WED 13 NOV 2024

REMINDERS FROM THE PREVIOUS LECTURE

We fix:
- time horizon $T > 0$
- space dimensions $d, \kappa \in \mathbb{N}$ (e.g. $d = \kappa = 1$)
- driving path $\underset{\substack{\| \\ (X_t)_{t\in[0,T]}}}{X} : [0,T] \to \mathbb{R}^d$ (e.g. X = Brownian motion)
- function $\sigma : \mathbb{R}^\kappa \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^\kappa)$ } $\mathbb{R}^\kappa \otimes (\mathbb{R}^d)^* \simeq \mathbb{R}^{\kappa \times d}$

For Hölder $X \in \mathcal{C}^\alpha$ (non differentiable) consider the ill-defined ODE

$$(*) \qquad \dot{Z}_t = \sigma(Z_t)\, \dot{X}_t \qquad\qquad Z : [0,T] \to \mathbb{R}^\kappa$$

UNKNOWN

Basic reformulation as a $\overbrace{\text{FINITE DIFFERENCE EQUATION}}^{\text{LATER: equivalent integral reformulation}}$:

$$(*') \qquad \underbrace{Z_t - Z_s}_{\delta Z_{st}} = \sigma(Z_s)\, \underbrace{(X_t - X_s)}_{\delta X_{st}} + o(t-s) \qquad \text{for } 0 \leqslant s < t \leqslant T$$

$$\delta Z_{st} = \sigma(Z_s)\, \delta X_{st} + o(t-s)$$

<u>Goal</u>: Prove well-posedness for $(*')$ for any $X \in \mathcal{C}^\alpha$, $\alpha \in (\frac{1}{2}, 1]$: existence, uniqueness, regularity of solution Z given $Z_0 = z_0 \in \mathbb{R}^\kappa$, under suitable regularity of σ (e.g. C^2).

(Spoiler: for $\alpha \in (0, \frac{1}{2}]$ we will need to "enrich" X and modify $(*')$.)

FUNCTION SPACES

- $[0,T]^n_{\leq} := \{ 0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq T \}$

- $C_n := \{$ continuous functions of n ordered variables $F : [0,T]^n_{\leq} \to \mathbb{R}^k \}$

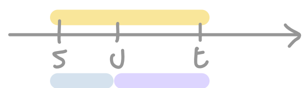
- Hölder-like norm: $\|F\|_\eta := \sup_{0 \leq t_1 < \dots < t_n \leq T} \frac{|F_{t_1 \cdots t_n}|}{(t_n - t_1)^\eta}$ $\quad F \in C_n, \; n = 2, 3, \dots$

  $\eta \in (0, \infty)$

  $C_n^\eta := \{ F \in C_n : \|F\|_\eta < \infty \}$ Banach

- Increment

$$\begin{array}{ccc} \delta : C_1 & \to & C_2 \\ \quad\; \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ f & \mapsto & \delta f_{st} := f_t - f_s \end{array} \qquad \begin{array}{ccc} \delta : C_2 & \to & C_3 \\ \quad\; \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ F & \mapsto & \delta F_{sut} := F_{st} - F_{su} - F_{ut} \end{array}$$



$\delta \circ \delta = 0$ $\qquad C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_3$

$\delta f = 0 \iff f_t = c = \text{const.}$ $\qquad \delta F = 0 \iff F = \delta f$ for some $f \in C_2$

- Classical Hölder space $\mathcal{C}^\alpha = \{ f \in C_1 : \delta f \in C_2^\alpha \}$ $\quad 0 < \alpha \leq 1$

  $\|\delta f\|_\alpha = \sup_{0 \leq s < t \leq T} \frac{|\delta f_{st}|}{(t-s)^\alpha}$ $\quad$ seminorm on $\mathcal{C}^\alpha$

  $\|f\|_{\mathcal{C}^\alpha} := \|\delta f\|_\alpha + \underbrace{\|f\|_\infty}_{= \sup_{0 \leq t \leq T} |f_t|} \simeq \|\delta f\|_\alpha + |f_0|$

