WED 4 DEC 2024

REMINDERS. Fix dimensions $d, k \in \mathbb{N}$ and time horizon $T \in (0,\infty)$.

- Fix an exponent $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ "ROUGH CASE"
- Fix a path $X : [0,T] \to \mathbb{R}^d$ of class $\mathcal{C}^\alpha$ $\quad \|\delta X\|_\alpha < \infty$
- Fix $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ an α-ROUGH PATH (RP) over X

$\mathbb{X}^1 : [0,T]^2_\leq \to \mathbb{R}^d \qquad \mathbb{X}^2 : [0,T]^2_\leq \to \mathbb{R}^d \otimes \mathbb{R}^d \simeq \mathbb{R}^{d\times d}$

▷ $|\mathbb{X}^1_{st}| \lesssim (t-s)^\alpha \qquad |\mathbb{X}^2_{st}| \lesssim (t-s)^{2\alpha} \qquad \|\mathbb{X}^1\|_\alpha < \infty$

$\leq C \cdot (t-s)^\alpha \qquad$ "$\int_s^t \delta X_{su} \dot{X}_u du$" $\leq C\,(t-s)^{2\alpha} \qquad \|\mathbb{X}^2\|_{2\alpha} < \infty$

▷ $\mathbb{X}^1_{st} = \delta X_{st} \qquad \delta \mathbb{X}^2_{sut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}$ (Chen)

- Fix $\sigma : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k)$ of class C^1, define $\sigma_2 : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^k)$

$$\sigma_2(z) := \nabla\sigma(z) \circ \sigma(z)$$

Rough Difference Equation for unknown path $Z : [0,T] \to \mathbb{R}^k$

$$(*) \qquad \delta Z_{st} = \sigma(Z_s)\,\mathbb{X}^1_{st} + \sigma_2(Z_s)\,\mathbb{X}^2_{st} + o(t-s) \qquad \text{unif. } 0 \leq s < t \leq T$$

Define
$$Z^{[2]}_{st} := \underbrace{\delta Z_{st}}_{Z^{[1]}_{st}} - \sigma(Z_s)\,\mathbb{X}^1_{st}$$

$$Z^{[3]}_{st} := \underbrace{\delta Z_{st} - \sigma(Z_s)\,\mathbb{X}^1_{st}}_{Z^{[2]}_{st}} - \sigma_2(Z_s)\,\mathbb{X}^2_{st} \qquad = o(t-s) \text{ iff } Z \text{ solves } (*)$$

Last time we discussed:

- UNIQUENESS OF SOLUTIONS when $\sigma \in C^3$
- A PRIORI ESTIMATES when $\|\nabla\sigma\|_\infty, \|\nabla\sigma_2\|_\infty < \infty$

▸ $$\|Z^{[3]}\|_{3\alpha} \le \underbrace{K_{3\alpha}}_{\text{SEWING BOUND}} \cdot c'_{\alpha,\sigma,\mathbb{X}} \cdot \left(\|\delta Z\|_\alpha + \|Z^{[2]}\|_{2\alpha}\right)$$

▸ For small time $T^\alpha \le \varepsilon'_{\alpha,\sigma,\mathbb{X}}$:

$$\|\delta Z\|_\alpha + \|Z^{[2]}\|_{2\alpha} \le 2\left(\sigma(Z_0)\|\mathbb{X}^1\|_\alpha + \sigma_2(Z_0)\|\mathbb{X}^2\|_{2\alpha}\right)$$

$$\Rightarrow \quad \boxed{\|\delta Z\|_\alpha + \|Z^{[2]}\|_{2\alpha} + \|Z^{[3]}\|_{3\alpha} \le f_{\alpha,\sigma,\mathbb{X}}(Z_0)}$$

Today we complete our treatment of the RDE (*'') discussing EXISTENCE and CONTINUITY OF THE SOLUTION MAP

EXISTENCE

Theorem. Fix $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, $X \in \mathcal{C}^\alpha$ and $\mathbb{X}$ α-RP over X. Assume that $\sigma \in C^1$ with σ and σ_2 globally Lipschitz ($\|\nabla\sigma\|_\infty, \|\nabla\sigma_2\|_\infty < \infty$). Then, for any $T \in (0,\infty)$ and any $z_0 \in \mathbb{R}^k$, there exists a solution $Z = (Z_t)_{t\in[0,T]}$ of the RDE (*'').

