \frac{|F_{st}|}{(t-s)^\eta}$$

$G \in C_3$: $\quad \|G\|_\eta := \sup_{0 \le s \le u \le t \le T} \frac{|G_{sut}|}{(t-s)^\eta}$

Fact: $(C_n, \|\cdot\|_\eta)$ is Banach (exercise!).

<u>Notation:</u> For $f \in C_1$, define $\delta f \in C_2$

$$(\delta f)_{st} := f_t - f_s \qquad 0 \le s < t \le T.$$

$$(*') \quad \delta Z_{st} = \sigma(Z_s)\, \delta X_{st} + o(t-s) \qquad 0 \le s < t \le T$$

Note that $\quad f \in \mathcal{C}^\alpha \iff \|\delta f\|_\alpha < \infty$

(The norm on $\mathcal{C}^\alpha$ is $\|f\|_{\mathcal{C}^\alpha} := \|f\|_\infty + \|\delta f\|_\alpha$.)

Observation: $\quad |f_t| \le |f_0| + |f_t - f_0|$

$$\le |f_0| + t^\alpha \cdot \frac{|f_t - f_0|}{t^\alpha}$$

$$\overset{!}{\le} |f_0| + T^\alpha \|\delta f\|_\alpha \qquad \forall t \in [0,T]$$

$$\Rightarrow \|f\|_\infty \le |f_0| + T^\alpha \|\delta f\|_\alpha$$

$\|f\|_{\mathcal{C}^\alpha}$ is equivalent to $|f_0| + \|\delta f\|_\alpha$.

BASIC LOCAL UNIQUENESS (LINEAR CASE)

Consider the case when $\sigma : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k)$ is a linear map. Our equation $(\ast')$ becomes

$$(\tilde{\ast}') \quad \delta Z_{st} = \sigma \cdot Z_s \cdot \delta X_{st} + o(t-s) \qquad 0 \le s < t \le T$$

$$Z : [0,T] \to \mathbb{R}^k \qquad X : [0,T] \to \mathbb{R}^d$$

<u>Theorem</u> (LOCAL UNIQUENESS) Let $X \in \mathcal{C}^\alpha$ with $\alpha > \frac{1}{2}$. Let $\sigma \in \mathcal{L}(\mathbb{R}^k, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k))$. If $T > 0$ is small enough, $\forall z_0 \in \mathbb{R}^k$ there is at most one solution $Z = (Z_t)_{t \in [0,T]}$ of $(\tilde{\ast}')$ with $Z_0 = z_0$

<u>Proof</u>. Assume that Z and Z' are two solutions with $Z_0 = Z'_0 = z_0$. Define $Y := Z - Z' = (Z_t - Z'_t)_{t \in [0,T]}$.

Then Y is also a solution of $(\tilde{\ast}')$ with $Y_0 = 0$. We need to show that $Y \equiv 0$, i.e. $\|Y\|_\infty = 0$. It is enough to show that $\|\delta Y\|_\alpha = 0$ because

$$\|Y\|_\infty \le |Y_0| + T^\alpha \|\delta Y\|_\alpha .$$

Define the "remainder"

$$R_{st} := \delta Y_{st} - \sigma \cdot Y_s \cdot \delta X_{st}$$

i.e. $$\delta Y_{st} = \sigma \cdot Y_s \cdot \delta X_{st} + \underbrace{R_{st}}_{o(t-s)}$$

Then $$\|\delta Y\|_\alpha \leq |\sigma| \cdot \underbrace{\|Y\|_\infty}_{\leq \cancel{|Y_0|} + T^\alpha \|\delta Y\|_\alpha} \cdot \|\delta X\|_\alpha + \|R\|_\alpha$$

$$\|R\|_\alpha = \sup_{s,t} \frac{|R_{st}|}{(t-s)^\alpha} \leq T^\alpha \cdot \sup_{s,t} \frac{|R_{st}|}{(t-s)^{2\alpha}} = T^\alpha \cdot \|R\|_{2\alpha}$$

$$\|\delta Y\|_\alpha \leq T^\alpha \cdot \left(|\sigma| \cdot \|\delta X\|_\alpha \cdot \|\delta Y\|_\alpha + \|R\|_{2\alpha} \right)$$

