WED 18 DEC 2024

REMINDERS

$$\mathbb{D} = \bigcup_{k=0}^{\infty} \left\{ \frac{i}{2^k} : 0 \le i \le 2^k \right\}$$ dyadic rationals in $[0,1]$

"CONSECUTIVE" $d \to d' \iff d = \frac{i}{2^k},\ d' = \frac{i+1}{2^k}$ for some i, k

GOAL: $|A_{st}| \le C (t-s)^\gamma \quad \forall t, s \in \mathbb{D}$

THEOREM [KOL 1] (KOLMOGOROV CRITERION, DETERMINISTIC PART)

Assume that $A : \mathbb{D}^2_{\le} \to \mathbb{R}$ satisfies, for some $0<\rho<\gamma$,

- $$Q_\gamma(A) := \sup_{d,d' \in \mathbb{D} : d \to d'} \frac{|A_{d,d'}|}{|d'-d|^\gamma} < \infty$$

$$\iff |A_{d,d'}| \le Q_\gamma (d'-d)^\gamma \quad \forall d \to d'$$

- $$K_{\rho,\gamma}(A) := \sup_{s \le u \le t \in \mathbb{D}} \frac{|\delta A_{sut}|}{\min\{u-s, t-u\}^\rho \, |t-s|^{\gamma-\rho}} < \infty$$

$$|\delta A_{sut}| \le K_{\rho,\gamma} \cdot \min\{u-s, t-u\}^\rho (t-s)^{\gamma-\rho} \le K_{\rho,\gamma} \cdot (t-s)^\gamma$$

Then there is a universal explicit $C_{\rho,\gamma} < \infty$ such that

$$|A_{st}| \le C_{\rho,\gamma} (Q_\gamma + K_{\rho,\gamma}) (t-s)^\gamma \quad \forall s<t \in \mathbb{D}.$$

We next bound $Q_\gamma(A)$ for a RANDOM A.

THEOREM [Kol 2] (KOLMOGOROV CRITERION, RANDOM PART)

Let $A : \mathbb{D}^2_\leq \to \mathbb{R}$ be random and satisfy

$$\mathbb{E}[|A_{st}|^p] \leq c\,(t-s)^{\underbrace{p\cdot\gamma_0}_{>1}} \quad \forall\ s<t \in \mathbb{D}$$

for some $p, \gamma_0, c \in (0,\infty)$. Then

$$\mathbb{E}[|Q_\gamma(A)|^p] < \infty \quad \forall\, 0<\gamma< \gamma_0 - \frac{1}{p},$$

$$Q_\gamma(A) = \sup_{d \to d'} \frac{|A_{dd'}|}{(d'-d)^\gamma}$$

hence $Q_\gamma(A) < \infty$ a.s. Hence, if also $K_{p,\gamma}(A)<\infty$,

a.s. $\quad |A_{st}| \lesssim (t-s)^\gamma \quad \forall\ s<t \in \mathbb{D}$.

We then prove:

Theorem ③: For $h = (h_u)_{u\in[0,T]}$ a continuous, adapted process, consider the Itô integral (continuous version of)

$$I_t = I_0 + \int_0^t h_u\, dB_u \qquad h_u \in \mathcal{L}(\mathbb{R}^d, \mathbb{R})$$

Fix any $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$

(a) a.s. $I \in \mathcal{C}^\alpha$, that is for $0 \leq s < t \leq T$

$$|\delta I_{st}| \lesssim (t-s)^\alpha \longrightarrow \underset{C<\infty \text{ a.s.}}{\leq C(\omega)}(t-s)^\alpha$$

Note now that $\delta I_{st} = \int_s^t h_u \, dB_u$

$$\Rightarrow \delta I_{st} - h_s \underbrace{(B_t - B_s)}_{\mathbb{B}^1_{st}} = \int_s^t \underbrace{(h_u - h_s)}_{\delta h_{su}} \, dB_u$$

