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Last times: for $X \in C^1$ and $\sigma \in C^2$, the solution Z of

$$(\ast) \qquad \dot{Z}_t = \sigma(Z_t)\, \dot{X}_t$$

admits the following expansion, uniformly in $0 \leq s < t \leq T$:

$$(\ast') \qquad \underbrace{\delta Z_{st}}_{(Z_t - Z_s)} = \sigma(Z_s)\, \mathbb{X}^1_{st} + o(t-s) \qquad O((t-s)^2)$$

with $\mathbb{X}^1_{st} := \delta X_{st} = X_t - X_s$

$$(\ast'') \qquad \delta Z_{st} = \sigma(Z_s)\, \mathbb{X}^1_{st} + \sigma_2(Z_s)\, \mathbb{X}^2_{st} + o(t-s) \qquad O((t-s)^3)$$

with $\sigma_2(z) := \nabla\sigma(z) \circ \sigma(z)$ $\qquad \mathbb{X}^2_{st} := \int_s^t \delta X_{su} \otimes \dot{X}_u \, du$

$\sigma_2(z) \in \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^K) \simeq \mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^K))$ $\qquad \mathbb{X}^2_{st} \in \mathbb{R}^d \otimes \mathbb{R}^d$, $\quad \mathbb{X}^2_{st} = \iint_{s<v<u<t} \dot{X}_v \otimes \dot{X}_u \, dv\, du$

Key property: CHEN'S RELATION for $s < u < t$

$$\delta \mathbb{X}^2_{sut} := \mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut} = (X_u - X_s) \otimes (X_t - X_u)$$

Also analytic bounds:

$$|\mathbb{X}^1_{st}| \lesssim (t-s)^{\alpha} \qquad |\mathbb{X}^2_{st}| \lesssim (t-s)^{2\alpha}$$

$= O(t-s)$

$\leq C \cdot (t-s)$

Definition (ROUGH PATH). Fix $d \in \mathbb{N}$ and $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. Given a path $X = (X_t)_{t \in [0,T]}$ in $\mathbb{R}^d$ of class $\mathcal{C}^\alpha$, we call α-ROUGH PATH OVER X a pair $(\mathbb{X}^1, \mathbb{X}^2)$ of functions

$$\mathbb{X}^1 : [0,T]^2_{\leq} \longrightarrow \mathbb{R}^d$$
$$\mathbb{X}^2 : [0,T]^2_{\leq} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$$

which satisfy the following properties:

(1) ALGEBRAIC RELATIONS: $\forall\, 0 \leq s < u < t \leq T$

$$\mathbb{X}^1_{st} = \delta X_{st} = X_t - X_s \qquad \underbrace{\delta \mathbb{X}^2_{sut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}}_{\text{CHEN'S RELATION}}$$

(2) ANALYTIC BOUNDS: uniformly in $0 \leq s < t \leq T$

$$|\mathbb{X}^1_{st}| \lesssim (t-s)^\alpha \qquad |\mathbb{X}^2_{st}| \lesssim (t-s)^{2\alpha}$$

$$\|\mathbb{X}^1\|_\alpha < \infty \qquad \|\mathbb{X}^2\|_{2\alpha} < \infty$$

Remark: For any path X of class C^α, $\frac{1}{3} < \alpha \leq \frac{1}{2}$, there always exists an α-rough path $(\mathbb{X}^1, \mathbb{X}^2)$ over X. This is a non-trivial result (Lyons-Victoir).

Lemma ((NON-)UNIQUENESS). Given two α-rough paths $(\mathbb{X}^1, \mathbb{X}^2)$ and $(\tilde{\mathbb{X}}^1, \tilde{\mathbb{X}}^2)$ over the same path $X \in \mathcal{C}^\alpha$, $\frac{1}{3} < \alpha \leq \frac{1}{2}$, there is a path $f \in \mathcal{C}^{2\alpha}$ such that $\mathbb{X}^2 - \tilde{\mathbb{X}}^2 = \delta f$.

Vice versa, if $(\mathbb{X}^1, \mathbb{X}^2)$ is an α-rough path over X, then $(\tilde{\mathbb{X}}^1 := \mathbb{X}^1, \tilde{\mathbb{X}}^2 := \mathbb{X}^2 + \delta f)$ with $f \in \mathcal{C}^{2\alpha}$ is also an α-rough path over X.