We will prove it assuming $T > 0$ small enough.

We proceed similarly to the Young case $\alpha > \frac{1}{2}$, i.e. we construct a sequence $Z^n = (Z^n_t)_{t \in \Pi^n}$ of "approximate solutions" defined on dyadic partitions Π^n of $[0,T]$, then we extend Z^n linearly to $[0,T]$, we extract a converging subsequence $Z^{n_k} \to Z$ and we show that Z is indeed a solution of ($*$").

However we enrich the Euler scheme used to construct Z^n in the Young case, using a MILSTEIN SCHEME.

Fix $\Pi = \{0 = t_0 < t_1 < \ldots < t_K = T\}$. Fix $z_0 \in \mathbb{R}^k$.
Define $Z^\Pi = (Z^\Pi_t)_{t \in \Pi}$ by

$$Z^\Pi_0 := z_0, \qquad Z^\Pi_{t_{i+1}} := Z^\Pi_{t_i} + \sigma(Z^\Pi_{t_i}) \cdot \mathbb{X}^1_{t_i t_{i+1}} + \sigma_2(Z^\Pi_{t_i}) \cdot \mathbb{X}^2_{t_i t_{i+1}}$$

If we define the "remainder"

$$(Z^\Pi)^{[3]}_{st} := \delta Z^\Pi_{st} - \sigma(Z^\Pi_s)\, \mathbb{X}^1_{st} - \sigma_2(Z^\Pi_s)\, \mathbb{X}^2_{st}$$

Then, by construction, $(Z^\Pi)^{[3]}_{t_i t_{i+1}} = 0 \quad \forall i = 0, \ldots, K-1$.

Then we can apply the DISCRETE SEWING BOUND

$$\| (Z^\Pi)^{[3]} \|^\Pi_{3\alpha} \leq C_{3\alpha} \| \delta (Z^\Pi)^{[3]} \|^\Pi_{3\alpha}$$

With an almost identical proof as we did, we obtain a priori estimates:

$$\boxed{\| \delta Z^{\pi} \|_{\alpha}^{\pi} + \| (Z^{\pi})^{[2]} \|_{2\alpha}^{\pi} + \| (Z^{\pi})^{[3]} \|_{3\alpha}^{\pi} \leq \tilde{f}_{\alpha,\sigma,X}(z_0)}$$

UNIFORM IN π !

Consider now $\pi_n := \{ \frac{i}{2^n} : i = 0, 1, 2, \ldots \} \cap [0,T]$

Define for simplicity $Z^n := Z^{\pi_n}$ and extend it to $[0,T]$ as a piecewise linear function $Z^n = (Z^n_t)_{t \in [0,T]}$.

By cheap arguments $\| \delta Z^n \|_{\alpha} \leq 3 \| \delta Z^n \|_{\alpha}^{\pi_n}$

possibly also $\| (Z^n)^{[3]} \|_{3\alpha} \leq 3 \| (Z^n)^{[3]} \|_{3\alpha}^{\pi_n}$

$$\Rightarrow \boxed{\| \delta Z^n \|_{\alpha} + \| (Z^n)^{[3]} \|_{3\alpha}^{\pi_n} \leq \underbrace{3 \tilde{f}_{\alpha,\sigma,X}(z_0)}_{C}} \quad \forall n \in \mathbb{N}.$$

Since $Z^n_0 = z_0$ and $\| \delta Z^n \|_{\alpha} \leq C$, $\forall n \in \mathbb{N}$, the sequence $(Z^n)_{n \in \mathbb{N}}$ is equi-bounded and equi-continuous in $C([0,T], \mathbb{R}^k)$, hence by Arzelà-Ascoli we can extract a converging subsequence $Z^{n_k} \to Z \in C([0,T], \mathbb{R}^k)$.

So we have $Z^{n_k}_t \xrightarrow[k \to \infty]{} Z_t \quad \forall t \in [0,T]$.

