WED 11 DEC 2024

Reminders.

- $B = (B_t)_{t\in[0,T]}$ standard Brownian motion in $\mathbb{R}^d$ (continuous paths)

def. $\mathbb{B}^1_{st} := \delta B_{st} = B_t - B_s$

- $\mathbb{B}^2_{0t} := \int_0^t B_u \otimes dB_u$ Ito integral (continuous paths)

def. $\mathbb{B}^2_{st} := \mathbb{B}^2_{0t} - \mathbb{B}^2_{0s} - B_s \otimes (B_t - B_s)$

$$= \int_s^t \delta B_{su} \otimes dB_u$$

Theorem (1): For any $\alpha \in (\frac{1}{3}, \frac{1}{2})$, a.s. $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$ is an α-rough path over B, i.e. $\forall s<u<t$

- $\mathbb{B}^1_{st} = \delta B_{st}$ $\qquad \delta \mathbb{B}^2_{sut} = \mathbb{B}^1_{su} \otimes \mathbb{B}^1_{ut}$ (CHEN'S RELATION)

- $|\mathbb{B}^1_{st}| \lesssim (t-s)^{\alpha}$ $\qquad |\mathbb{B}^2_{st}| \lesssim (t-s)^{2\alpha}$

Fix $\sigma : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k)$ of class C^1 and def.

$$\sigma_2(z) := \nabla\sigma(z)\,\sigma(z) \qquad \forall z \in \mathbb{R}^k$$

Consider the ROUGH DIFFERENTIAL EQ. (RDE)

$$\text{(RDE)} \qquad \delta Y_{st} = \sigma(Y_s)\,\mathbb{B}^1_{st} + \sigma_2(Y_s)\,\mathbb{B}^2_{st} + o(t-s)$$

unif. for $0 \le s \le t \le T$

Compare it to the STOCHASTIC DIFFERENTIAL EQ (SDE)

$$dY_t = \sigma(Y_t)\, dB_t \iff$$

(SDE) $$Y_t = Y_0 + \int_0^t \sigma(Y_u)\, dB_u \qquad \text{for } 0 \leq s \leq T$$

Theorem (2) (SDE & RDE): If $\sigma \in C^2$, then given a solution $Y = (Y_t)_{t \in [0,T]}$ of (SDE), a.s. Y solves (RDE).

Moreover, if $\sigma \in C^3$ wit σ and σ_2 globally Lipschitz, then (SDE) and (RDE) have both existence and uniqueness of solutions, for any given initial datum $y_0 \in \mathbb{R}^k$, and these solutions coincie a.s.

Everything is based on the following Key result..

Theorem (3): For $h = (h_u)_{u \in [0,T]}$ a continuous, adapted process, consider the Ita integral (continuous version of)

$$I_t = I_0 + \int_0^t h_u\, dB_u \qquad h_u \in \mathcal{L}(\mathbb{R}^d, \mathbb{R})$$

Fix any $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$

(a) a.s. $I \in \mathcal{C}^\alpha$, that is for $0 \leq s < t \leq T$

$$|\delta I_{st}| \lesssim (t-s)^\alpha \qquad \leq C(\omega)\,(t-s)^\alpha$$

$C < \infty$ a.s.

Note now that $\delta I_{st} = \int_s^t h_u \, dB_u$

$$\Rightarrow \quad \delta I_{st} - h_s \underbrace{(B_t - B_s)}_{\mathbb{B}^1_{st}} = \int_s^t \underbrace{(h_u - h_s)}_{\delta h_{su}} \, dB_u$$

(b) If a.s. $h \in \mathcal{C}^\beta$, i.e. $|\delta h_{su}| \lesssim (u-s)^\beta$, for some $\beta \in (0,1]$, then a.s.

$$|\delta I_{st} - h_s \mathbb{B}^1_{st}| \lesssim (t-s)^{\alpha+\beta}$$

Now we go one step further:

(c) If a.s. $|\delta h_{su} - h^1_s \mathbb{B}^1_{su}| \lesssim (u-s)^{\alpha+\eta}$, for some $\eta \in (0,1]$ and for some adapted $h^1 = (h^1_u)_{u \in [0,T]} \in \mathcal{C}^\eta$, then a.s.

$$\Big|\underbrace{\delta I_{st} - h_s \mathbb{B}^1_{st}}_{\int_s^t \delta h_{su} \, dB_u} - \underbrace{h^1_s \mathbb{B}^2_{st}}_{h^1_s \int_s^t \delta B_{su} \, dB_u}\Big| \lesssim (t-s)^{2\alpha+\eta}$$

$$\underbrace{\int_s^t \delta h_{su} \, dB_u - h^1_s \int_s^t \delta B_{su} \, dB_u}_{\int_s^t \{\underbrace{\delta h_{su} - h^1_s \delta B_{su}}_{\lesssim (u-s)^{\alpha+\eta} \le (t-s)^{\alpha+\eta}}\} \, dB_u}$$

Proof of Theorem 1 We show that, a.s., $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$ is an α-rough path, for any $\alpha \in (\frac{1}{2}, \frac{1}{3})$.

- ALGEBRAIC RELATIONS: $\mathbb{B}^1_{st} = \delta B_{st}$ by definition

CHEN'S RELATION: Fix $i, j \in \{1, \dots, d\}$ and $s < u < t$:

$$
\begin{aligned}
\delta(\mathbb{B}^2)^{ij}_{sut} &= (\mathbb{B}^2)^{ij}_{st} - (\mathbb{B}^2)^{ij}_{su} - (\mathbb{B}^2)^{ij}_{ut} \\
&= \int_s^t \delta B^i_{sr}\, dB^j_r - \int_s^u \delta B^i_{sr}\, dB^j_r - \int_u^t \delta B^i_{ur}\, dB^j_r \\
&= \int_u^t \delta B^i_{sr}\, dB^j_r - \int_u^t \delta B^i_{ur}\, dB^j_r \\
&= \int_u^t \big(\underbrace{\delta B^i_{sr} - \delta B^i_{ur}}_{\delta B^i_{su}}\big)\, dB^j_r \\
&= \delta B^i_{su} \cdot \int_u^t 1\, dB^j_r = \underbrace{\delta B^i_{su}}_{(\mathbb{B}^1)^i_{su}} \cdot \underbrace{\delta B^j_{ut}}_{(\mathbb{B}^1)^j_{ut}}
\end{aligned}
$$

This holds a.s., given any fixed $s < u < t$.
Then it holds a.s. simultaneously $\forall\, s < u < t \in \mathbb{Q}$, hence $\forall\, s < u < t \in \mathbb{R}$ by continuity of both sides in s, u, t.

ANALYTIC BOUNDS:

- $|\mathbb{B}^1_{st}| = |\delta B_{st}| \lesssim (t-s)^\alpha$ by Theorem ③ (a) choosing $h \equiv 1$.

- $|(\mathbb{B}^2)^{ij}_{st}| = \left|\int_s^t \delta B^i_{su}\, dB^j_u\right| = \left|\int_s^t B^i_u\, dB^j_u - B^i_s\, \delta B^j_{st}\right|$

$\lesssim (t-s)^{2\alpha}$ by Theorem ③ (b) choosing $h = B^i$

$\square$

<u>Proof of Theorem</u> ② It suffices to show the first part: assume $\sigma \in C^2$, fix Y a solution of (SDE) and we show that, a.s., Y solves (RDE).