Proof. If $(\mathbb{X}^1, \mathbb{X}^2)$ and $(\tilde{\mathbb{X}}^1, \tilde{\mathbb{X}}^2)$ are α-rough paths over the same path X, then $\mathbb{X}^1 = \tilde{\mathbb{X}}^1 = \delta X$ and

$$\delta \mathbb{X}^2 = \delta \tilde{\mathbb{X}}^2 \quad \text{by Chen's relation}$$

$$\Rightarrow \quad \delta(\mathbb{X}^2 - \tilde{\mathbb{X}}^2) = 0$$

$$\Rightarrow \quad \mathbb{X}^2 - \tilde{\mathbb{X}}^2 = \delta f \quad \text{for some } f : [0,T] \to \mathbb{R}^{d\times d}$$

Finally, the analytic bounds in the def. of RP give

$$\|\delta f\|_{2\alpha} = \|\mathbb{X}^2 - \tilde{\mathbb{X}}^2\|_{2\alpha} \leq \|\mathbb{X}^2\|_{2\alpha} + \|\tilde{\mathbb{X}}^2\|_{2\alpha} < \infty$$

that is $f \in \mathcal{C}^{2\alpha}$. ∎

Next time we will prove that, if $X = B$ is Brownian motion in $\mathbb{R}^d$, we obtain a rough path over B (a.s.) by

$$(\mathbb{B}^{ITO})^1 = \delta B \qquad (\mathbb{B}^{ITO})^2_{st} := \int_s^t (B_u - B_s) \otimes dB_u \quad (\text{ITO})$$

Another (different!) RP over B is

$$(\mathbb{B}^{STRAT})^1 = \delta B \qquad (\mathbb{B}^{STRAT})^2_{st} := \int_s^t (B_u - B_s) \otimes \circ\, dB_u \quad (\text{Strat.})$$

$$\stackrel{!}{=} (\mathbb{B}^{ITO})^2_{st} + \tfrac{1}{2}(t-s)\, I$$

Remark (ROUGH PATHS AS "PATHS") If $(\mathbb{X}^1, \mathbb{X}^2)$ is a RP over X, then $\mathbb{X}^2_{st}$ is determined by X and by the one-variable function $t \mapsto \mathbb{X}^2_{0t}$. Indeed let us set

$$\mathbb{I}_t := \mathbb{X}^2_{0t} + X_0 \otimes (X_t - X_0)$$

$$= \text{"} \int_0^t X_u \otimes \dot{X}_u \, du \text{"}$$

then $\mathbb{X}^2_{st} = \mathbb{X}^2_{0t} - \mathbb{X}^2_{0s} - \underbrace{\delta \mathbb{X}_{0st}}_{\mathbb{X}^1_{0s} \otimes \mathbb{X}^1_{st} = (X_s - X_0) \otimes (X_t - X_s)}$, hence $\mathbb{X}^2_{st}$ is determined by $\mathbb{I}_t$ and X

Vice versa, a function $\mathbb{I}_t$ determines a rough path if

$$|\underbrace{\underbrace{\mathbb{I}_t - \mathbb{I}_s}_{\text{"} \int_s^t X_u \otimes \dot{X}_u \, du \text{"}} - X_s \otimes (X_t - X_s)}_{\text{def } \mathbb{X}^2_{st}}| \lesssim (t-s)^{2\alpha}$$

Henceforth we fix dimensions $d, k \in \mathbb{N}$, exp. $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, and

- $X : [0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$
- $(\mathbb{X}^1, \mathbb{X}^2)$ an α-RP over X
- $\sigma : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k)$ of class C^1.

- Define $\sigma_2 : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^k)$ by

$$\sigma_2(z) := \nabla\sigma(z) \circ \sigma(z)$$

we then consider the finite difference equation for an unknown path $Z : [0,T] \to \mathbb{R}^k$ given by

$$(*'') \qquad \delta Z_{st} = \sigma(Z_s)\,\mathbb{X}^1_{st} + \sigma_2(Z_s)\,\mathbb{X}^2_{st} + o(t-s)$$

(unif. for $0 \leq s < t \leq T$)

Given any path Z, we define

$$Z^{[3]}_{st} := \delta Z_{st} - \sigma(Z_s)\,\mathbb{X}^1_{st} - \sigma_2(Z_s)\,\mathbb{X}^2_{st}$$

$$Z^{[2]}_{st} := \delta Z_{st} - \sigma(Z_s)\,\mathbb{X}^1_{st}$$

$$Z^{[1]}_{st} := \delta Z_{st}$$

Note that Z solves $(*'')$ iff $Z^{[3]}_{st} = o(t-s)$.