- Basic bounds: (1) $\quad \|f\|_\infty \leq |f_0| + T^\alpha \|\delta f\|_\alpha$

  (2) $\quad \|F\|_\alpha \leq T^\beta \|F\|_{\alpha+\beta} \quad \forall \beta > 0$

THE SEWING BOUND

$Z = (Z_t)_{t\in[0,T]}$ is a solution of $(*')$ $\Longleftrightarrow$

$$\underbrace{R_{st}}_{\text{"REMAINDER"}} := \delta Z_{st} - \sigma(Z_s)\,\delta X_{st} = o(t-s)$$

It is useful to estimate $\|R\|_\eta = \sup_{0\le s<t\le T} \frac{|R_{st}|}{(t-s)^\eta}$ for $\eta > 1$.

(E.g. we have uniqueness (for linear $\sigma(\cdot)$) if $\|R\|_{2\alpha} \le C\,\|\delta Z\|_\alpha$)

Theorem (SEWING BOUND). For any $R = (R_{st})$ with $R_{st} = o(t-s)$:

$$\forall \eta > 1: \qquad \|R\|_\eta \le K_\eta \,\|\delta R\|_\eta \qquad \left(K_\eta = \frac{1}{1-2^{1-\eta}}\right)$$

Let us apply the sewing bound to prove that, when $\sigma(\cdot)$ is linear, we have $\|R\|_{2\alpha} \le C \cdot \|\delta Z\|_\alpha \Rightarrow$ local uniqueness.

Lemma: If $R_{st} = W_s\,\delta X_{st}$ then $\delta R_{sut} = -\delta W_{su}\,\delta X_{ut}$

Proof: $\qquad s<u<t$

$$\begin{aligned}
\delta R_{sut} &= R_{st} - R_{su} - R_{ut} \\
&\overset{!}{=} W_s\,\delta X_{st} - W_s\,\delta X_{su} - W_u\,\delta X_{ut} \\
&\overset{!}{=} W_s \underbrace{(\delta X_{st} - \delta X_{su})}_{X_t - X_s - (X_u - X_s) = X_t - X_u = \delta X_{ut}} - W_u\,\delta X_{ut} = -\delta W_{su}\,\delta X_{ut}
\end{aligned}$$

∎

In our case $\delta R_{sut} = \delta(\cancel{\delta Z})_{sut} + \delta\sigma(Z)_{su}\,\delta X_{ut}$

$$\overset{!}{=} \sigma \cdot \delta Z_{su} \cdot \delta X_{ut}$$

$$\Rightarrow \| \delta R \|_{2\alpha} = \sup_{0 \leq s < u < t \leq T} \frac{|\delta R_{sut}|}{(t-s)^{2\alpha}} \leq |\sigma| \cdot \| \delta Z \|_{\alpha} \cdot \| \delta X \|_{\alpha} = C \cdot \| \delta Z \|_{\alpha}$$

with $C = |\sigma| \cdot \| \delta X \|_{\alpha}$

Sewing bound: $\| R \|_{2\alpha} \leq K_{2\alpha} \cdot \| \delta R \|_{2\alpha} \leq K_{2\alpha} \cdot C \cdot \| \delta Z \|_{\alpha}$

PROVING THE SEWING BOUND

Fix $[a,b] \subseteq [0,T]$ and a partition $\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_m = b\}$

Define

$$I_{\mathcal{P}}(R) := \sum_{i=1}^{m} R_{t_{i-1} t_i} \qquad |\mathcal{P}| = \max_{i=1,\ldots,m} (t_i - t_{i-1})$$

<u>Lemma</u>. If $R_{st} = o(t-s)$, then

$$\lim_{n\to\infty} I_{\mathcal{P}_n}(R) = 0 \qquad \forall \text{ partitions } (\mathcal{P}_n)_{n\in\mathbb{N}_0} \text{ with } |\mathcal{P}_n| \to 0$$

In particular, if $\mathcal{P}_0 = \{a,b\}$, then

$$R_{ab} = \sum_{n=0}^{\infty} \{ I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R) \}$$

