WED 20 NOV 2024

Reminders

Fix: • $X : [0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$, $\alpha \in (\frac{1}{2}, 1]$ (YOUNG CASE)

• $\sigma : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k)$ (sufficiently regular)

Finite difference equation for $Z : [0,T] \to \mathbb{R}^k$

$$(*') \qquad \underbrace{\delta Z_{st}}_{Z_t - Z_s} = \sigma(Z_s) \underbrace{\delta X_{st}}_{X_t - X_s} + \underbrace{o(t-s)}_{Z^{[2]}_{st}} \qquad 0 \leq s < t \leq T$$

i.e. $Z^{[2]}_{st} := \delta Z_{st} - \sigma(Z_s)\delta X_{st} = o(t-s)$ (remainder)

Theorem (SEWING BOUND) If $R_{st} = o(t-s)$ then

$$\forall \eta > 1 : \quad \|R\|_\eta \leq K_\eta \|\delta R\|_\eta \qquad K_\eta = (1-2^{1-\eta})^{-1}$$

$$\sup_{0 \leq s < t \leq T} \frac{|R_{st}|}{(t-s)^\eta} \qquad \delta R_{sut} := R_{st} - R_{su} - R_{ut}$$

Basic bounds:

(1) $\|f\|_\infty \leq |f_0| + T^\alpha \|\delta f\|_\alpha$

(2) $\|F\|_\alpha \leq T^\beta \|F\|_{\alpha+\beta} \qquad \forall \beta > 0$

(3) $F_{st} = g_s H_{st}$ or $F_{st} = g_t H_{st} \Rightarrow \|F\|_\eta \leq \|g\|_\infty \cdot \|H\|_\eta$

(4) $F_{sut} = G_{su} H_{ut} \Rightarrow \|F\|_{\eta+\eta'} \leq \|G\|_\eta \cdot \|H\|_{\eta'}$

Uniqueness

Theorem. Let $X \in \mathscr{C}^\alpha$ with $\alpha \in (\frac{1}{2}, 1]$. Let $\sigma \in C^2$. (or $\nabla\sigma$ Lipschitz)
Then $\forall z_0 \in \mathbb{R}^k$ there is at most one solution Z of (*').

Proof. Fix Z, Z' two solutions of (*') and set

$$Y_t := Z_t - Z'_t \qquad\qquad Y^{[2]}_{st} := Z^{[2]}_{st} - Z'^{[2]}_{st}$$

We show that for $T>0$ small enough (...)

$$\|Y\|_\infty \leq 2\,|Y_0| \quad \Rightarrow \quad Z = Z' \text{ if } Z_0 = Z'_0 = z_0.$$

How? For some $c_1, c'_1, c_2, c'_2 < \infty$ (depending on everything but T)

(a) $$\|\delta Y\|_\alpha \leq c_1 \|Y\|_\infty + c'_1\, T^\alpha\, \|Y^{[2]}\|_{2\alpha}$$

(b) $$\|Y^{[2]}\|_{2\alpha} \leq c_2 \|Y\|_\infty + c'_2\, T^\alpha\, \|Y^{[2]}\|_{2\alpha}$$

Then for $T>0$ small enough:

$$\|Y^{[2]}\|_{2\alpha} \leq 2 c_2 \|Y\|_\infty$$

$$\|\delta Y\|_\alpha \leq 2 c_1 \|Y\|_\infty$$

(1) $$\|Y\|_\infty \leq |Y_0| + \underbrace{T^\alpha \|\delta Y\|_\alpha}_{2 c_1 T^\alpha \|Y\|_\infty} \leq 2\,|Y_0|$$

It remains to prove (a) and (b).