(b) If a.s. $h \in \mathcal{C}^\beta$, i.e. $|\delta h_{su}| \lesssim (u-s)^\beta$, for some $\beta \in (0,1]$, then a.s.

$$|\delta I_{st} - h_s \mathbb{B}^1_{st}| \lesssim (t-s)^{\alpha+\beta}$$

(c) If a.s. $|\delta h_{su} - h^1_s \mathbb{B}^1_{su}| \lesssim (u-s)^{\alpha+\eta}$, for some $\eta \in (0,1]$ and for some adapted $h^1 = (h^1_u)_{u \in [0,T]} \in \mathcal{C}^\eta$, then a.s.

$$|\delta I_{st} - h_s \mathbb{B}^1_{st} - h^1_s \mathbb{B}^2_{st}| \lesssim (t-s)^{2\alpha+\eta}$$

Last time we proved part (a). We now focus on part (b).

Proof of (b). Recall $I_t = \int_0^t h_u \, dB_u$

Set $R_{st} := \delta I_{st} - h_s \underbrace{\mathbb{B}'_{st}}_{(B_t - B_s)} = \int_s^t (h_u - h_s) \, dB_u$

Our goal is $|R_{st}| \lesssim (t-s)^{\alpha+\beta}$ a.s.

assuming $|h_u - h_s| \lesssim (s-u)^{\beta}$ i.e. $\|\delta h\|_{\beta} < \infty$ a.s.

We set $\gamma = \alpha + \beta$. and $\rho := \alpha \wedge \beta$

By Kol 1 it is enough to show that

$$Q_{\gamma}(R) < \infty \qquad \& \qquad K_{\rho,\gamma}(R) < \infty \quad \text{a.s.}$$

Let us focus first on $K_{\rho,\gamma}(R)$. We note that

$$\delta R_{sut} = \delta(\delta I - h \, \delta B)_{sut} = \delta h_{su} \cdot \delta B_{ut}$$

$$\Rightarrow K_{\rho,\gamma}(R) = \sup_{s<u<t} \frac{|\delta R_{sut}|}{\{(u-s) \wedge (t-u)\}^{\rho} (t-s)^{\gamma-\rho}}$$

$$= \sup_{s<u<t} \frac{|\delta h_{su}| \cdot |\delta B_{ut}|}{(u-s)^{\beta} \quad (t-u)^{\alpha}} \underbrace{\left\{ \frac{(u-s)^{\beta} (t-u)^{\alpha}}{\{(u-s) \wedge (t-u)\}^{\rho} (t-s)^{\gamma-\rho}} \right\}}_{\leq 1 \quad \text{(CLAIM)}}$$

$$\leq \|\delta h\|_{\beta} \cdot \|\delta B\|_{\alpha} < \infty \quad \text{a.s.}$$

To prove the claim, set $a := \frac{u-s}{t-s}$, $b := \frac{t-u}{t-s}$

then $a>0$, $b>0$, $a+b=1$ and we can write

$$\frac{(u-s)^\beta (t-u)^\alpha}{\{(u-s)\wedge(t-u)\}^\rho (t-s)^{\gamma-\rho}} = \frac{a^\beta b^\alpha}{(a\wedge b)^\rho \cancel{(a+b)^{\gamma-\rho}}} = \frac{a^\beta b^\alpha}{(a\wedge b)^{\alpha\wedge\beta}}$$

$$(\text{SAY } a<b) \qquad = a^{\beta-\alpha\wedge\beta} \cdot b^\alpha \leq 1$$

$$(\text{SAY } a>b) \qquad = a^\beta b^{\alpha-\alpha\wedge\beta} \leq 1$$

It remains to show that $Q_\gamma(R)<\infty$ a.s.