Let us rewrite $\|(Z^n)^{[3]}\|_{3\alpha}^{\Pi_n} \leq C_1$ as

$$|(Z^n)^{[3]}_{st}| = |\delta Z^n_{st} - \sigma(Z^n_s)\mathbb{X}^1_{st} - \sigma(Z^n_s)\mathbb{X}^2_{st}| \leq C_1 (t-s)^{3\alpha}$$

$\forall n \in \mathbb{N}, \quad \forall s,t \in \Pi_n.$

Fix $\bar{n} \in \mathbb{N}$ and $s,t \in \Pi_{\bar{n}}$. Consider $n = n_k \geq \bar{n}$. Since $s,t \in \Pi_n \supseteq \Pi_{\bar{n}}$, as $k \to \infty$ we have

$$|\delta Z_{st} - \sigma(Z_s)\mathbb{X}^1_{st} - \sigma(Z_s)\mathbb{X}^2_{st}| \leq C_1 (t-s)^{3\alpha}$$

$$\forall s,t \in \Pi_{\bar{n}} \Rightarrow \forall s,t \in \bigcup_{\bar{n} \in \mathbb{N}} \Pi_{\bar{n}} = \mathbb{D}.$$

Since $\mathbb{D}$ is dense in $[0,T]$, and since Z is continuous, the same bound holds $\forall s,t \in [0,T]$, which means that Z solves (*''). ∎

CONTINUITY OF THE SOLUTION MAP

Assume now that $\sigma \in C^3$ with

$$\|\nabla\sigma\|_\infty, \|\nabla^2\sigma\|_\infty, \|\nabla^3\sigma\|_\infty, \|\nabla\sigma_2\|_\infty, \|\nabla^2\sigma_2\|_\infty \leq D < \infty$$

These assumptions entail GLOBAL EXISTENCE + UNIQUENESS of solutions $Z = (Z_t)_{t \in [0,T]}$ of (*''), for any initial datum $z_0 \in \mathbb{R}^k$.

If we fix $K, d \in \mathbb{N}$, $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, $T \in (0, \infty)$,

then we can consider the SOLUTION MAP

$$\Phi : \mathbb{R}^K \times \mathcal{R}_{\alpha,d} \to \mathcal{C}^\alpha$$

$$(Z_0, \mathbb{X}) \mapsto Z = (Z_t)_{t \in [0,T]} \text{ solution of } (*'')$$

where we denote by $\mathcal{R}_{\alpha,d}$ the space of α-ROUGH PATHS.

We can show that this map is CONTINUOUS, in fact LOCALLY LIPSCHITZ, when we endow the space of RP $\mathcal{R}_{\alpha,d}$ with the distance for $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$, $\tilde{\mathbb{X}} = (\tilde{\mathbb{X}}^1, \tilde{\mathbb{X}}^2)$

$$d_\alpha(\mathbb{X}, \tilde{\mathbb{X}}) := \|\mathbb{X}^1 - \tilde{\mathbb{X}}^1\|_\alpha + \|\mathbb{X}^2 - \tilde{\mathbb{X}}^2\|_{2\alpha}$$

<u>Theorem</u> (CONTINUITY OF THE SOLUTION MAP)

Fix $D, M_0, M < \infty$. Assume that

$$\|\nabla\sigma\|_\infty, \|\nabla^2\sigma\|_\infty, \|\nabla^3\sigma\|_\infty, \|\nabla\sigma_2\|_\infty, \|\nabla^2\sigma_2\|_\infty \leq D < \infty$$

and consider starting points $Z_0, \tilde{Z}_0$ such that

$$|\sigma(Z_0)|, |\sigma(\tilde{Z}_0)|, |\sigma_2(Z_0)|, |\sigma_2(\tilde{Z}_0)| \leq M_0$$

and consider rough paths $\mathbb{X}, \tilde{\mathbb{X}}$ such that

$$\|\mathbb{X}^1\|_\alpha, \ \|\mathbb{X}^2\|_{2\alpha}, \ \|\tilde{\mathbb{X}}^1\|_\alpha, \|\tilde{\mathbb{X}}^2\|_{2\alpha} \leq M.$$

Then the corresponding solutions $Z, \tilde{Z}$ of (*'') satisfy

$$\|Z - \tilde{Z}\|_\infty + \|\delta Z - \delta\tilde{Z}\|_\alpha + \|Z^{[2]} - \tilde{Z}^{[2]}\|_{2\alpha}$$
$$\leq 32(DM+1)\,|Z_0 - \tilde{Z}_0| + 30 M_0 \cdot d_\alpha(\mathbb{X}, \tilde{\mathbb{X}}).$$

provided $T > 0$ is small enough:

$$0 < T \leq \tau'_{\alpha, D, M_0, M} < \infty.$$

We omit the proof.