We recall two results stated last time

Lemma: If $R_{st} = W_s \delta X_{st}$ then $\delta R_{sut} = -\delta W_{su} \delta X_{ut}$

Lemma: Let $\sigma \in C^2$ and set

$$C'_R := \sup\{|\nabla\sigma(x)| : |x| \leq R\} \qquad C''_R := \sup\{|\nabla^2\sigma(x)| : |x| \leq R\}$$

Then for $|x|, |\bar{x}|, |y|, |\bar{y}| \leq R$:

$$|(\sigma(x)-\sigma(y)) - (\sigma(\bar{x})-\sigma(\bar{y}))| \leq C'_R \,|(x-y)-(\bar{x}-\bar{y})|$$
$$+ C''_R \{|x-y| + |\bar{x}-\bar{y}|\} \,|y-\bar{y}|.$$

Proof of (a) $\quad \|\delta Y\|_\alpha \leq c_1 \|Y\|_\infty + c'_1 T^\alpha \|Y^{[2]}\|_{2\alpha}$

$$\delta Z_{st} = \sigma(Z_s)\delta X_{st} + Z^{[2]}_{st} \qquad + \text{ same for } Z'$$

$$\delta Y_{st} = \underbrace{(\sigma(Z_s) - \sigma(Z'_s))}_{|\cdot| \leq C_1 \cdot |Z_s - Z'_s| = C_1 \cdot |Y_s|} \delta X_{st} + Y^{[2]}_{st}$$

$$\|\delta Y\|_\alpha \leq \underbrace{\|\sigma(Z) - \sigma(Z')\|_\infty}_{\leq C_1 \cdot \|Y\|_\infty} \cdot \|\delta X\|_\alpha + \underbrace{\|Y^{[2]}\|_\alpha}_{T^\alpha \cdot \|Y^{[2]}\|_{2\alpha}}$$

Define $\quad C' = C'_R = \sup\{|\nabla\sigma(x)| : |x| \leq R\} \qquad R = \|Z\|_\infty \vee \|Z'\|_\infty$

$\quad C'' = C''_R = \ldots$

Proof of (b). $\quad \|Y^{[2]}\|_{2\alpha} \leq c_2 \|Y\|_\infty + c'_2 T^\alpha \|Y^{[2]}\|_{2\alpha}$

Sewing bound: $\|Y^{[2]}\|_{2\alpha} \leq K_{2\alpha} \|\delta Y^{[2]}\|_{2\alpha}$

$2\alpha > 1$

$$\delta Y^{[2]}_{sut} = \underbrace{\left(\delta\sigma(z)_{su} - \delta\sigma(z')_{su}\right)}_{|\cdot| \leq C' \cdot |\delta z_{su} - \delta z'_{su}| + C'' \cdot \{|\delta z_{su}| + |\delta z'_{su}|\} \cdot |\underbrace{z_s - z'_s}_{Y_s}|} \delta X_{ut}$$

δY_{su}

$$\Rightarrow \|\delta Y^{[2]}\|_{2\alpha} \leq \underbrace{\|\delta\sigma(z) - \delta\sigma(z')\|_\alpha}_{} \|\delta X\|_\alpha$$

$$\leq C' \cdot \|\delta Y\|_\alpha + C'' \cdot \{\|\delta z\|_\alpha + \|\delta z'\|_\alpha\} \cdot \|Y\|_\infty$$

$$\Rightarrow \|Y^{[2]}\|_{2\alpha} \leq K_{2\alpha} \|\delta X\|_\alpha \cdot \left\{ C' \|\delta Y\|_\alpha + C''\{\|\delta z\|_\alpha + \|\delta z'\|_\alpha\} \|Y\|_\infty \right\}$$

$$= c_2 \cdot \|Y\|_\infty + \tilde{c}_2 \|\delta Y\|_\alpha$$

However $\|\delta Y\|_\alpha \leq c_1 \|Y\|_\infty + c_1' T^\alpha \|Y^{[2]}\|_{2\alpha}$ (a)

Finally $\|Y^{[2]}\|_{2\alpha} \leq \underbrace{\hat{c}_2}_{c_2 + \tilde{c}_2 c_1} \|Y\|_\infty + \tilde{c}_2 \cdot c_1' \cdot T^\alpha \|Y^{[2]}\|_{2\alpha}$

A priori estimates $(\alpha \in (\frac{1}{2}, 1])$

For any solution Z of $(*')$ $\qquad \delta Z_{st} = \sigma(Z_s)\,\delta X_{st} + \underbrace{Z^{[2]}_{st}}_{o(t-s)}$

- $\| Z^{[2]} \|_{2\alpha} \leq C_{\alpha, X, \sigma} \, \| \delta Z \|_{\alpha}$ $\qquad C_{\alpha,X,\sigma} = K_{2\alpha} \|\delta X\|_\alpha \|\nabla\sigma\|_\infty$