By Kol 2 it is enough to bound

$$\mathbb{E}[|R_{st}|^p] \leq c\,(t-s)^{\gamma_0 p}$$

$$\Rightarrow \quad Q_\gamma(R)<\infty \text{ a.s.} \qquad \forall\, \gamma<\gamma_0-\frac{1}{p}$$

Recall: $R_{st} = \int_s^t (h_u - h_s)\, dB_u$

By BDG:

$$\mathbb{E}\left[|R_{st}|^p\right] \leq c_p\, \mathbb{E}\left[|R_{st}|^2\right]^{p/2}$$

$$= c_p \left\{\int_s^t \mathbb{E}\left[(h_u-h_s)^2\right] du\right\}^{p/2}$$

$$\leq c_p\, \mathbb{E}\left[\|\delta h\|_\beta^2\right]^{p/2} \underbrace{\left\{\int_s^t (u-s)^{2\beta}\, du\right\}^{p/2}}_{\leq (t-s)^{p(\beta+\frac{1}{2})}}$$

Thus we proved the bound with any $p>0$ and $\gamma_0 = \beta + \frac{1}{2}$

$$\Rightarrow \quad \gamma < \gamma_0 - \frac{1}{p} = \beta + \frac{1}{2} - \frac{1}{p}$$

OK since $\gamma = \beta + \alpha$ and for $p \gg 1$ we have $\alpha < \frac{1}{2} - \frac{1}{p}$ since $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$.

The proof works under the assumption that

$$\mathbb{E}\left[\|\delta h\|_\beta^2\right] < \infty$$

which is stronger than our original assumption

$$\|\delta h\|_\beta < \infty \quad \text{a.s.}$$

However we can repeat the proof for the process

$$h_u^{(n)} := h_{u \wedge \tau_n}$$

$$\text{where} \quad \tau_n := \inf\{t : \|\delta h\|_{\beta,[0,t]} > n\}$$

By construction $\|\delta h^{(n)}\|_\beta \leq n$ so the proof applies.

Finally letting $n \to \infty$ yields the final statement.

SDEs / RDEs WITH A DRIFT

Consider the SDE with a drift

$$dY_t = b(Y_t)\,dt + \sigma(Y_t)\,dB_t \qquad \text{(SDE+)}$$

$$\iff \quad Y_t = Y_0 + \int_0^t b(Y_u)\,du + \int_0^t \sigma(Y_u)\,dB_u \quad \text{a.s.}$$

where $b : \mathbb{R}^K \to \mathbb{R}^K$ and $\sigma : \mathbb{R}^K \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^K)$

<u>Theorem</u>: If $\sigma \in C^2$ and $b \in C^0$, then a.s. any solution $Y = (Y_t)_{t\in[0,T]}$ of (SDE+) is also a solution of

$$\delta Y_{st} = b(Y_s)(t-s) + \sigma(Y_s)\mathbb{B}^1_{st} + \sigma_2(Y_s)\mathbb{B}^2_{st} + o(t-s) \quad \text{(RDE+)}$$

$$0 \leq s \leq t \leq T$$

If furthermore $\sigma, b \in C^3$ and σ, σ_2, b are globally Lipschitz, then a.s. both (SDE+) and (RDE+) have a unique solution, for any given $Y_0 \in \mathbb{R}^K$, and they coincide.

<u>Proof (SKETCH)</u> – We enrich the driving path $B = (B^1, \ldots, B^d)$ with a further "time component": $\tilde{B} : [0,T] \to \mathbb{R}^{d+1}$

$$\tilde{B}_t := (B_t, t) = (B^1_t, \ldots, B^d_t, t) \in \mathbb{R}^{d+1}$$

We correspondingly extend the Ito rough path $\mathbb{B}$:

$$\tilde{\mathbb{B}}^1_{st} := \delta\tilde{B}_{s,t} = (\delta B_{st}, t-s)$$

$$\tilde{\mathbb{B}}^2_{st} := \left(\begin{array}{c|c} \mathbb{B}^2_{st} = \int_s^t \delta B_{su} \otimes dB_u & \int_s^t \delta B_{su}\, du \\ \hline \int_s^t (u-s)\, dB_u & \int_s^t (u-s)\, du = \frac{(t-s)^2}{2} \end{array}\right)$$

We can show that $\tilde{B} \in \mathcal{C}^\alpha \quad \forall\, \alpha \in (\frac{1}{3}, \frac{1}{2})$

and that $\tilde{\mathbb{B}}$ is an α-rough path over $\tilde{B}$, a.s.