<u>Proof</u>. Note that

$$\sum_{n=0}^{N} \{ I_{\mathcal{P}_n}(R) - I_{\mathcal{P}_{n+1}}(R) \} = I_0(R) - I_{N+1}(R) = R_{ab} - I_{N+1}(R) \xrightarrow[N\to\infty]{} R_{ab}$$

$$I_{\mathcal{P}_n}(R) = \sum_{i=1}^{m_n} \frac{R_{t^n_{i-1} t^n_i}}{(t^n_i - t^n_{i-1})} (t^n_i - t^n_{i-1}) \le g(|\mathcal{P}_n|) \cdot \underbrace{\sum_{i=1}^{m_n} (t^n_i - t^n_{i-1})}_{t^n_{m_n} - t^n_0 = b-a}$$

$$\downarrow n\to\infty$$

$$0$$

$$g(\varepsilon) := \sup_{0<t-s<\varepsilon} \frac{R_{st}}{(t-s)}$$

$$g(\varepsilon) \xrightarrow[\varepsilon \downarrow 0]{} 0$$

∎

Observation: if $P=\{t_0<t_1<\dots<t_m\}$ and $P'=P\setminus\{t_i\}$,

then

$$I_{P'}(R)-I_P(R) = R_{t_{i-1}t_{i+1}} - \left(R_{t_{i-1}t_i}+R_{t_it_{i+1}}\right) \overset{!}{=} \delta R_{t_{i-1}t_it_{i+1}}$$

Fix $[a,b]\subseteq[0,T]$.

Dyadic partition $P_n=\left\{t_i^n = a+\frac{i}{2^n}(b-a) : 0\le i\le 2^n\right\}$

$$|P_n| = \frac{(b-a)}{2^n}$$

$$\begin{aligned}
\left|I_{P_n}(R)-I_{P_{n+1}}(R)\right| &\le \sum_{j=0}^{2^n-1} \left|\delta R_{t^{n+1}_{2j}\, t^{n+1}_{2j+1}\, t^{n+1}_{2j+2}}\right| \\
&\le 2^n \cdot \|\delta R\|_\eta \cdot \big(\underbrace{t^{n+1}_{2j+2}-t^{n+1}_{2j}}_{\frac{\cancel{2}\cdot(b-a)}{2^{n+\cancel{1}}}}\big)^\eta \\
&\le \frac{(b-a)^\eta}{2^{(\eta-1)n}}\,\|\delta R\|_\eta
\end{aligned}$$

$$\Rightarrow |R_{ab}| \le \sum_{n=0}^{\infty} \left|I_{P_n}(R)-I_{P_{n+1}}(R)\right| \le \|\delta R\|_\eta\cdot(b-a)^\eta\cdot \underbrace{\sum_{n=0}^{\infty}\frac{1}{(2^{\eta-1})^n}}_{K_\eta}$$

i.e. $\|R\|_\eta \le K_\eta\,\|\delta R\|_\eta$. $\blacksquare$

Remark: $\|R\|_\eta \leq C_1 \iff |R_{ab}| \leq C_1 (b-a)^\eta$.

DISCRETE SEWING BOUND.

Fix a finite set $\Pi \subseteq [0,T]$, say $\Pi = \{t_0 < t_1 < \ldots < t_m\}$.

Given $R = (R_{st})_{s<t,\, s,t \in \Pi}$

$$\|R\|_\eta^\Pi := \max_{s,t\in\Pi,\, s<t} \frac{|R_{st}|}{(t-s)^\eta}$$

$$\|\delta R\|_\eta^\Pi := \max_{s,u,t\in\Pi,\, s<u<t} \frac{|\delta R_{sut}|}{(t-s)^\eta}$$

Theorem (DISCRETE SEWING BOUND) If $R_{st} = 0$ whenever s,t are consecutive points in Π (i.e. $s = t_i$, $t = t_{i+1}$ for some i) then

$$\forall \eta > 1 \quad \|R\|_\eta^\Pi \leq C_\eta \|\delta R\|_\eta^\Pi \quad \text{with } C_\eta = 2^\eta \sum_{n=1}^\infty \frac{1}{n^\eta}$$

Remark The sewing bound is only useful if $\|\delta R\|_\eta < \infty$ for some $\eta > 1$. The assumption of the sewing bound is that $R_{st} = o(t-s)$. If we have $R_{st} = O((t-s)^\eta)$ for some $\eta > 1$, then also $\underbrace{\delta R_{sut}}_{R_{st} - R_{su} - R_{ut}} = O((t-s)^\eta) \Rightarrow \|\delta R\|_\eta < \infty$.