4- STOCHASTIC DIFFERENTIAL EQUATIONS

We now connect the ROUGH DIFFERENCE EQUATIONS (RDE) studied in the last chapter with the STOCHASTIC DIFFERENTIAL EQUATIONS (SDE) driven by Brownian Motion (BM) B. Indeed, both RDEs and SDEs are ways to give a meaning to the ill-defined differential equation

$$(*) \qquad \dot{Y}_t = \sigma(Y_t)\,\dot{B}_t$$

Setting. Fix dimensions $d, k \in \mathbb{N}$, time horizon $T \in (0,\infty)$. Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which is defined a standard Brownian Motion (BM) in $\mathbb{R}^d$ $B = (B_t)_{t\in[0,T]} = (B_t^{(i)})_{t\in[0,T],\, i=1,\dots,d}$ relative to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. We fix a version of BM with continuous paths, so B is a (random) element of C^0, in fact of $\mathcal{C}^\alpha$ for any $\alpha \in (0,\frac{1}{2})$.
We fix $\alpha \in (\frac{1}{3},\frac{1}{2})$.

Given $\sigma : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^d,\mathbb{R}^k) \simeq \mathbb{R}^{d\times k}$ globally Lipschitz, i.e. $\|\nabla\sigma\|_\infty < \infty$, and given $y_0 \in \mathbb{R}^k$, we can consider the unique strong solution $Y = (Y_t)_{t\in[0,T]}$ of the SDE

$$\text{(SDE)} \qquad Y_t = y_0 + \int_0^t \sigma(Y_u)\, dB_u \qquad \text{for } t\in[0,T]$$

(i.e. $Y_0 = y_0$, $dY_t = \sigma(Y_t)\, dB_t$)

where the integral is a STOCHASTIC ITO INTEGRAL.
We fix a continuous version of $Y = (Y_t)_{t\in[0,T]}$.

In order to connect this solution to the RDE (*''),
we need to introduce the ITO ROUGH PATH $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$
associated to the BM B:

$$\mathbb{B}^1_{st} = \delta B_{st} = B_t - B_s \qquad \mathbb{B}^2_{st} := \int_s^t \underbrace{\delta B_{su}}_{B_u - B_s} \otimes dB_u$$

$$:= I_t - I_s - B_s \otimes (B_t - B_s)$$

where $I_t = \int_0^t B_u \otimes dB_u$ (Ito integral) and we fix a version of $I = (I_t)_{t\in[0,T]}$ with continuous paths.

Theorem (ITO ROUGH PATH) Fix $\alpha \in (\frac{1}{3}, \frac{1}{2})$ and let B be BM in $\mathbb{R}^d$. Then, a.s., $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$ is an α-ROUGH PATH over B, i.e.

- $\mathbb{B}^1_{st} = \delta B_{st} \qquad \delta \mathbb{B}^2_{sut} = \mathbb{B}^1_{su} \otimes \mathbb{B}^1_{ut} \quad \forall\, 0 \le s \le u \le t$
- $|\mathbb{B}^1_{st}| \lesssim (t-s)^\alpha \qquad |\mathbb{B}^2_{st}| \lesssim (t-s)^{2\alpha}$

  $\le C \cdot (t-s)^\alpha \qquad \le C' (t-s)^{2\alpha}$

(where the implicit constants in $\lesssim$ are random).

Note that $\mathbb{B}^2_{st} \in \mathbb{R}^d \otimes \mathbb{R}^d$ i.e.

$$\left(\mathbb{B}^2_{st}\right)^{ij} = \int_s^t \left(B_v^{(i)} - B_s^{(i)}\right) dB_v^{(j)}$$

Diagonal components:

$$\begin{aligned}
\left(\mathbb{B}^2_{st}\right)^{ii} &= \int_s^t B_v^{(i)} dB_v^{(i)} - B_s^{(i)}\left(B_t^{(i)} - B_s^{(i)}\right) \\
&\overset{!}{=} \frac{\left(B_t^{(i)}\right)^2 - \left(B_s^{(i)}\right)^2 - (t-s)}{2} - B_s^{(i)}\left(B_t^{(i)} - B_s^{(i)}\right) \\
&\overset{!}{=} \frac{\left(B_t^{(i)} - B_s^{(i)}\right)^2 - (t-s)}{2} \quad \lesssim (t-s)^{2\alpha} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left(\alpha < \tfrac{1}{2}\right)
\end{aligned}$$