We then rewrite (RDE+) as the "usual" (RDE) w.r.t. $\tilde{\mathbb{B}}$:

$$\delta Y_{st} = \tilde{\sigma}(Y_s)\tilde{\mathbb{B}}^1_{st} + \tilde{\sigma}_2(Y_s)\tilde{\mathbb{B}}^2_{st} + o(t-s)$$

where $\tilde{\sigma}(y)(x,t) := \sigma(y)x + b(y)t$

so $\tilde{\sigma} : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^{d+1}, \mathbb{R}^k)$.

Ito vs. STRATONOVICH

Let $X = (X_t)_{t \in [0,T]} \in E \simeq \mathbb{R}^\ell$ be an adapted process and a semi-martingale, say a Ito process:

$$dX_t = \varphi_t \, dB_t + \psi_t \, dt$$

$$\iff \quad X_t = X_0 + \int_0^t \varphi_u \, dB_u + \int_0^t \psi_u \, du$$

for some $\varphi = (\varphi_u)_{u \in [0,T]} \in \mathcal{M}^2_{loc}(\mathcal{L}(\mathbb{R}^d, E))$,

$\psi = (\psi_u) \in \mathcal{M}^1_{loc}(E)$

When $E = \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k)$, we define the STRATONOVICH INTEGRAL

$$\int_0^t X_u \circ dB_u := \int_0^t X_u \, dB_u + \frac{1}{2} \langle X, B \rangle_t$$

$\circ$ → STRATONOVICH

where $\langle X, B \rangle_t := \int_0^t \mathrm{Tr}[\varphi_u] \, du$

In particular

$$\int_0^t B_u^i \circ dB_u^j = \int_0^t B_u^i \, dB_u^j + \frac{1}{2} \underbrace{\langle B^i, B^j \rangle_t}_{\delta_{ij} \cdot t}$$

We now consider the STRATONOVICH SDE

$$dY_t = b(Y_t) \, dt + \sigma(Y_t) \circ dB_t \qquad \text{(Strat-SDE)}$$

$$\iff \quad Y_t = Y_0 + \int_0^t b(Y_u) \, du + \int_0^t \sigma(Y_u) \circ dB_u$$

Last time we showed:

$$\sigma(Y_t) = \sigma(Y_0) + \int_0^t \sigma_2(Y_u)\,dB_u + \int_0^t \overbrace{p(Y_u)}^{\frac{1}{2}\nabla^2\sigma(Y_u)\cdot\sigma(Y_u)\sigma(Y_u)}\,du$$

$$\iff \quad d\sigma(Y_t) = \sigma_2(Y_t)\,dB_t + p(Y_t)\,dt$$

$$\Rightarrow \quad \langle \sigma(Y), B\rangle_t = \int_0^t \mathrm{Tr}_{\mathbb{R}^d}[\sigma_2(Y_u)]\,du$$

We thus rewrite (Strat-SDE) as a usual Itô (SDE) with a modified drift:

$$Y_t = Y_0 + \int_0^t \underbrace{\left\{ b(Y_u) + \frac{1}{2}\mathrm{Tr}_{\mathbb{R}^d}[\sigma_2(Y_u)]\right\}}_{\hat{b}(Y_u)} du + \int_0^t \sigma(Y_u)\,dB_u \qquad (*)$$

By what we showed previously, it follows that if $\sigma(\cdot)$ and $b(\cdot)$ are regular enough, $Y = (Y_t)_{t\in[0,T]}$ solves the SDE $(*)$ iff it solves the following RDE

$$\delta Y_{st} = \left(b(Y_s) + \frac{1}{2}\mathrm{Tr}_{\mathbb{R}^d}[\sigma_2(Y_s)]\right)(t-s) + \sigma(Y_s)\,\mathbb{B}^1_{st} + \sigma_2(Y_s)\,\mathbb{B}^2_{st} + o(t-s) \qquad (\S)$$