- $\| \delta Z \|_\alpha \leq 2 \, \| \delta X \|_\alpha \, |\sigma(Z_0)|$ if $T^\alpha \leq \varepsilon_{\alpha,X,\sigma} = \dfrac{1}{2(K_{2\alpha}+3)\|\delta X\|_\alpha \|\nabla\sigma\|_\infty}$

Existence

Theorem (EXISTENCE) If σ is globally Lipschitz $(\|\nabla\sigma\|_\infty < \infty)$
Then $\forall\, z_0 \in \mathbb{R}^K$ there exists a solution $Z = (Z_t)_{t\in[0,T]}$ to $(*')$

We will prove it assuming $T > 0$ small enough.

Fix a partition $\Pi = \{0 = t_0 < t_1 < \dots < t_{|\Pi|} = T\}$ of $[0,T]$.

We define an approximate solution $Z^\Pi = (Z^\Pi_t)_{t\in\Pi}$ by

$$Z^\Pi_0 = z_0, \qquad Z^\Pi_{t_{i+1}} := Z^\Pi_{t_i} + \sigma(Z^\Pi_{t_i}) \cdot \delta X_{t_i t_{i+1}}$$

Define the remainder $R_{st} = (Z^\Pi)^{[2]}_{st}$ by

$$R_{st} = \delta Z^\Pi_{st} - \sigma(Z^\Pi_s)\,\delta X_{st} \qquad \text{for } s < t \in \Pi.$$

By construction $R_{t_i t_{i+1}} \equiv 0 \quad \forall i = 0, \dots, \#\Pi - 1$.

Then $\quad \|R\|_{2\alpha}^{\Pi} \leq C_{2\alpha} \|\delta R\|_{2\alpha}^{\Pi}$ (DISCRETE SEWING BOUND)

Theorem (A PRIORI ESTIMATES FOR APPROXIMATE SOLUTIO)

Assume $X \in \mathcal{C}^\alpha$, with $\alpha \in (\frac{1}{2}, 1]$, and $\|\nabla \sigma\|_\infty < \infty$. Then

- $\|R\|_{2\alpha}^{\Pi} \leq \tilde{C}_{\alpha, X, \sigma} \|\delta Z^\Pi\|_\alpha^\Pi$
- $\|\delta Z^\Pi\|_\alpha^\Pi \leq 2 \|\delta X\|_\alpha \cdot |\sigma(z_0)|$ for $T^\alpha \leq \tilde{\varepsilon}_{\alpha, X, \sigma}$.

Proof of existence.

For $n \in \mathbb{N}$ define $\Pi_n := \{ \frac{i}{2^n} : i \in \mathbb{N}_0 \} \cap [0, T]$.

We set for short $Z^n = Z^{\Pi_n}$ and $R^n = (Z^{\Pi_n})^{[2]}$, i.e.

$$R^n_{st} = \delta Z^n_{st} - \sigma(Z_{st}) \delta X_{st}$$

We extend $Z^n = (Z^n_t)_{t \in \Pi_n}$ by linear interpolation to $Z^n = (Z^n_t)_{t \in [0,T]}$. Then we can show (exercise) that

$$\|\delta Z^n\|_\alpha \leq 3 \|\delta Z^n\|_\alpha^{\Pi_n} \qquad \alpha \in (\tfrac{1}{2}, 1].$$

We claim that $(Z^n)_{n \in \mathbb{N}}$ is relatively compact in $C([0,T], \mathbb{R}^k)$.

Indeed, by the a priori estimate just stated,

$$\|\delta Z^n\|_\alpha \leq 6 \cdot \|\delta X\|_\alpha \cdot |\sigma(z_0)| \quad \text{Assuming } T^\alpha \leq \tilde{\varepsilon}_{\alpha, X, \sigma}.$$

Therefore $|Z^n_t - Z^n_s| \leq \|\delta Z^n\|_\alpha (t-s)^\alpha$

$\Rightarrow (Z^n)_{n\in\mathbb{N}}$ are EQUI-CONTINUOUS.

They are also equi-bounded because $Z^n_0 = z_0 = \text{const}$.