We can now consider the RDE driven by $\mathbb{B}$:

$$\text{(RDE)} \quad \delta Y_{st} = \sigma(Y_s)\,\mathbb{B}^1_{st} + \sigma_2(Y_s)\,\mathbb{B}^2_{st} + o(t-s), \quad Y_0 = y_0$$

Theorem (SDE & RDE) If $\sigma \in C^2$, then a.s, any solution $Y = (Y_t)_{t\in[0,T]}$ of (RDE) is also a solution of (SDE).

If moreover $\sigma \in C^3$ and σ and σ_2 are globally Lipschitz, then a.s. both (SDE) and (RDE) have a unique solution $Y = (Y_t)_{t\in[0,T]}$ and these solutions coincide.

Corollary. Assume $\sigma \in C^3$ and $\|\nabla \sigma\|_\infty, \|\nabla \sigma_2\|_\infty < \infty$. Then, a.s., the solution of (SDE) is the only path $Y = (Y_t)_{t \in [0,T]}$ which solves (RDE). Thus there exists an event A with $\mathbb{P}(A) = 1$ such that $\forall \omega \in A$ there is a unique path $Y = (Y_t)_{t\in[0,T]}$ such that

$$\delta Y_{st} = \sigma(Y_s)\, \mathbb{B}^1_{st}(\omega) + \sigma_2(Y_s)\, \mathbb{B}^2_{st}(\omega) + o(t-s)$$

unif. for $0 \le s < t \le T$

and such path $Y = Y(\omega)$ solves (SDE).

In general, the solution Y of a SDE is only a measurable function of the driving BM B. The gain we obtain from the theory of RDE is to factorize such a measurable map as a composition of two maps:

MEASURABLE   CONTINUOUS

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, mapsto, "\substack{\text{RDE} \\ \text{solution map}}"] & \\ B \arrow[ur, mapsto, "\substack{\text{ITO} \\ \text{rough path}}"] \arrow[rr, mapsto, "\substack{\text{SDE} \\ \text{solution map}}"'] & & Y \end{tikzcd}$$

MEASURABLE

The key tool is the following result.

<u>Theorem</u> (LOCAL EXPANSION OF STOCHASTIC INTEGRALS)

Let $B = (B_t)_{t\in[0,T]}$ be a BM in $\mathbb{R}^d$, Let $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$ be the associatet Ito rough path. Fix $\alpha \in (\frac{1}{3}, \frac{1}{2})$.

Let $h = (h_t)_{t\in[0,T]}$ a continuous, adapted process, $h : [0,T] \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k)$. Define, for $I_0 \in \mathbb{R}^k$,

$$I_t := I_0 + \int_0^t h_u \, dB_u \qquad \text{(continuous version)}$$

(1) A.s. $I = (I_t)_{t\in[0,T]} \in \mathcal{C}^\alpha$, i.e.

$$\delta I_{st} \underbrace{\lesssim}_{\text{random constant}} (t-s)^\alpha \qquad 0 \leq s < t \leq T.$$

(2) Assume that $\overset{\text{a.s.}}{\vee}$ $\delta h_{sr} \lesssim (r-s)^\beta$ for some $\beta \in (0,1]$, i.e. a.s. $h \in \mathcal{C}^\beta$. Then, a.s.

$$|\delta I_{st} - h_s \cdot \mathbb{B}^1_{st}| = \left| \int_s^t \delta h_{s,u} \, dB_u \right| \lesssim (t-s)^{\alpha+\beta}.$$

(3) Assume that a.s. $|\delta h_{sr} - h^1_s \mathbb{B}^1_{st}| \lesssim (r-s)^{\alpha+\eta}$, $\eta \in (0,1]$, for some $h^1 = (h^1_t)_{t\in[0,T]} \in \mathcal{C}^\eta$. Then, a.s.

$$|\delta I_{st} - h_s \cdot \mathbb{B}^1_{st} - h^1_s \mathbb{B}^2_{st}| = \left| \int_s^t (\delta h_{su} - h^1_s \mathbb{B}^1_{su}) dB_u \right| \lesssim (t-s)^{2\alpha+\eta}$$