2. DIFFERENCE EQUATIONS: YOUNG CASE

We now apply the tools developed so far to prove WELL-POSEDNESS for the difference equation

$$(*') \qquad \delta Z_{st} = \sigma(Z_s)\,\delta X_{st} + o(t-s) \qquad (0 \le s < t \le T)$$

when $X \in \mathcal{C}^\alpha$ with $\alpha \in (\frac{1}{2}, 1]$, the so-called YOUNG CASE.

$$(*) \qquad \dot{Z}_t = \sigma(Z_t)\,\dot{X}_t$$

THEOREM (WELL-POSEDNESS, YOUNG CASE)

Fix $X : [0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$ for some $\alpha \in (\frac{1}{2}, 1]$. Fix $\sigma : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k)$. Then we have:

- LOCAL EXISTENCE: if σ locally Lipschitz (e.g. if $\sigma \in C^1$) then $\forall z_0 \in \mathbb{R}^k$ $\exists\, T' = T'_{\alpha, X, \sigma}(z_0) \in (0,T]$ such that $(*')$ admits a solution $Z = (Z_t)_{t \in [0,T']}$.

  (It actually suffices that $\sigma \in \mathcal{C}^\gamma_{loc}$ for some $\gamma \in (\frac{1}{\alpha} - 1, 1]$).

- GLOBAL EXISTENCE: if σ is globally Lipschitz (e.g. $\sigma \in C^1$ with $\|\nabla \sigma\|_\infty < \infty$) then we can take $T' = T$.

  (It suffices that σ is globally $\mathcal{C}^\gamma$ for some $\gamma \in (\frac{1}{\alpha} - 1, 1]$).

- **UNIQUENESS**: if $\sigma \in C^1$ with locally Lipschitz $\nabla\sigma$ (e.g. $\sigma \in C^2$) then $\forall z_0 \in \mathbb{R}^k$ there is exactly one solution Z of $(*')$ with $Z_0 = z_0$.

(It suffices that $\sigma \in \mathcal{C}^\gamma$ with $\gamma \in (\frac{1}{\alpha}, 2]$).

- **CONTINUITY OF THE SOLUTION MAP**: if $\sigma \in C^1$ with $\nabla\sigma$ bounded and globally Lipschitz (e.g. if $\sigma \in C^2$ with $\|\nabla\sigma\|_\infty < \infty$, $\|\nabla^2\sigma\| < \infty$), the (unique) solution $Z = (Z_t)_{t \in [0,T]}$ of $(*')$ is a continuous function of the starting point z_0 and of the driving path X, i.e. the map $(z_0, X) \mapsto Z$ is $\underbrace{\text{continuous}}_{\text{locally Lipschitz}}$ from $\mathbb{R}^k \times \mathcal{C}^\alpha \to \mathcal{C}^\alpha$.

(It suffices that $\sigma \in C^1$ with $\|\nabla\sigma\|_\infty < \infty$ and $\nabla\sigma \in \mathcal{C}^{\gamma-1}$ for some $\gamma \in (\frac{1}{\alpha}, 2]$).

- Basic bounds: (1) $\|f\|_\infty \leq |f_0| + T^\alpha \|\delta f\|_\alpha$

(2) $\|F\|_\alpha \leq T^\beta \|F\|_{\alpha+\beta}$ $\quad \forall \beta > 0$

(3) If $F_{st} = g_s H_{st}$ or $= g_t H_{st}$ $\Rightarrow$ $\|F\|_\eta \leq \|g\|_\infty \cdot \|H\|_\eta$

(4) If $F_{sut} = G_{su} H_{ut}$ then $\|F\|_{\eta+\eta'} \leq \|G\|_\eta \cdot \|H\|_{\eta'}$

A PRIORI ESTIMATES

We consider the equation (*')

$$(*') \qquad \delta Z_{st} = \sigma(Z_s)\,\delta X_{st} + o(t-s)$$

We assume that there is a solution Z.