$\Rightarrow$ by Arzelà-Ascoli $(Z^n)_{n\in\mathbb{N}}$ are relatively compact.

We can thus extract a converging subsequence

$$Z^{n_k} \longrightarrow Z \quad \text{in } C([0,T],\mathbb{R}).$$

It remains to show that the limit function Z is a solution.

By a priori estimate: $\|R^n\|^{\Pi_n}_{2\alpha} \leq \underbrace{\tilde{C}_{\alpha,X,\sigma}\, 2\, \|\delta X\|_\alpha \cdot |\sigma(z_0)|}_{\hat{C}<\infty}$

$$|R^n_{st}| = |\delta Z^n_{st} - \sigma(Z^n_s)\delta X_{st}| \leq \hat{C}(t-s)^{2\alpha} \quad \forall\, s<t\in\Pi_n.$$

(as $n\to\infty$) $|\delta Z_{st} - \sigma(Z_s)\delta X_{st}| \leq \hat{C}(t-s)^{2\alpha}$ $\forall s,t$ dyadic rationals.

By continuity of Z, the same bound holds $\forall\, 0\leq s<t\leq T$.

Thus $|Z^{[2]}_{st}| \doteq |\delta Z_{st} - \sigma(Z_s)\delta X_{st}| \leq \hat{C}(t-s)^{2\alpha} = o(t-s)$

that is $\delta Z_{st} = \sigma(Z_s)\delta X_{st} + o(t-s)$ because $2\alpha>1$

i.e. Z is a solution of $(*')$. ∎

Continuity of the solution map.

Fix $X = (X_t)_{t\in[0,T]} \in \mathcal{C}^\alpha$ with $\alpha \in (\frac{1}{2}, 1]$.

Consider σ of class C^2 with $\|\nabla\sigma\|_\infty < \infty$, $\|\nabla^2\sigma\|_\infty < \infty$.

Then we have global existence and uniqueness for $(*')$.

We can then define a SOLUTION MAP

$$\Phi : \mathbb{R}^k \times \mathcal{C}^\alpha \longrightarrow \mathcal{C}^\alpha$$

$$(z_0, X) \longmapsto Z = (Z_t)_{t\in[0,T]} = \text{unique solution of } (*') \text{ with } Z_0 = z_0.$$

We can show that $\Phi(\cdot,\cdot)$ is a locally Lipschitz map.

Theorem (CONTINUITY OF THE SOLUTION MAP)

Fix $T>0$, $\alpha \in (\frac{1}{2}, 1]$. Consider σ such that

$$\|\nabla\sigma\|_\infty \vee \|\nabla^2\sigma\|_\infty \leq D < \infty.$$

Consider starting points $z_0, z_0' \in \mathbb{R}^k$ such that

$$|\sigma(z_0)| \vee |\sigma(z_0')| \leq M_0 < \infty$$

Consider $X, X' \in \mathcal{C}^\alpha$ such that

$$\|\delta X\|_\alpha \vee \|\delta X'\|_\alpha \leq M < \infty.$$

Then, for any $D, M_0, M < \infty$, the unique solution Z, Z' of $(*')$ satisfy

$$\underbrace{\|Z - Z'\|_\infty + \|\delta Z - \delta Z'\|_\alpha}_{\|Z - Z'\|_{\mathcal{C}^\alpha}} \leq \overset{2(DM+1)}{\widetilde{C_M}} \cdot |Z_0 - Z'_0| + 6 M_0 \|\delta X - \delta X'\|_\alpha$$

provided $T < \hat{\tau}_{\alpha, D, M_0, M} < \infty$.

<u>Remark</u>. Recall our equation $\delta Z_{st} = \sigma(Z_s) \delta X_{st} + Z^{[2]}_{st}$

with $Z^{[2]}_{st} = o(t-s)$.

If $X \in \mathcal{C}^\alpha$, $\alpha \in (\frac{1}{2}, 1]$, the a priori estimate that we proved shows that

$$\|Z^{[2]}\|_{2\alpha} \leq C_{\alpha, X, \sigma} \cdot \|\delta X\|_\alpha < \infty .$$

that is $Z^{[2]}_{st} = O\left((t-s)^{2\alpha}\right) = o(t-s)$.

3 - DIFFERENCE EQUATIONS: ROUGH CASE.