A PRIORI ESTIMATES

<u>Lemma</u>. Assuming $\sigma \in C^2$, Any solution Z of $(*'')$ is of class $\mathscr{C}^\alpha$ (like X).

<u>Proof</u>. By $(*'')$ we have

$$|\delta Z_{st}| \leq \|\sigma(Z)\|_\infty \cdot \underbrace{|\mathbb{X}^1_{st}|}_{\lesssim (t-s)^\alpha} + \|\sigma_2(Z)\|_\infty \cdot \underbrace{|\mathbb{X}^2_{st}|}_{\lesssim (t-s)^{2\alpha} \lesssim (t-s)^\alpha} + \underbrace{o(t-s)}_{\lesssim (t-\alpha)^\alpha}$$

This shows that $|\delta Z_{st}| \lesssim (t-s)^\alpha$, i.e. $Z \in \mathcal{C}^\alpha$, provided $\|\sigma(Z)\|_\infty < \infty$, $\|\sigma_2(Z)\|_\infty < \infty$. This follows if we show that $\|Z\|_\infty < \infty$.

By Hp $\forall \varepsilon > 0 \; \exists \delta_\varepsilon > 0$: $\underbrace{|a(t-s)|}_{Z^{[3]}_{st}} \leq \varepsilon(t-s) \leq \varepsilon T$

$\forall\, 0 \leq s < t \leq T$ with $|t-s| \leq \delta_\varepsilon$.

Fix $\varepsilon = 1$ and partition $[0,T]$ into intervals $[a,b]$ with $b-a < \delta_1$. It is enough to show that

$$\sup_{t\in[a,b]} |Z_t| \leq |Z_a| + \sup_{t\in[a,b]} |\delta Z_{at}|$$

Now
$$|\delta Z_{at}| = \left| Z^{[3]}_{at} - \sigma(Z_a)\mathbb{X}^1_{at} - \sigma_2(Z_a)\mathbb{X}^2_{at} \right|$$
$$\leq 1\cdot T + |\sigma(Z_a)|\cdot\|\mathbb{X}^1_{a\cdot}\|_\infty + |\sigma_2(Z_a)|\cdot\|\mathbb{X}^2_{a\cdot}\|_\infty$$
$$< \infty.$$

∎

Theorem (ROUGH A PRIORI ESTIMATES) Let $X \in \mathcal{C}^\alpha$ with $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. Let $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ be an α-RP over X. Let σ and σ_2 be globally Lipschitz: $\|\nabla\sigma\|_\infty < \infty$, $\|\nabla\sigma_2\|_\infty < \infty$.

Any solution Z of $(*')$ satisfies

(i) $\|Z^{[3]}\|_{3\alpha} \leq K_{3\alpha} \cdot C'_{\alpha,\sigma,\mathbb{X}} \cdot \left(\|\delta Z\|_\alpha + \|Z^{[2]}\|_{2\alpha} \right)$

with $K_{3\alpha} = (1 - 2^{1-3\alpha})^{-1}$ and

$$c^1_{\alpha,\sigma,\mathbb{X}} := \|\nabla\sigma\|_\infty \|\mathbb{X}^1\|_\alpha + \|\nabla\sigma_2\|_\infty \|\mathbb{X}^2\|_{2\alpha} + \left(\|\nabla\sigma\|^2_\infty + \|\nabla\sigma_2\|_\infty\right) \|\mathbb{X}^1\|^2_\alpha$$

Moreover, for $T>0$ small enough ($T^\alpha < \varepsilon^1_{\alpha,\sigma,\mathbb{X}}$),

(iii) $\|\delta Z\|_\alpha + \|Z^{[2]}\|_{2\alpha} \leq 2\left(\sigma(Z_0)\|\mathbb{X}^1\|_\alpha + \sigma_2(Z_0)\|\mathbb{X}^2\|_{2\alpha}\right)$

with $\varepsilon^1_{\alpha,\sigma,\mathbb{X}} = \dfrac{1}{4(K_{3\alpha}+3)(c^1_{\alpha,\sigma,\mathbb{X}}+1)}$

<u>Proof</u>. By def. Z is a solution of $(*'')$ iff $Z^{[3]}_{st} = o(t-s)$.