Let us now define the STRATONOVICH ROUGH PATH $\overline{\mathbb{B}}$ above Brownian motion B:

$$\bar{\mathbb{B}}^1_{st} = \delta B_{st} = B_t - B_s = \mathbb{B}^1_{st}$$

$$\bar{\mathbb{B}}^2_{st} = \int_s^t (B_u - B_s) \otimes \circ\, dB_u = \underbrace{\int_s^t (B_u - B_s) \otimes dB_u}_{\mathbb{B}^2_{st}} + \frac{1}{2} I_{\mathbb{R}^d} (t-s)$$

In components:

$$\left(\bar{\mathbb{B}}^S_{st}\right)^{ij} = \int_s^t (B^i_u - B^i_s)\, dB^j_u + \frac{1}{2}\delta_{ij}(t-s) = \begin{cases} \dfrac{(B^i_t - B^i_s)^2}{2} & (i=j) \\ \displaystyle\int_s^t \delta B^i_{su}\, dB^j_{su} \end{cases}$$

Then (§) can be written, replacing $\mathbb{B}^2$ by $\bar{\mathbb{B}}^2$, as

$$\delta Y_{st} = b(Y_s)(t-s) + \sigma(Y_s)\bar{\mathbb{B}}^1_{st} + \sigma_2(Y_s)\bar{\mathbb{B}}^2_{st} + o(t-s)$$

which is exactly the usual (RDE) where we replaced the Itô rough path $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$ by the Stratonovich rough path $\bar{\mathbb{B}} = (\bar{\mathbb{B}}^1, \bar{\mathbb{B}}^2)$.

WONG - ZAKAI

Consider the usual SDE (without a drift)

$$Y_t = Y_0 + \int_0^t \sigma(Y_u)\, dB_u \qquad \text{(Itô)}$$

$$Y_t = Y_0 + \int_0^t \sigma(Y_u) \circ dB_u \qquad \text{(Strat.)}$$

If we replace $B = (B_u)_{u \in [0,T]}$ by a smooth approximation $B^\varepsilon = (B^\varepsilon_u)_{u \in [0,T]}$, we can consider a usual ODE:

$$\dot{Y}^\varepsilon_t = \sigma(Y^\varepsilon_t)\,\dot{B}^\varepsilon_t \qquad (\varepsilon\text{-ODE})$$

$$\iff Y^\varepsilon_t = Y^\varepsilon_0 + \int_0^t \sigma(Y^\varepsilon_u)\,\dot{B}^\varepsilon_u\,du$$

Consider a smooth probability density $\rho : [-1,+1] \to [0,\infty)$

Define
$$\rho^\varepsilon(t) := \frac{1}{\varepsilon}\rho\left(\frac{t}{\varepsilon}\right) \qquad \rho^\varepsilon : [-\varepsilon,+\varepsilon] \to [0,\infty)$$

$$\int_{\mathbb{R}} \rho^\varepsilon(t)\,dt = 1$$

Define $$B^\varepsilon_t := (B * \rho^\varepsilon)_t = \int_{-\infty}^{+\infty} \rho^\varepsilon(t-s)\,B^\varepsilon_s\,ds$$

(Define $(B_t)_{t \in \mathbb{R}}$ as a two-sided BM, i.e. $(B_{-t})_{t \geq 0}$ is a BM indep. of $(B_t)_{t \geq 0}$.)

It is not difficult to show that, $\forall \alpha \in (\frac{1}{3}, \frac{1}{2})$,

as: $$B^\varepsilon \xrightarrow[\varepsilon \downarrow 0]{} B \text{ in } \mathcal{C}^\alpha$$

i.e. $$\|B^\varepsilon - B\|_\infty + \|\delta B^\varepsilon - \delta B\|_\alpha \xrightarrow[\varepsilon \downarrow 0]{} 0 \qquad (\text{on } [0,T]).$$

Natural question: does the (well-defined!) solution Y^ε of (ε-ODE) converge as $\varepsilon \downarrow 0$ to same limit Y?