So far we have studied the equation

$$(*') \qquad \delta Z_{st} = \sigma(Z_s)\,\delta X_{st} + \underbrace{o(t-s)}_{Z^{[2]}_{st}}$$

and we proved well-posedness for $X \in \mathcal{C}^\alpha$ with $\alpha \in (\frac{1}{2}, 1]$.

When $\alpha \leq \frac{1}{2}$ the formulation $(*')$ is no longer appropriate because, in general, it admits no solution!

Indeed, if Z solves $(*')$, taking δ of both sides gives

$$\delta\sigma(Z)_{su} \cdot \delta X_{ut} = \underbrace{\delta Z^{[2]}_{sut}}_{Z^{[2]}_{st} - Z^{[2]}_{su} - Z^{[2]}_{ut}} = o(t-s) \qquad s<u<t$$

If $X \in \mathcal{C}^\alpha$ then $\delta X_{ut} = O((t-u)^\alpha)$ but not better, in general.

We showed that $Z \in \mathcal{C}^\alpha$, but not better, in general, hence $\delta Z_{su} = O((u-s)^\alpha) \Rightarrow \delta\sigma(Z_{su}) = O((u-s)^\alpha)$.

Then $\delta\sigma(Z)_{su} \cdot \delta X_{ut} = O\big((u-s)^\alpha (t-u)^\alpha\big) = O\big((t-s)^{2\alpha}\big)$ but not better, in general. Then for $\alpha \leq \frac{1}{2}$ we cannot hope that the LHS is $o(t-s)$.

How to proceed?

Consider again the case when $X \in C^1$ and the equation

$$* \qquad \dot{Z}_t = \sigma(Z_t)\dot{X}_t$$

$$\iff \qquad Z_t = Z_0 + \int_0^t \sigma(Z_u)\dot{X}_u \, du$$

For $0 \leq s \leq t \leq T$ we can write

$$Z_t = Z_s + \int_s^t \sigma(Z_u)\dot{X}_u \, du$$

$$= Z_s + \sigma(Z_s)\cdot \delta X_{st} + \int_s^t \{\sigma(Z_u) - \sigma(Z_s)\}\dot{X}_u \, du$$

Let us define

$$\sigma_2(z) := \nabla\sigma(z)\cdot\sigma(z)$$

$\sigma : \mathbb{R}^K \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^K)$ $\qquad$ $\nabla\sigma : \mathbb{R}^K \to \mathcal{L}(\mathbb{R}^K, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^K))$

$\sigma(z) = \left(\sigma(z)^i_j\right)_{\substack{i=1,\dots,K \\ j=1,\dots,d}}$ $\qquad$ $\nabla\sigma(z) = \left(\partial_{i'}\sigma(z)^i_j\right)_{\substack{i,i'=1,\dots,K \\ j=1,\dots,d}}$

$$\sigma_2(z) = \nabla\sigma(z)\,\sigma(z)$$

$$= \left(\sigma_2(z)^i_{jl} = \sum_{i'=1}^K \partial_{i'}\sigma(z)^i_j \; \sigma(z)^{i'}_l \right)_{\substack{i=1,\dots,K \\ j,l=1,\dots,d}}$$

$$\sigma(Z_u) - \sigma(Z_s) = \int_s^u \frac{d}{dr}\sigma(Z_r)\, dr$$

$$\frac{d}{dr}\sigma(Z_r) = \nabla\sigma(Z_r)\cdot\underbrace{\dot{Z}_r}_{\sigma(Z_r)\dot{X}_r}$$

$$= \int_s^u \sigma_2(Z_r)\dot{X}_r \, dr$$

$$= \sigma_2(Z_s)\,\delta X_{su} + \int_s^u \{\sigma_2(Z_r) - \sigma_2(Z_s)\}\dot{X}_r\,dr$$

We thus get

$$Z_t = Z_s + \sigma(Z_s)\,\delta X_{st} + \int_s^t \left[\sigma_2(Z_s)\,\delta X_{su}\right]\dot{X}_u\,du + R_{st}$$

$$R_{st} = \int_s^t \left(\int_s^u \{\sigma_2(Z_r) - \sigma_2(Z_s)\}\,\dot{X}_r\,dr\right)\dot{X}_u\,du$$

$$= o\left((t-s)^2\right)$$

Let us set, for $a, b \in \mathbb{R}^K$, $a \otimes b = (a_i b_j)_{\substack{1 \le i \le K \\ 1 \le j \le K}} \in \underbrace{\mathbb{R}^K \otimes \mathbb{R}^K}_{\mathbb{R}^{2K}}$