We can apply the SEWING BOUND: $\|Z^{[3]}\|_{3\alpha} \leq K_{3\alpha} \cdot \|\delta Z^{[3]}\|_{3\alpha}$

Recall: $Z^{[3]}_{st} = \delta Z_{st} - \sigma(Z_s)\underbrace{\mathbb{X}^1_{st}}_{\delta X_{st}} - \sigma_2(Z_s)\mathbb{X}^2_{st}$

<u>Lemma</u>. $\delta Z^{[3]}_{sut} = \underbrace{\left(\sigma(Z_u) - \sigma(Z_s) - \sigma_2(Z_s)\mathbb{X}^1_{su}\right)}_{B_{su}} \mathbb{X}^1_{ut} + \underbrace{\left(\sigma_2(Z_u) - \sigma_2(Z_s)\right)}_{\delta\sigma_2(Z)_{su}} \mathbb{X}^2_{ut}$

Then $\|\delta Z^{[3]}\|_{3\alpha} \leq \|B\|_{2\alpha}\|\mathbb{X}^1\|_\alpha + \underbrace{\|\delta\sigma_2(Z)\|_\alpha}_{\leq \|\nabla\sigma_2\|_\infty \cdot \|\delta Z\|_\alpha} \|\mathbb{X}^2\|_{2\alpha}$

Lemma: if $\sigma \in C^1$, then $\forall\, z_1, z_2 \in \mathbb{R}^k$:

$$\sigma(z_2) - \sigma(z_1) - \sigma_2(z_1)x = \int_0^1 \left[\sigma_2(z_1 + r\,\delta z_{12}) - \sigma_2(z_1)\right] x \, dr$$
$$+ \int_0^1 \left[\nabla\sigma(z_1 + r\,\delta z_{12})\left(\delta z_{12} - \sigma(z_1)x\right)\right] dr$$
$$- \int_0^1 \nabla\sigma(z_1 + r\,\delta z_{12}) \left(\int_0^r \left[\nabla\sigma(z_1 + v\,\delta z_{12})\, \delta z_{12}\, x\right] dv \right) dr$$

Then $$\|B\|_{2\alpha} \leq \left(\|\nabla\sigma_2\|_\infty + \|\nabla\sigma\|_\infty^2\right) \|\mathbb{X}^1\|_\alpha \|\delta Z\|_\alpha$$
$$+ \|\nabla\sigma\|_\infty \|Z^{[2]}\|_{2\alpha}$$

These estimates lead directly to (i).

Let us now prove (ii). Recall

$$Z^{[2]}_{st} = \delta Z_{st} - \sigma(Z_s)\,\mathbb{X}^1_{st}$$

$$Z^{[3]}_{st} = \delta Z_{st} - \sigma(Z_s)\,\mathbb{X}^1_{st} - \sigma_2(Z_s)\,\mathbb{X}^2_{st} \overset{!}{=} Z^{[2]}_{st} - \sigma_2(Z_s)\,\mathbb{X}^2_{st}$$

Then $$\|Z^{[2]}\|_{2\alpha} \overset{!}{=} \|\sigma_2(Z)\,\mathbb{X}^2 + Z^{[3]}\|_{2\alpha}$$
$$\leq \underbrace{\|\sigma_2(Z)\|_\infty}_{\leq |\sigma_2(Z_0)| + T^\alpha \underbrace{\|\delta\sigma_2(Z)\|_\alpha}_{\leq \|\nabla\sigma_2\|_\infty \|\delta Z\|_\alpha}} \|\mathbb{X}^2\|_{2\alpha} + \underbrace{\|Z^{[3]}\|_{2\alpha}}_{\leq T^\alpha \|Z^{[3]}\|_{3\alpha}}$$

$$\|Z^{[2]}\|_{2\alpha} \leqslant |\sigma_2(Z_0)|\,\|\mathbb{X}^2\|_{2\alpha} + T^{\alpha}\left\{\|\nabla\sigma_2\|_\infty\,\|\mathbb{X}^2\|_{2\alpha}\,\|\delta Z\|_\alpha + \|Z^{[3]}\|_{3\alpha}\right\}$$