THEOREM (WONG-ZAKAI) Assume that $\sigma \in C^3$ with
$\|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty + \|\nabla^3\sigma\|_\infty + \|\nabla\sigma_2\|_\infty + \|\nabla^2\sigma_2\|_\infty < \infty$.
Then, $\forall \alpha \in (\frac{1}{3}, \frac{1}{2})$, we have

$$\underbrace{d_{\varphi\alpha}(Y^\varepsilon, Y)}_{\|Y^\varepsilon - Y\|_\infty + \|\delta Y^\varepsilon - \delta Y\|_\alpha} \xrightarrow[\varepsilon \downarrow 0]{} 0 \quad \text{in probability}$$

where Y is the solution of (Strat-SDE).

Proof. Define the CANONICAL ROUGH PATH $\mathbb{B}^\varepsilon$ associated to the path B^ε:

$$(\mathbb{B}^\varepsilon)^1_{st} = B^\varepsilon_t - B^\varepsilon_s \qquad (\mathbb{B}^\varepsilon)^2_{st} = \int_s^t (B^\varepsilon_u - B^\varepsilon_s) \otimes \dot{B}^\varepsilon_u \, du$$

We showed that the solution Y^ε of (ε-ODE) satisfies the following RDE:

$$\delta Y^\varepsilon_{st} = \sigma(Y^\varepsilon_s)(\mathbb{B}^\varepsilon)^1_{st} + \sigma_2(Y^\varepsilon_s)(\mathbb{B}^\varepsilon)^2_{st} + o(t-s)$$

We proved that the general RDE

$$\delta Z_{st} = \sigma(Z_s)\, \mathbb{X}^1_{st} + \sigma_2(Z_s)\, \mathbb{X}^2_{st} + o(t-s)$$

admits existence & uniqueness of solutions and also CONTINUOUS DEPENDENCE: $Z = \Phi(z_0, \mathbb{X})$ is a continuous function of $z_0 \in \mathbb{R}^k$ and $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$.

If we show that $\mathbb{B}^\varepsilon \longrightarrow \bar{\mathbb{B}}$ (Stratonovich rough path) then $Y^\varepsilon \to Y$ solution of Strat-RDE $\Longleftrightarrow$ Strat-SDE.

Why $\mathbb{B}^\varepsilon \to \bar{\mathbb{B}}$ Stratonovich, and not Ito?

$$(\mathbb{B}^\varepsilon)^1_{st} = B^\varepsilon_t - B^\varepsilon_s \longrightarrow B_t - B_s = \bar{\mathbb{B}}^1_{st}$$

$$\left((\mathbb{B}^\varepsilon)^2_{st}\right)^{ii} = \int_s^t (B^\varepsilon_u - B^\varepsilon_s)^i \cdot (\dot{B}^\varepsilon_u)^i du = \frac{\left[(B^\varepsilon_t)^i - (B^\varepsilon_s)^i\right]^2}{2}$$

$$\downarrow \varepsilon \downarrow 0$$

$$\int_s^t (f(u) - f(s))\, \dot{f}(u)\, du$$

with $f(t) = (B^\varepsilon_t)^i$

$$\frac{\left[B^i_t - B^i_s\right]^2}{2} = \left(\bar{\mathbb{B}}^2_{st}\right)^{ii}$$

STRATONOVICH!

$\neq$ ITO

For $i \neq j$ one can show that

$$\left((\mathbb{B}^\varepsilon)^2_{st}\right)^{ij} = \int_s^t (B^\varepsilon_u - B^\varepsilon_s)^i (\dot{B}^\varepsilon_u)^j du \xrightarrow[\varepsilon \downarrow 0]{} \int_s^t (B_u - B_s)^i dB^j_u$$

$$= \left(\bar{\mathbb{B}}^2_{st}\right)^{ij} = \left(\mathbb{B}^2_{st}\right)^{ij}$$

STRATONOVICH = ITO.