Fix $0 \leq s < t \leq T$ and compute

$$\underbrace{\delta Y_{st}}_{Y_t - Y_s \underset{(SDE)}{=} \int_s^t \sigma(Y_u)\, dB_u} - \underbrace{\sigma(Y_s)\,\mathbb{B}^1_{st}}_{= \sigma(Y_s)(B_t - B_s) = \int_s^t \sigma(Y_s)\, dB_u} - \sigma_2(Y_s)\,\mathbb{B}^2_{st}$$

$$= \int_s^t \{\sigma(Y_u) - \sigma(Y_s)\}\, dB_u - \underbrace{\sigma_2(Y_s) \cdot \int_s^t \delta B_{su} \otimes dB_u}_{\int_s^t \{\sigma_2(Y_s)\, \delta B_{su}\}\, dB_u}$$

$$= \int_s^t \{ \underbrace{\delta\sigma(Y)_{su} - \sigma_2(Y_s)\,\delta B_{su}} \}\, dB_u = ✪$$

We claim that a.s. $\{\cdots\} \lesssim (s-u)^{2\alpha}$ $\forall\ 0 \le s < u \le T$.

Then we can apply Theorem 3 (c) with $h = \sigma(Y)$ and $h^1 = \sigma_2(Y)$, with $\eta = \alpha$, to get

a.s. $$✪ \lesssim (t-s)^{3\alpha} = o(t-s) \quad \text{since } \alpha > \tfrac{1}{3}.$$

If a.s. $|\delta h_{su} - h^1_s \mathbb{B}^1_{su}| \lesssim (u-s)^{\alpha+\eta}$, for some $\eta \in (0,1]$ and for some adapted $h^1 = (h^1_u)_{u\in[0,T]} \in \mathcal{C}^\eta$, then a.s.

$$|\delta I_{st} - h_s \mathbb{B}^1_{st} - h^1_s \mathbb{B}^2_{st}| \lesssim (t-s)^{2\alpha+\eta}$$

It remain to prove the claim:

a.s. $$|\delta\sigma(Y)_{st} - \sigma_2(Y_s)\mathbb{B}^1_{st}| \lesssim (t-s)^{2\alpha}$$

By Ito formula $\quad [\, dY_u = \sigma(Y_u)\, dB_u \,]$

$$\sigma(Y_t) = \sigma(Y_0) + \int_0^t \nabla\sigma(Y_u)\, dY_u + \frac{1}{2}\int_0^t \nabla^2\sigma(Y_u)\cdot d\langle Y, Y\rangle_u$$

$$= \sigma(Y_0) + \int_0^t \sigma_2(Y_u)\, dB_u + \frac{1}{2}\int_0^t \underbrace{\nabla^2\sigma(Y_u)\cdot(\sigma\sigma^t)(Y_u)}_{p(u)}\, du$$

Then
$$\delta\sigma(Y)_{st} = \sigma(Y_t) - \sigma(Y_s) \overset{!}{=} \int_s^t \sigma_2(Y_u)\, dB_u + \frac{1}{2}\int_s^t p(u)\, du$$

Finally
$$\delta\sigma(Y)_{st} - \sigma_2(Y_s)\, \mathbb{B}^1_{st} = \underbrace{\int_s^t \{\sigma_2(Y_u) - \sigma_2(Y_s)\}\, dB_u}_{A_{st}} + \frac{1}{2}\underbrace{\int_s^t p(u)\, du}_{O(t-s) \lesssim (t-s)^{2\alpha}}$$

because p is continuous, hence locally bounded

We can now apply again Theorem 3 (b) with $h = \sigma_2(Y)$ which satisfies, a.s., $h \in \mathcal{C}^\alpha$: indeed

$$|\delta h_{st}| = |\sigma_2(Y_t) - \sigma_2(Y_s)| \leq \|\nabla\sigma_2\|_\infty\, |\delta Y_{st}| \lesssim (t-s)^\alpha$$

$$|\delta Y_{st}| = \left| \int_s^t \sigma(Y_u)\, dB_u \right| \lesssim (t-s)^\alpha \text{ by Theorem 3 (a)}$$

$$\Rightarrow |A_{st}| = \left| \int_s^t (\delta h_{su})\, dB_u \right| \lesssim (t-s)^{2\alpha}. \qquad \blacksquare$$

We now explain how to prove Theorem 3 –
All the results (a) (b) (c) are of the form

$$|A_{st}| \lesssim (t-s)^\gamma$$

for suitable choice of A_{st} and γ.