Imagine that we can prove the following bound:

$$\|R\|_{2\alpha} \leq C_1 \cdot \|\delta Y\|_\alpha$$

Then our proof is easily completed:

$$\|\delta Y\|_\alpha \leq T^\alpha \cdot \left(|\sigma| \cdot \|\delta X\|_\alpha + C_1 \right) \cdot \|\delta Y\|_\alpha$$

If we fix $T > 0$ small enough (depending on σ, X, α) we obtain $\|\delta Y\|_\alpha \leq \frac{1}{2} \|\delta Y\|_\alpha \Rightarrow \|\delta Y\|_\alpha = 0$.

[We already know that $\|\delta Y\|_\alpha < \infty$ because

$$\frac{|\delta Y_{st}|}{(t-s)^\alpha} \leq C \cdot \frac{|\delta X_{st}|}{(t-s)^\alpha} + \frac{a(t-s)}{(t-s)^\alpha} \quad (\alpha \leq 1)$$
]

∎

It remains to show the claimed bound

$$\|R\|_{2\alpha} \leq C_1 \|\delta Y\|_\alpha \qquad (\text{if } \alpha > \tfrac{1}{2})!$$

THE SEWING BOUND

Consider a function $R \in C_2$ (i.e. a continuous function $(R_{st})_{0 \leq s \leq t \leq T}$) such that $R_{st} = o(t-s)$.

Fix $[a,b] \subseteq [0,T]$ and a partition $P = \{a = t_0 < t_1 < \dots < t_m = b\}$

Define $$I_P(R) = \sum_{i=1}^{m} R_{t_{i-1} t_i}$$

Set $|P| = \max_{i=1,\dots,m} |t_i - t_{i-1}|$.

<u>Lemma</u>. Let $R \in C_2$ with $R_{st} = o(t-s)$.
For any $[a,b] \subseteq [0,T]$ and for any sequence of partitions $(P_n)_{n \in \mathbb{N}}$ of $[a,b]$ with $|P_n| \longrightarrow 0$ we have

$$\lim_{n \to \infty} I_{P_n}(R) = 0.$$

Fix a partition $\mathcal{P} = \{a = t_0 < t_1 < .. < t_m = b\}$.

Removing a point t_i we obtain a new partition

$\mathcal{P}' = \{a = t_0 < t_1 < \_ < t_{i-1} < t_{i+1} < \_ < t_m = b\}$.

$$I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R) = R_{t_{i-1} t_{i+1}} - R_{t_{i-1} t_i} - R_{t_i t_{i+1}}$$
$$= \delta R_{t_{i-1} t_i t_{i+1}}$$

Def For $R \in C_2$ we define $\delta R \in C_R$ by

$$\delta R_{sut} := R_{st} - R_{su} - R_{ut} \quad (0 \leq s \leq u \leq t \leq T)$$

($\delta R = 0 \iff R = \delta f$ for some $f \in C_1$)

$$\Rightarrow |I_{\mathcal{P}'}(R) - I_{\mathcal{P}}(R)| \leq (t_{i+1} - t_{i-1})^{\eta} \cdot \|\delta R\|_{\eta}$$

Theorem (SEWING BOUND). Let $R \in C_2$ with $R_{st} = o(t-s)$.
Then the following bound holds: $\forall \eta \in (1, \infty)$.

$$\|R\|_{\eta} \leq K_{\eta} \cdot \|\delta R\|_{\eta} \quad \left(\text{with } K_{\eta} = \frac{1}{1 - 2^{1-\eta}}\right)$$