Now $\delta Z_{st} = \sigma(Z_s)\,\mathbb{X}^1_{st} + Z^{[2]}_{st}$

$$\Rightarrow \|\delta Z\|_\alpha \leqslant \underbrace{\|\sigma(Z)\|_\infty}_{\leqslant |\sigma(Z_0)| + T^{\alpha}\underbrace{\|\delta\sigma(Z)\|_\alpha}_{\leqslant \|\nabla\sigma\|_\infty\,\|\delta Z\|_\alpha}}\,\|\mathbb{X}^1\|_\alpha + \underbrace{\|Z^{[2]}\|_\alpha}_{\leqslant T^{\alpha}\|Z^{[2]}\|_{2\alpha}}$$

$$\|\delta Z\|_\alpha \leqslant |\sigma(Z_0)|\,\|\mathbb{X}^1\|_\alpha + T^{\alpha}\left\{\|\nabla\sigma\|_\infty\,\|\mathbb{X}^1\|_\alpha\,\|\delta Z\|_\alpha + \|Z^{[2]}\|_{2\alpha}\right\}$$

We obtained

$$\|\delta Z\|_\alpha + \|Z^{[2]}\|_{2\alpha} \leqslant |\sigma(Z_0)|\,\|\mathbb{X}^1\|_\alpha + |\sigma_2(Z_0)|\,\|\mathbb{X}^2\|_{2\alpha} + \tilde{C}_{\alpha,\sigma,\mathbb{X}}\cdot T^{\alpha}\left\{\|\delta Z\|_\alpha + \|Z^{[2]}\|_{2\alpha}\right\}$$

$\blacksquare$

UNIQUENESS

Theorem. Let $X \in \mathcal{C}^\alpha$ with $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. Let $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ be an α-RP over X. Let σ be of class $\mathcal{C}^\gamma$ with $\gamma > \frac{1}{\alpha}$ (e.g. $\sigma \in C^3$). Then $\forall z_0 \in \mathbb{R}^k$ there is at most one solution Z of (*").

Proof. Let Z, Z' be two solutions and set

$$Y := Z - Z'$$

Since $\|Y\|_\infty \leq |Y_0| + T^\alpha \|\delta Y\|_\alpha$ it is enough to show that, for $T > 0$ small, $\|\delta Y\|_\alpha \lesssim |Y_0|$.

Define $Y^{[2]} := Z^{[2]} - Z'^{[2]}$ $\quad Y^{[3]} := Z^{[3]} - Z'^{[3]}$

Strategy: prove that

$$\|\delta Y\|_\alpha \leq c_1 \|Y\|_\infty + c_1' \, T^\alpha \|Y^{[2]}\|_{2\alpha}$$

$$\|Y^{[2]}\|_{2\alpha} \leq c_2 \|Y\|_\infty + c_2' \, T^\alpha \|Y^{[3]}\|_{3\alpha}$$

$$\|Y^{[3]}\|_{3\alpha} \leq c_3 \|Y\|_\infty + c_3' \, T^\alpha \|Y^{[3]}\|_{3\alpha}$$

with constants c_i, c_i' which are INDEPENDENT of T.

From the last equation, for $T>0$ small enough, we get

$$\frac{1}{2} \|Y^{[3]}\|_{3\alpha} \leq C_3 \|Y\|_\infty$$

Then from the second eq., for $T>0$ small enough,

$$\|Y^{[2]}\|_{2\alpha} \leq 2 C_2 \|Y\|_\infty$$

which plugged into the first equation gives

$$\|\delta Y\|_\alpha \leq 2 C_1 \|Y\|_\infty \qquad \text{for } T>0 \text{ small enough.}$$

Finally:
$$\|Y\|_\infty \leq |Y_0| + T^\alpha \|\delta Y\|_\alpha$$
$$\leq |Y_0| + \underbrace{T^\alpha 2C_1}_{\leq \frac{1}{2} \text{ for } T>0 \text{ small enough}} \|Y\|_\infty$$

$$\Rightarrow \|Y\|_\infty \leq 2|Y_0|.$$

$\blacksquare$