We then fix a (later random) continuous function $A: [0,T]^2_{\leq} \to \mathbb{R}$, that is $A = (A_{st})_{0 \leq s \leq t \leq T}$.
For notational simplicity $T=1$.

By continuity, it is enough to prove that

$$(\star) \qquad \boxed{|A_{st}| \leq C_1 \cdot (t-s)^{\gamma}} \qquad \forall \; s \leq t \in \mathbb{D}$$

where $\mathbb{D} \subseteq [0,1]$ is the set of dyadic rationals:

$$\mathbb{D} = \bigcup_{k=0}^{\infty} \mathbb{D}_k \qquad \mathbb{D}_k = \left\{ t_i^k := \frac{i}{2^k} \,,\; 0 \leq i \leq 2^k \right\}$$

Let us write $d \to d'$ to mean that $d, d' \in \mathbb{D}$ are consecutive dyadic rationals in same $\mathbb{D}_k$, i.e.

$$d \to d' \iff \exists\, k \in \mathbb{N}_0,\; i \in \{0, \dots, 2^k - 1\} : \; d = \frac{i}{2^k},\; d' = \frac{i+1}{2^k}$$

It turns out that, if we can prove $(\star)$ for $s=d$, $t=d'$ with $d \to d'$, and if we can bound δA_{sut} $\forall\, s<u<t \in \mathbb{D}$, then we can deduce $(\star)$ $\forall\, s<t \in \mathbb{D}$.

More precisely, we have:

Theorem (KOLMOGOROV CRITERION, DETERMINISTIC PART) (Kol 1)

Assume that $A : \mathbb{D}^2_< \to \mathbb{R}$ satisfies, for some $0<\rho<\gamma$,

- $$Q_\gamma := \sup_{d,d' \in \mathbb{D} :\, d \to d'} \frac{|A_{d,d'}|}{|d'-d|^\gamma} < \infty$$

  $$\Leftrightarrow \quad |A_{d,d'}| \leq Q_\gamma (d'-d)^\gamma \quad \forall\, d \to d'$$

- $$K_{\rho,\gamma} := \sup_{s<u<t \in \mathbb{D}} \frac{|\delta A_{sut}|}{\min\{u-s, t-u\}^\rho \, |t-s|^{\gamma-\rho}} < \infty$$

  $$|\delta A_{sut}| \leq K_{\rho,\gamma} \cdot \min\{u-s, t-u\}^\rho (t-s)^{\gamma-\rho} \leq K_{\rho,\gamma} \cdot (t-s)^\gamma$$

Then there is a universal explicit $C_{\rho,\gamma} < \infty$ such that

$$|A_{st}| \leq C_{\rho,\gamma} \left(Q_\gamma + K_{\rho,\gamma} \right) (t-s)^\gamma \quad \forall\, s<t \in \mathbb{D}.$$

We complement the previous result with a criterion to check that $Q_\gamma < \infty$, that is to prove $|A_{dd'}| \lesssim (d'-d)^\gamma$ for consecutive dyadic rationals $d \to d'$, when A is random and we can bound its moments.

(This implies the classical Kolmogorov criterion)

Theorem (KOLMOGOROV CRITERION, RANDOM PART). (Kol2)

Let $A : \mathbb{D}^2_{\leq} \to \mathbb{R}$ be random and satisfy

$$\mathbb{E}[|A_{st}|^p] \leq c\,(t-s)^{p\cdot\gamma_0} \quad \forall\ s<t\in\mathbb{D}$$

for some $p, \gamma_0, c \in (0,\infty)$. Then

$$\mathbb{E}[|Q_\gamma(A)|^p] < \infty \quad \forall\ \gamma < \gamma_0 - \frac{1}{p},$$

$$= \sup_{d\to d'} \frac{|A_{dd'}|}{(d'-d)^\gamma}$$

hence $Q_\gamma(A) \leq \infty$ a.s. Hence, if also $K_{p,\gamma}(A)<\infty$,

a.s. $|A_{st}| \lesssim (t-s)^\gamma \quad \forall\ s<t\in\mathbb{D}$.