<u>Lemma</u> If $X \in \mathcal{C}^\alpha$ with $\alpha \in (0,1]$ and $\sigma(\cdot)$ is continuous, then any solution Z of (*') satisfies $Z \in \mathcal{C}^\alpha$.

<u>Proof</u>. We proved last time that Z is continuous

$\Rightarrow$ Z is bounded: $\|Z\|_\infty = \sup_{0 \le t \le T} |Z_t| < \infty$.

$\Rightarrow$ $\sigma(Z)$ is bounded: $\|\sigma(Z)\|_\infty < \infty$

$$\Rightarrow \quad |\delta Z_{st}| \le \|\sigma(Z)\|_\infty \underbrace{|\delta X_{st}|}_{O((t-s)^\alpha)} + \underbrace{|o(t-s)|}_{O((t-s)^\alpha)} = O((t-s)^\alpha).$$

$\blacksquare$

<u>Theorem (A PRIORI ESTIMATE)</u> Let $X \in \mathcal{C}^\alpha$ with $\alpha \in (\frac{1}{2}, 1]$.
Let σ be globally Lipschitz, say $|\sigma(x) - \sigma(y)| \le C|x-y|$
Let Z be a solution of (*') and set $\qquad \|\nabla\sigma\|_\infty < \infty.$

$$Z^{[2]}_{st} := \delta Z_{st} - \sigma(Z_s)\,\delta X_{st} \quad = o(t-s).$$

Then

$$\|Z^{[2]}\|_{2\alpha} \le C_{\alpha,X,\sigma} \cdot \|\delta Z\|_\alpha \qquad C_{\alpha,X,\sigma} = K_{2\alpha} \cdot \|\nabla\sigma\|_\infty \cdot \|\delta X\|_\alpha$$

Moreover

$$\boxed{\|\delta Z\|_\alpha \leq 2\|\delta X\|_\alpha \cdot |\sigma(Z_0)|}$$

provided T is small enough, i.e.

$$T^\alpha \leq \varepsilon_{\alpha,X,\sigma} := \frac{1}{2(K_{2\alpha}+3)\|\delta X\|_\alpha \|\nabla\sigma\|_\infty}$$

Proof. By assumption $Z^{[2]}_{st} = o(t-s)$ (i.e. Z is a solution).

Sewing bound: $\|Z^{[2]}\|_{2\alpha} \leq K_{2\alpha} \cdot \|\delta Z^{[2]}\|_{2\alpha}$.

$\gamma > 1$ since $\alpha > \frac{1}{2}$

We already computed (lemma)

$$\delta Z^{[2]}_{sut} = \delta\sigma(Z)_{su}\,\delta X_{ut}$$

$$\Rightarrow \|\delta Z^{[2]}\|_{2\alpha} \leq \underbrace{\|\delta\sigma(Z)\|_\alpha}_{\leq \|\nabla\sigma\|_\infty \cdot \|\delta Z\|_\alpha} \cdot \|\delta X\|_\alpha$$

This proves the first bound.

Let us now estimate $\|\delta Z\|_\alpha$.

$$|\delta Z_{st}| \underset{\leq}{=} |\sigma(Z_s)\cdot\delta X_{st}| + |Z^{[2]}_{st}|$$

$$\|\delta Z\|_\alpha \le \underbrace{\|\sigma(Z)\|_\infty}_{\le |\sigma(Z_0)| + T^\alpha \underbrace{\|\delta\sigma(Z)\|_\alpha}_{\le \|\nabla\sigma\|_\infty \cdot \|\delta Z\|_\alpha}} \cdot \|\delta X\|_\alpha + \underbrace{\|Z^{[2]}\|_\alpha}_{\le T^\alpha \cdot \|Z^{[2]}\|_{2\alpha}}$$

$$\|\delta Z\|_\alpha \le |\sigma(Z_0)|\cdot\|\delta X\|_\alpha + T^\alpha\Big\{\|\nabla\sigma\|_\infty \|\delta X\|_\alpha \cdot \|\delta Z\|_\alpha + C_{\alpha,X,\sigma}\cdot\|\delta Z\|_\alpha\Big\}$$

$$\|\delta Z\|_\alpha \le |\sigma(Z_0)|\cdot\|\delta X\|_\alpha + T^\alpha \cdot C'_{\alpha,X,\sigma}\cdot\|\delta Z\|_\alpha$$