Then we can write

$$\left[\sigma_2(Z_s)\,\delta X_{su}\right]\dot{X}_u = \sigma_2(Z_s)\left(\delta X_{su} \otimes \dot{X}_u\right)$$

$$\left[\sigma_2(Z_s)\left(\delta X_{su} \otimes \dot{X}_u\right)\right]^i = \sum_{j,l=1}^{d} \sigma_2(Z_s)^i_{jl}\,\delta X^l_{su}\,\dot{X}^j_u$$

Finally we get

$$\delta Z_{st} = \sigma(Z_s)\underbrace{\delta X_{st}}_{\mathbb{X}^1_{st}} + \sigma_2(Z_s)\cdot\underbrace{\int_s^t (X_u - X_s)\otimes\dot{X}_u\,du}_{\mathbb{X}^2_{st}} + R_{st}$$

where $R_{st} = o\left((t-s)^2\right)$ if $X \in C^1$.

The idea is now to replace R_{st} by $o(t-s)$ and consider

$$(*'')\quad \delta Z_{st} = \sigma(Z_s)\,\mathbb{X}^1_{st} + \sigma_2(Z_s)\,\mathbb{X}^2_{st} + o(t-s)$$

where $\mathbb{X}^1_{st} = X_t - X_s$ and $\mathbb{X}^2_{st} = \int_s^t (X_u - X_s) \otimes \dot{X}_u \, du$

We have a problem when X is not differentiable, because $\mathbb{X}^2_{st}$ still contains the derivative $\dot{X}$.
The idea is now to ASSUME that the object $\mathbb{X}^2_{st}$ is given, together with X. Then the equation $(*'')$ is meaningful, i.e. we can look for solutions $Z = (Z_t)$ which solve it.

How to define or "assign" $\mathbb{X}^2_{st}$?

If $X = B$ is Brownian motion, we have a notion of integral:

$$\mathbb{X}^2_{st} = \mathbb{B}^2_{st} = \int_s^t (B_u - B_s)\, dB_u$$

Actually we have multiple (non canonical) notions of integration which lead to different $\mathbb{X}^2_{st}$. This is not a problem! Any "reasonable" choice of $\mathbb{X}^2$ leads to a different meaningful equation $(*'')$.

What if $X \in \mathcal{C}^\alpha$ is a generic Hölder path with $\alpha \leq \frac{1}{2}$?

In this case we can define a "reasonable" notion of the iterated integral $\mathbb{X}^2_{st}$ by requiring two natural properties:

(1) ALGEBRAIC IDENTITY (CHEN'S RELATION)

$$\delta \mathbb{X}^2_{sut} := \mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \underbrace{\delta X_{su}}_{\mathbb{X}^1_{su}} \otimes \underbrace{\delta X_{ut}}_{\mathbb{X}^1_{ut}}$$

Indeed when $X \in C^1$ we have

$$\delta \mathbb{X}^2_{sut} = \int_s^t (\cancel{X_r} - X_s) \otimes \dot{X}_r \, dr - \int_s^u (\cancel{X_r} - X_s) \otimes \dot{X}_r \, dr - \int_u^t (\cancel{X_r} - X_u) \otimes \dot{X}_r \, dr$$

$$= -X_s (X_t - X_s) + X_s (X_u - X_s) + X_u (X_t - X_u)$$

$$\overset{!}{=} (X_u - X_s) \otimes (X_t - X_u)$$

(2) ANALYTIC BOUNDS: we have

$$|\mathbb{X}^1_{st}| = |X_t - X_s| = O((t-s)^\alpha)$$

$$|\mathbb{X}^2_{st}| = O((t-s)^{2\alpha})$$

$$\mathbb{X}^2_{st} = \int_s^t \underbrace{(X_r - X_s)}_{(r-s)^\alpha \leq (t-s)^\alpha} \dot{X}_r \, dr$$