Proof.

$$|Q_\gamma(A)|^p \leq \sum_{\substack{d<d'\in\mathbb{D}\\ d\to d'}} \left(\frac{|A_{dd'}|}{(d'-d)^\gamma}\right)^p$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{2^k-1} \frac{\left|A_{\frac{i}{2^k}\frac{i+1}{2^k}}\right|^p}{\left(\frac{1}{2^k}\right)^{\gamma p}}$$

Then

$$\mathbb{E}[|Q_\gamma(A)|^p] \leq \sum_{k=0}^{\infty} \sum_{i=0}^{2^k-1} \frac{\left(\frac{1}{2^k}\right)^{p\gamma_0}}{\left(\frac{1}{2^k}\right)^{\gamma p}} = \sum_{k=0}^{\infty} \left(\frac{1}{2^k}\right)^{p(\gamma_0-\gamma)-1} < \infty$$

as long as $p(\gamma_0-\gamma)-1>0 \iff \gamma_0-\gamma > \frac{1}{p} \iff \gamma < \gamma_0 - \frac{1}{p}$. $\blacksquare$

We finally apply this refined Kolmogorov criterion to prove Theorem ③.

Proof of Theorem ③.

(a) $$I_t - I_s = \int_s^t h_u \, dB_u$$

Consider $A_{st} := I_t - I_s = \delta I_{st}$

Plainly $\delta A \equiv 0$ hence $K_{p,\gamma}(A) = 0$ thus

$\left[\begin{array}{l}\text{by Kol ①} \\ \text{with } \gamma = \alpha\end{array}\right]$ $$|I_t - I_s| = |A_{st}| \lesssim Q_\alpha(A) \cdot (t-s)^\alpha$$

It remains to show that $Q_\alpha(A) < \infty$ a.s., which we deduce by Kol②.

We may assume that $|h_u(\omega)| \leq C < \infty$.
(Indeed, in the general case we argue by localisation, defining the stopping times

$$\tau_n := \inf\{u \in [0,T] : |h_u| > n\}$$

and working with the bounded process $h_u^{\tau_n} := h_u \cdot \mathbb{1}_{\{u \leq \tau_n\}}$)

Assuming $|h_u(\omega)| \leq C_1 < \infty$, we apply Kol(2):

$$\mathbb{E}\left[|Q_\gamma(A)|^p \right] < \infty \quad \text{if} \quad \mathbb{E}\left[|A_{st}|^p \right] \lesssim (t-s)^{p\gamma_0}$$

for $\gamma < \gamma_0 - \frac{1}{p}$

To estimate $\mathbb{E}\left[|A_{st}|^p \right] = \mathbb{E}\left[|\delta I_{st}|^p \right] = \mathbb{E}\left[\left| \int_s^t h_u \, dB_u \right|^p \right]$

we apply the Burkholder-Davis-Gundy inequality:

$$\mathbb{E}\left[\left| \int_s^t h_u \, dB_u \right|^p \right] \simeq \mathbb{E}\left[\left| \int_s^t h_u \, dB_u \right|^2 \right]^{p/2}$$

$$= \mathbb{E}\left[\int_s^t h_u^2 \, du \right]^{p/2}$$

$\left[|h_u| \leq C_1 \right]$

$$\leq C^p \cdot \underbrace{(t-s)^{p/2}}_{(t-s)^{p\gamma_0}} \qquad \forall\, p \in [2, \infty)$$

$\gamma_0 = \frac{1}{2}$

We then have $Q_\gamma(A) < \infty$ a.s., $\forall\, \gamma < \gamma_0 - \frac{1}{p} = \frac{1}{2} - \frac{1}{p}$

Taking p large, we get $Q_\alpha(A) < \infty$ a.s. $\forall\, \alpha < \frac{1}{2}$.