If $T>0$ is small enough so that $T^\alpha C'_{\alpha,X,\sigma} \le \frac{1}{2}$, then

$$\|\delta Z\|_\alpha \le |\sigma(Z_0)|\cdot\|\delta X\|_\alpha + \frac{1}{2}\|\delta Z\|_\alpha$$

Since $\|\delta Z\|_\alpha < \infty \Rightarrow \quad \|\delta Z\|_\alpha \le 2\,|\sigma(Z_0)|\cdot\|\delta X\|_\alpha .$

$\blacksquare$

UNIQUENESS

Lemma. Let φ be of class C^1 with locally Lipschitz gradient: $\forall R<\infty$: $C'_R := \sup_{|x|\le R} |\nabla\varphi(x)| < \infty$

$$C''_R := \sup_{|x|,|y|\leq R} \frac{|\nabla\psi(x)-\nabla\psi(y)|}{|x-y|} < \infty.$$

Then for $x, y, \bar{x}, \bar{y}$ with $|x|, |y|, |\bar{x}|, |\bar{y}| \leq R$

$$|(\psi(x)-\psi(y)) - (\psi(\bar{x})-\psi(\bar{y}))|$$
$$\leq C'_R |(x-y)-(\bar{x}-\bar{y})| + C''_R \{|x-y|+|\bar{x}-\bar{y}|\} |y-\bar{y}|.$$

Theorem (UNIQUENESS) Let $X \in \mathcal{C}^\alpha$ for some $\alpha \in (\frac{1}{2}, 1]$. Let σ be of class C^1 with locally Lipschitz $\nabla\sigma$ (e.g. $\sigma \in C^2$). Then $\forall z_0 \in \mathbb{R}^K$ there is at most one solution Z of $(*')$ with $Z_0 = z_0$.

Proof. Let Z, Z' be two solutions of $(*')$. Define

$$Y := Z - Z'$$

We will show that $\|Y\|_\infty \leq 2|Y_0|$ if $T>0$ is small enough.

Note that $\|Y\|_\infty \leq |Y_0| + T^\alpha \|\delta Y\|_\alpha$

so it is enough to show that $\|\delta Y\|_\alpha \leq |Y_0|$.

We define $Y^{[2]}_{st} := Z^{[2]}_{st} - Z'^{[2]}_{st}$

$$= \delta Y_{st} - \{\sigma(Z_s) - \sigma(Z'_s)\} \delta X_{st}$$

We will show that

(a) $\|\delta Y\|_\alpha \leqslant c_1 \|Y\|_\infty + T^\alpha \|Y^{[2]}\|_{2\alpha}$

(b) $\|Y^{[2]}\|_{2\alpha} \leqslant c_2 \|Y\|_\infty + c_2' T^\alpha \|Y^{[2]}\|_{2\alpha}$

for some constants c_1, c_2, c_2' which depend on everything $(\alpha, X, \sigma, Z, Z')$ but not on T.

For $T>0$ small enough, (b) $\Rightarrow$ $\|Y^{[2]}\|_{2\alpha} \leqslant 2c_2 \|Y\|_\infty$

(a) $\Rightarrow$ $\|\delta Y\|_\alpha \leqslant c_1 \|Y\|_\infty + 2c_2 T^\alpha \|Y\|_\infty$

For $T>0$ small enough $\|\delta Y\|_\alpha \leqslant 2c_1 \underbrace{\|Y\|_\infty}_{|Y_0| + T^\alpha \|\delta Y\|_\alpha}$

For $T>0$ small enough $\|\delta Y\|_\alpha \leqslant 3c_1 |Y_0|$. $\blacksquare$